MATHEMATICS OF COMPUTATION
VOLUME 58, NUMBER 198
APRIL 1992, PAGES 815–830

COMPUTATION OF THE ZEROS OF p-ADIC L-FUNCTIONS

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ABSTRACT. The authors have computed the zeros of the Kubota-Leopoldt p-adic L-functions $L_p(s, \chi)$ for some small odd primes p and for a number of Dirichlet characters χ. The zeros of the corresponding Iwasawa power series $f_\theta(T)$ are also computed. The characters χ (associated with quadratic extensions of the pth cyclotomic field) are chosen so as to cover as many different splitting types of $f_\theta(T)$ as possible. The λ-invariant of this power series, equal to its number of zeros, assumes values up to 8.

The article is a report on these computations and their results, including the required theoretical background. Much effort is devoted to a study of the accuracy of the computed approximations.

1. INTRODUCTION

This article is a report on our computations of the zeros of the Kubota-Leopoldt p-adic L-functions $L_p(s, \chi)$, where p is an odd prime and χ a Dirichlet character. Some theoretical results about the zeros are also established.

Let $f_\theta(T)$ denote the Iwasawa power series representing $L_p(s, \chi)$; here θ, the so-called first factor of χ (for p), is assumed to be nonprincipal. A major part of our work consists in computing the zeros of $f_\theta(T)$. These, besides being of interest in their own right, have a close relationship to the zeros of $L_p(s, \chi)$. They are both theoretically and computationally easier to treat than the latter; in particular, their number equals λ_θ, the λ-invariant of the power series.

Several mathematicians have previously computed zeros of $L_p(s, \theta)$ and $f_\theta(T)$; see [7, 10, 11, 12, 13]. Our computations followed the same main lines as those by Wagstaff in [13]. However, apart from a few examples (in [13]), all the earlier computations deal with the case $\lambda_\theta = 1$, whereas we let λ_θ vary from 2 to 6. We also have partial results with $\lambda_\theta = 7$ and 8. The characters θ we used in the computations are mainly selected from our recent work [3]. They turned out to represent quite a great variety of different cases, as regards the splitting type and splitting field of $f_\theta(T)$. It was our intention to elaborate a systematic approach applicable to all of these cases, including a thorough study of the accuracy of the computed approximations.

In the material computed by us there are also many examples leading to zeros of $L_p(s, \chi)$ for characters $\chi \neq \theta$, i.e., for χ not of the "first kind". These

Received July 3, 1990; revised March 8, 1991.

1991 *Mathematics Subject Classification*. Primary 11S40, 11Y05, 11Y40, 11R20, 11R23.

Key words and phrases. p-adic L-functions, computation of zeros, factorization of polynomials, Newton's tangent method, Abelian fields, Iwasawa theory.

results, as well as some other results pertaining to more complicated cases, will be presented in the forthcoming Part II of the article.

The computations for the present research were performed by the first author. The second author did most of the theoretical work while spending the academic year 1988–89 at the Ohio State University in Columbus, Ohio. He would like to thank the Department of Mathematics at the OSU for its hospitality and the Academy of Finland for financial support.

2. Preliminaries

For an odd prime p let us fix an embedding of the field of algebraic numbers into $\mathbb{C}_p$, the completion of an algebraic closure of the p-adic field $\mathbb{Q}_p$. Let v_p denote the p-adic exponential valuation on $\mathbb{C}_p$, normalized so that $v_p(p) = 1$.

The p-adic L-function $L_p(s, \chi)$ attached to a Dirichlet character χ, always assumed primitive, was first introduced by Kubota and Leopoldt in [9]. Recall that $L_p(s, \chi)$ is a meromorphic function on

$$D_s = \left\{ s \in \mathbb{C}_p : v_p(s) > -1 + \frac{1}{p-1} \right\}$$

satisfying

$$L_p(1-m, \chi) = (1 - \chi_m(p)\, p^{m-1})\, L(1-m, \chi_m) \qquad (m = 1, 2, \ldots),$$

where $L(s, \chi)$ denotes the Dirichlet L-function and $\chi_m = \chi\omega^{-m}$, ω being the Teichmüller character mod p. In fact, $L_p(s, \chi)$ is analytic on D_s, except that there is a pole at $s = 1$ when $\chi = 1$. We assume that χ is even, i.e., $\chi(-1) = 1$, since otherwise $L_p(s, \chi)$ vanishes identically.

As in the Introduction, let θ be the first factor of χ. Write $\chi = \theta\psi_n$, where ψ_n is of order p^n $(n \geq 0)$. Note that the conductor of θ is of the form d or dp with d prime to p, and the conductor of ψ_n equals p^{n+1} (or 1 if $n = 0$).

Let $\mathcal{O}_\theta$ denote the ring of integers in the extension of $\mathbb{Q}_p$ generated by all the values of θ. If θ is nonprincipal, there is a power series

$$f_\theta(T) = \sum_{j=0}^{\infty} a_j T^j \in \mathcal{O}_\theta[[T]]$$

such that

$$L_p(s, \chi) = f_\theta(\rho_n(1+dp)^s - 1) \tag{2.1}$$

with $\rho_n = \psi_n(1+dp)^{-1}$. For $\theta = 1$, $f_\theta(T)$ is to be replaced by a quotient of two power series.

Assume that $\theta \neq 1$. Viewing $f_\theta(T)$ as a function of T we see that it is analytic on the open unit disc

$$D_T = \{T \in \mathbb{C}_p \colon v_p(T) > 0\}.$$

By the Ferrero-Washington theorem, $v_p(a_j)$ vanishes for some $j \geq 0$. Thus the p-adic Weierstrass preparation theorem enables one to write

$$f_\theta(T) = u_\theta(T)(b_0 + b_1 T + \cdots + b_{\lambda-1}T^{\lambda-1} + T^\lambda), \tag{2.2}$$

where $u_\theta(T)$ is an invertible power series in $\mathcal{O}_\theta[[T]]$, the coefficients b_j belong to the maximal ideal of $\mathcal{O}_\theta$, and $\lambda = \lambda_\theta$ is the λ-invariant of $f_\theta(T)$,

$$\lambda_\theta = \min\{j \geq 0 \colon v_p(a_j) = 0\}.$$

From this it follows that $f_\theta(T)$ has exactly λ zeros $T_1, \dots, T_\lambda$ in D_T (not necessarily distinct).

If $\theta = 1$, then

$$f_\theta(T) = \frac{g_\theta(T)}{1-(1+p)/(1+T)}$$

with $g_\theta(T) \in \mathcal{O}_\theta[[T]]$, $g_\theta(T)$ invertible (e.g., [14, p. 125]). This formal identity of power series indeed gives an equation in $\mathbb{C}_p$ for every $T \in D_T$ except for $T = p$ (corresponding to the pole of $L_p(s, 1)$). Consequently, $f_\theta(T)$ has no zeros for $\theta = 1$, and also $L_p(s, \psi_n)$ is nonzero in D_s.

Henceforth we will exclude the case $\theta = 1$.

To determine the zeros of $L_p(s, \chi)$, we first compute the λ zeros of $f_\theta(T)$. Hence, one has to understand the relationship between the zeros of these functions. This will be discussed in the next section.

3. Relationship between the zeros of L_p and f_θ

The results of this section are partly due to Childress and Gold [2]. Our presentation is somewhat different, however.

By (2.1), the mapping

$$\kappa_\chi : D_s \to D_T, \qquad \kappa_\chi(s) = \rho_n(1+dp)^s - 1 \tag{3.1}$$

sends any zero of $L_p(s, \chi)$ to a zero of $f_\theta(T)$. To study κ_χ more closely, set

$$D_T^0 = \left\{T \in \mathbb{C}_p : v_p(T) > \frac{1}{p-1}\right\},$$

$$C_n = \left\{T \in \mathbb{C}_p : v_p(T) = \frac{1}{(p-1)p^{n-1}}\right\} \qquad (n = 1, 2, \dots).$$

Proposition 1. *Let* $\chi = \theta\psi_n$. *If* $s \in D_s$, *then*

$$T = \kappa_\chi(s) \in \begin{cases} D_T^0 & \text{for } n = 0, \\ C_n & \text{for } n \geq 1. \end{cases}$$

Proof. Introducing the p-adic exponential and logarithm functions, we may write

$$T = \rho_n(\exp(s\log(1+dp)) - 1) + \rho_n - 1.$$

Since

$$v_p(\exp(s\log(1+dp)) - 1) = v_p(s\log(1+dp)) = v_p(s) + 1 > \frac{1}{p-1},$$

$$v_p(\rho_n - 1) = \frac{1}{(p-1)p^{n-1}} \qquad (n = 1, 2, \dots),$$

and since $\rho_0 = 1$, the assertion follows. □

Let T_0 be a zero of $f_\theta(T)$ in D_T. If there exists a character $\chi = \theta\psi_n$ and a zero s_0 of $L_p(s, \chi)$ such that $\kappa_\chi(s_0) = T_0$, we say that the zero s_0—or, more precisely, the pair (ψ_n, s_0)—*corresponds* to T_0. As observed by Washington [15, p. 351], such a pair (ψ_n, s_0), if it exists, is unique. Indeed, an equation of the form

$$\rho_{n_1}(1+dp)^{s_1} = \rho_{n_2}(1+dp)^{s_2}$$

implies, upon raising to the $p^{n_1+n_2}$ th power, that $s_1 = s_2$. Hence, moreover, $\rho_{n_1} = \rho_{n_2}$, and so $\psi_{n_1} = \psi_{n_2}$.

By Proposition 1, a necessary condition for the existence of a zero s_0 corresponding to T_0 is that T_0 lies in D_T^0 or on one of the C_n. The next propositions provide some sufficient conditions.

Remark. In [2] it is shown that there *always* exist zeros of more general L-functions corresponding to T_0 (in the above sense). These functions, p-adic L-functions over totally real fields, are not discussed in the present work.

Proposition 2. *If $f_\theta(T_0) = 0$ and $T_0 \in D_T^0$, then there is a zero s_0 of $L_p(s, \theta)$ corresponding to T_0. In fact,*

$$s_0 = \frac{\log(1+T_0)}{\log(1+dp)}. \tag{3.2}$$

Proof. Properties of the exp and log function imply that the mapping $\kappa_\theta\colon D_s \to D_T^0$ has the inverse

$$\kappa_\theta^{-1}\colon D_T^0 \to D_s, \qquad \kappa_\theta^{-1}(T) = \frac{\log(1+T)}{\log(1+dp)}. \quad \square$$

Proposition 3. *Let $T_0 \in C_n$ be a zero of $f_\theta(T)$. There is a zero s_0 of some $L_p(s, \theta\psi_n)$ corresponding to T_0 if and only if $\log(1+T_0)/\log(1+dp) \in D_s$. If this is the case, then ψ_n is determined by*

$$\rho_n = \psi_n(1+dp)^{-1}, \qquad v_p(1-\rho_n+T_0) > \frac{1}{p-1}, \tag{3.3}$$

and s_0 is given by (3.2).

Proof. If s_0 corresponds to T_0, it follows from the equation $T_0 = \rho_n(1+dp)^{s_0} - 1$ that s_0 satisfies (3.2). Thus, in particular, $\log(1+T_0)/\log(1+dp) \in D_s$.

Conversely, assume that $s_0 = \log(1+T_0)/\log(1+dp) \in D_s$. Then

$$\log((1+dp)^{s_0}) = s_0 \log(1+dp) = \log(1+T_0),$$

and so, since $(1+dp)^{s_0}$ and $1+T_0$ are p-adic units,

$$1+T_0 = \zeta(1+dp)^{s_0} \qquad (\zeta \text{ a root of } 1). \tag{3.4}$$

By writing this in the form

$$1-\zeta+T_0 = \zeta((1+dp)^{s_0}-1) \tag{3.5}$$

and by using the fact that $T_0 \in C_n$, we find (cf. the proof of Proposition 1) that $v_p(1-\zeta) = v_p(T_0) = 1/(p-1)p^{n-1}$. Consequently, ζ is of order p^n, and we may set $\zeta = \rho_n = \psi_n(1+dp)^{-1}$ for some character ψ_n. By (3.4), $L_p(s_0, \theta\psi_n) = f_\theta(T_0) = 0$. Moreover, by (3.5),

$$v_p(1-\rho_n+T_0) = v_p((1+dp)^{s_0}-1) > \frac{1}{p-1}.$$

It remains to show that the inequality in (3.3) determines ρ_n uniquely. Let ρ_n' be any root of unity with p-power order such that $v_p(1-\rho_n'+T_0) > 1/(p-1)$. Then $v_p(\rho_n - \rho_n') > 1/(p-1)$. Therefore, $\rho_n' = \rho_n$. $\square$

With the assumption of Proposition 3, when does $\log(1+T_0)/\log(1+dp)$ lie in D_s? Some results about this are presented in the following Propositions 4 and 5.

For the rest of this section we suppose that $T_0 \in C_n$ is a zero of $f_\theta(T)$ and s_0 is given by (3.2). Let ζ_m denote a primitive mth root of 1.

Proposition 4. *If $s_0 \in D_s$, then $\zeta_{p^n} \in \mathbb{Q}_p(T_0)$.*

Proof. Since ρ_n is a primitive p^nth root of 1, its conjugates ρ satisfy $v_p(\rho-\rho_n) = v_p(1-\rho^{-1}\rho_n) \le 1/(p-1)$ whenever $\rho \ne \rho_n$. Hence the inequality in (3.3) yields by Krasner's lemma that $\rho_n \in \mathbb{Q}_p(1+T_0)$. □

Proposition 5. (i) *Let θ satisfy $\mathcal{O}_\theta = \mathbb{Z}_p$ and $\lambda_\theta = (p-1)p^{n-1}$. If $s_0 \in D_s$, then $T_0 \in \mathbb{Q}_p(\zeta_{p^n})$.*

(ii) *Conversely, if $n = 1$ and $T_0 \in \mathbb{Q}_p(\zeta_p)$, then $s_0 \in D_s$.*

Proof. (i) Recall by (2.2) that T_0 is a zero of a polynomial in $\mathbb{Z}_p[T]$ of degree $(p-1)p^{n-1}$. Thus, if $s_0 \in D_s$, it follows from Proposition 4 that $\mathbb{Q}_p(T_0) = \mathbb{Q}_p(\zeta_{p^n})$.

(ii) Suppose that $v_p(T_0) = 1/(p-1)$ and $T_0 \in \mathbb{Q}_p(\zeta_p)$, say

$$T_0 \equiv a\pi \pmod{\pi^2} \qquad (a \in \mathbb{Z} \setminus p\mathbb{Z}),$$

where $\pi = 1 - \zeta_p$. We have $p = \varepsilon\pi^{p-1}$ with a p-adic unit ε,

$$\varepsilon = \prod_{k=1}^{p-1} \frac{1-\zeta_p^k}{1-\zeta_p} \equiv \prod_{k=1}^{p-1} k \equiv -1 \pmod{\pi}.$$

Consequently,

$$\log(1+T_0) \equiv T_0 + \frac{T_0^p}{p} \equiv a\pi + \frac{a^p\pi}{\varepsilon} \equiv (a-a^p)\pi \equiv 0 \pmod{\pi^2},$$

and so $v_p(s_0) = v_p(\log(1+T_0)) - 1 > 1/(p-1) - 1$. □

With this in mind, a first interesting open question is whether Proposition 5(ii) can be generalized to $n > 1$. In the numerical material so far generated by us there are only few zeros T_0 lying in some C_n for $n > 1$. These are examples with $p = 3$, $n = 2$, and the zeros T_0 never belong to $\mathbb{Q}_3(\zeta_9)$.

4. Plan of the computation

Tables of zeros of $f_\theta(T)$ and $L_p(s, \theta)$ were first published by Iwasawa and Sims [7] and by Wagstaff [12] who were concerned with $\theta = \omega^t$, a power of the Teichmüller character mod p, for a number of irregular primes p. Sunseri [11] expanded these results by computing considerably more places for the zeros up to $p < 1000$.

In [13] Wagstaff reported on new computations with $\theta = \theta_m$, the quadratic character of the field $\mathbb{Q}(\sqrt{m})$, for several positive primes $m \equiv 1 \pmod 4$ and for $p = 3$ and 5. His results were partly extended by Lamprecht and Zimmer [10] to $p \le 13$. Also, [15] contains some interesting data about the zeros of $L_3(s, \theta_m)$.

With a few exceptions in [13], all the above results are about the case $\lambda_\theta = 1$. Hence, for the characters θ in question, there is always a unique zero $T_0 \in p\mathbb{Z}_p$ of $f_\theta(T)$ and a zero s_0 of $L_p(s, \theta)$ corresponding to T_0. Moreover, in all cases one has $v_p(T_0) = 1$ and, thus, $v_p(s_0) = 0$.

In the present work we are concerned with (even) characters

$$\theta = \theta_m \omega^t \qquad (p \nmid m, \ 0 \le t \le p-2)$$

for $p = 3$, 5, 7, and 11, allowing both positive and negative values, not necessarily prime, for m. For characters of this form, values of λ_θ were computed in [3]. We selected examples with $\lambda_\theta > 1$, trying to find as many different types of cases as possible. The computation procedure was organized to follow the same main lines as that in [13]. Thus, there were the following four principal steps:

(1) computing $L_p(s, \theta)$ as a power series in s from Washington's formula (see (5.1) below),

(2) converting this function by (2.1) into the power series $f_\theta(T)$,

(3) computing the zeros of $f_\theta(T)$ by Newton's tangent method, and

(4) converting these zeros to the zeros of $L_p(s, \theta\psi_n)$ according to Propositions 2 and 3.

Note that $f_\theta(T)$ is not canonically defined. Following Washington, we adopt definition (2.1) which is quite natural, while [13] and [10] define

$$L_p(s, \theta) = -f_\theta((1+p)^{-s} - 1).$$

The p-adic values of the coefficients and of the zeros of $f_\theta(T)$ of course remain invariant.

We performed several additional computations to check the results at each step. For example, when there were zeros s_0 of $L_p(s, \theta\psi_n)$ with $\psi_n \neq 1$ (which in fact happened for $p = 3$ and $n = 1$ only), we also computed this L-function to verify the vanishing of $L_p(s_0, \theta\psi_n)$.

To come up with a representative sample of examples, we divided the computing process into two parts. First, in Program A, all characters θ with $\lambda_\theta > 1$ from our earlier work [3] were run through the steps (1) and (2). Here, the power series were computed just to a few places of the very first coefficients. In most cases this was sufficient to determine the p-adic values of the zeros of $f_\theta(T)$; if it was not, the computations were repeated with a greater number of places.

In Program B, we performed the entire procedure (1)–(4) for characters chosen on the basis of Program A. Some examples were also taken from Kobayashi [8]. The number of computed places varied case by case depending on the type of the example. We point out that step (1) takes much more time to execute than the other steps, and the time grows rapidly when the number of places is increased.

Program B contains cases with λ_θ up to 6. From Program A we also have some results about a few cases in which $\lambda_\theta = 7$ or 8.

The main part of the computations was carried out in 1988–89 on an IBM 3033 computer at the University of Turku.

A detailed description of the computation procedure appears in the following sections. In §§5–9 problems that arose at each step are studied from a theoretical point of view, while §11 in the Supplements section at the end of this issue gives more information about the actual computations. Results are presented in §10. Finally, §12 of the Supplements section is devoted to the proofs of some propositions of a more technical nature.

Henceforth in this paper, we let θ denote a nonprincipal character of the form $\theta_m \omega^t$.

5. Computing a p-adic L-function

Washington's formula [14, p. 57] reads

$$L_p(s, \chi) = \frac{1}{s-1}\frac{1}{\Phi}\sum_{\substack{a=1 \\ p\nmid a}}^{\Phi} \chi(a)\langle a\rangle^{1-s}\sum_{j=0}^{\infty}\binom{1-s}{j}B_j\left(\frac{\Phi}{a}\right)^j, \tag{5.1}$$

where Φ is any common multiple of p and the conductor of χ; the notation $\langle a\rangle$ stands for $a\omega^{-1}(a)$, and B_j denotes the jth Bernoulli number (with $B_1 = -1/2$). When using this formula for the computation of $L_p(s, \theta)$, the optimal choice of Φ is

$$\Phi = dp \quad \text{with } d = \text{ conductor of } \theta_m \text{ (either } |m| \text{ or } 4|m|).$$

As in [13], expand the right-hand side of (5.1) in a power series of s by writing

$$\langle a\rangle^{1-s} = \sum_{\nu=0}^{\infty}\binom{1-s}{\nu}(\langle a\rangle - 1)^{\nu}$$

and expanding the binomial coefficients in powers of s. This will be discussed more closely in §11 of the Supplements section. As a result, one obtains

$$L_p(s, \theta) = \sum_{i=0}^{\infty} u_i s^i \tag{5.2}$$

with coefficients u_i in $\mathbb{Q}_p$. It follows from [14, Theorem 5.12] (see also Proposition 6 below) that in fact $u_i \in \mathbb{Z}_p$.

By the computation of $L_p(s, \theta)$ we mean the calculation of rational integers $\overline{u}_i$ such that

$$u_i \equiv \overline{u}_i \pmod{p^M} \qquad \text{for } i = 0, \dots, i_M, \tag{5.3}$$

where M is a suitably chosen integer and i_M satisfies

$$i_M \geq M - 1, \qquad u_i \equiv 0 \pmod{p^M} \quad \text{for all } i > i_M. \tag{5.4}$$

To obtain a value for i_M, we need a lower bound for $v_p(u_i)$. The following estimate is sharper and more convenient than those given in [13] and [10].

Proposition 6. *We have*

$$v_p(u_i) \geq i - \frac{i-1}{p-1} \qquad (i = 1, 2, \dots).$$

Proof. The idea is to compare the expansion in (5.2) with that deriving from (2.1). First observe that

$$(1+dp)^s - 1 = \exp(s\log(1+dp)) - 1 = \sum_{i=1}^{\infty} d_i s^i$$

with $d_i = (\log(1+dp))^i/i!$. We have

$$v_p(d_i) = i - v_p(i!) \geq i - \frac{i-1}{p-1}. \tag{5.5}$$

Hence,

$$v_p(d_i s^i) > v_p(d_i) + i\left(-1 + \frac{1}{p-1}\right) > 0$$

for all $s \in D_s$, and so a general principle (e.g., [1, p. 284]) allows us to write the composite function

$$f_\theta((1+dp)^s - 1) = \sum_{j=0}^{\infty} a_j \left(\sum_{i=1}^{\infty} d_i s^i\right)^j$$

formally as a power series in s. Compared to (5.2), this implies that $u_0 = a_0$ and

$$u_i = \sum_{j=1}^{i} a_j d_{ij} \qquad \text{with} \qquad d_{ij} = \sum_{t_1+\cdots+t_j=i} d_{t_1}\cdots d_{t_j} \qquad (i = 1, 2, \ldots).$$

Since $v_p(a_j) \geq 0$, we have to check that $v_p(d_{ij}) \geq i - (i-1)/(p-1)$ for $j = 1, \ldots, i$. In view of (5.5) this is indeed the case. □

Remark. An analogous result for the coefficients in the expansion of $L_p(s, \theta\psi_n)$ will be proved in Part II.

6. Computing the Iwasawa power series

If $T \in D_T^0$, then $\log(1+T)/\log(1+dp) \in D_s$ and we have, by (2.1),

$$f_\theta(T) = L_p\left(\frac{\log(1+T)}{\log(1+dp)}, \theta\right). \tag{6.1}$$

We now obtain the coefficients of the power series $f_\theta(T) = \sum_{j=0}^{\infty} a_j T^j$ from this formula by using our expansion (5.2) for $L_p(s, \theta)$. Write

$$\frac{\log(1+T)}{\log(1+dp)} = \frac{1}{p}\sum_{j=1}^{\infty} e_j T^j \quad \text{with} \quad e_j = \frac{(-1)^{j-1}}{j}\frac{p}{\log(1+dp)}, \tag{6.2}$$

and note that $v_p(p^{-1}e_j T^j) > -1 + 1/(p-1)$ for all $j \geq 1$, provided $v_p(T) > v_0$, where v_0 is a suitable constant, $v_0 \geq 1/(p-1)$. Consequently, by the substitution principle employed in the proof of Proposition 6, we find that

$$L_p\left(\frac{\log(1+T)}{\log(1+dp)}, \theta\right) = \sum_{i=0}^{\infty} u_i \left(\frac{1}{p}\sum_{j=1}^{\infty} e_j T^j\right)^i = \sum_{j=0}^{\infty} a'_j T^j,$$

say, where $a'_0 = u_0$ and

$$a'_j = \frac{1}{p^j}\sum_{i=1}^{j} p^{j-i} u_i \sum_{t_1+\cdots+t_i=j} e_{t_1}\cdots e_{t_i} \qquad (j = 1, 2, \ldots). \tag{6.3}$$

Comparison with (6.1) now yields (and here the restriction $v_p(T) > v_0$ becomes superfluous)

$$a_j = a'_j \qquad (j = 0, 1, \ldots).$$

As mentioned in §2, the a_j are p-adic integers. Our computation of $f_\theta(T)$ consists of the calculation of rational integers $\overline{a}_j$ approximating a_j for $j =$

$0, \dots, M-1$. We do this by using (6.3), where u_i is replaced by its approximation $\overline{u}_i \bmod p^M$ as described in §5, and similarly e_j by an approximation $\overline{e}_j \bmod p^M$. Note that, in contrast to u_i, the numbers e_j do not tend (p-adically) to zero; in fact, e_j is generally not even a p-adic integer. By the congruence $e_j \equiv \overline{e}_j \pmod{p^M}$ we mean that $v_p(e_j - \overline{e}_j) \geq M$.

For the approximations $\overline{a}_j$ obtained in this way we may formulate the following result. The proof is postponed to §12 (see Supplements section).

Proposition 7. *We have* $a_j \equiv \overline{a}_j \pmod{p^{M-j}}$ *for* $j = 0, \dots, M-1$.

7. Computing zeros of a power series

Let

$$f(T) = \sum_{j=0}^{\infty} c_j T^j \in \mathscr{O}_F[[T]],$$

where $\mathscr{O}_F$ is the integer ring of F, a finite extension of $\mathbb{Q}_p$ (all extensions of $\mathbb{Q}_p$ are considered in $\mathbb{C}_p$). We will study the problem of computing zeros of $f(T)$, viewed as a function of T on D_T. This is then applied to the case when $f(T)$ is either $f_\theta(T)$ or the quotient of $f_\theta(T)$ by some of its linear factors.

Suppose we know that there is a simple zero $T_0 \in D_T$ in an extension E of F. Then $T_0 \in \pi\mathscr{O}_E$, where π is a prime element of $\mathscr{O}_E$. We let v_π denote the valuation on $\mathbb{C}_p$ to the base π, that is, $v_\pi = e v_p$, where e is the ramification index of $E/\mathbb{Q}_p$. Note that $v_\pi(f'(T_0))$ is a nonnegative integer (f' stands for the derivative of f).

Once a sufficiently good initial approximation for T_0 is known, one can compute T_0 to any desired accuracy by using Newton's tangent method. This is based on the following result.

Proposition 8 (Newton algorithm). *Let* $t_0 \in \pi\mathscr{O}_E$ *satisfy* $f(t_0) \equiv 0 \pmod{\pi^{2\gamma+1}}$, *where* $\gamma = v_\pi(f'(t_0))$. *Then there exists a unique* $T_0 \in \pi\mathscr{O}_E$ *such that*

$$f(T_0) = 0, \quad T_0 \equiv t_0 \pmod{\pi^{\gamma+1}}.$$

In fact, $T_0 = \lim_{n\to\infty} t_n$, *where* t_n *is defined by*

$$t_n = t_{n-1} - \frac{f(t_{n-1})}{f'(t_{n-1})} \qquad (n = 1, 2, \dots).$$

Proof. For the uniqueness which is crucial in the present work, one can give the following simple proof (omitted in several standard references).

Write $f(T) = (T - T_0)g(T)$ with $g(T) \in \mathbb{C}_p[[T]]$. Since $g(T_0) = f'(T_0)$, we have $\gamma = v_\pi(g(T_0)) = v_\pi(g(t_0))$. Thus, if U_0 is any zero congruent to $t_0 \pmod{\pi^{\gamma+1}}$, we see that $v_\pi(g(U_0)) = \gamma$ and so $g(U_0) \neq 0$. But $(U_0 - T_0)g(U_0) = f(U_0) = 0$, hence $U_0 = T_0$.

From the well-known existence proof we just quote the following formulas, valid for all $n \geq 1$:

$$t_n \equiv t_{n-1} \pmod{\pi^{\gamma+n}}, \tag{7.1}$$

$$f(t_n) \equiv 0 \pmod{\pi^{2\gamma+n+1}}, \tag{7.2}$$

$$v_\pi(f'(t_n)) = v_\pi(f'(T_0)) = \gamma. \quad \square \tag{7.3}$$

As regards the accuracy of the approximation t_n, (7.1) shows that

$$T_0 \equiv t_n \pmod{\pi^{\gamma+n+1}}. \tag{7.4}$$

In our real situation, the power series $f(T)$ is given approximately, in fact, as a polynomial. However, its zeros T_0 are approximately the same as the zeros of this polynomial because $T_0^j \to 0$ as $j \to \infty$. We formulate this in a quantitative form.

Let γ and t_n $(n \geq 0)$ be as in Proposition 8. Fix $N \in \mathbb{Z}$ so that $N > \gamma$. There is a polynomial

$$\overline{f}(T) = \sum_{j=0}^{N'} \overline{c}_j T^j \in \mathscr{O}_F[T] \tag{7.5}$$

with the following properties:

(i) $v_\pi(\overline{f}'(t_0)) = \gamma$, $\overline{f}(t_0) \equiv 0 \pmod{\pi^{2\gamma+1}}$;

(ii) if $\overline{t}_0, \overline{t}_1, \ldots$ is the sequence obtained for $\overline{f}(T)$ by the Newton algorithm, with the initial approximation $\overline{t}_0 = t_0$ and each $\overline{t}_n$ computed mod $\pi^{\gamma+n+1}$, then the zero $T_0 = \lim_{n\to\infty} t_n$ of $f(T)$ satisfies

$$T_0 \equiv \overline{t}_{N-\gamma} \pmod{\pi^{N+1}}. \tag{7.6}$$

In fact, such a polynomial $\overline{f}(T)$ is given by the following proposition, which will be proved in §12 of the Supplements section.

Proposition 9. *Choose $w \in \mathbb{Z}$ so that $1 \leq w \leq \gamma + 1$ and $t_0 \equiv 0 \pmod{\pi^w}$. Let $N' = [(N+\gamma)/w]$, where $[x]$ denotes the largest integer $\leq x$, and assume that $\overline{c}_j \in \mathscr{O}_F$ satisfies*

$$c_j \equiv \overline{c}_j \pmod{\pi^{N+\gamma+1-wj}} \qquad (j = 0, \ldots, N'). \tag{7.7}$$

Then the polynomial $\overline{f}(T)$ defined in (7.5) *has the properties* (i) *and* (ii).

8. Computing zeros of f_θ and L_p

Now consider the Iwasawa power series $f_\theta(T) = \sum_{j=0}^{\infty} a_j T^j \in \mathbb{Z}_p[[T]]$. In view of Proposition 7, we will assume that the a_j are approximated by $\overline{a}_j \in \mathbb{Z}$, so that $a_j \equiv \overline{a}_j \pmod{p^{M-j}}$. Set

$$\overline{f}_\theta(T) = \sum_{j=0}^{M-1} \overline{a}_j T^j.$$

Let T_0 be a simple zero of $f_\theta(T)$. As in the previous section, let E denote an extension of $\mathbb{Q}_p$ containing T_0; also keep the notations π and e associated with E.

Proposition 10. *Suppose that $t_0 \in \pi^c \mathscr{O}_E$ $(c \geq 1)$ satisfies*

$$T_0 \equiv t_0 \pmod{\pi^{\gamma+1}}, \qquad f_\theta(t_0) \equiv 0 \pmod{\pi^{2\gamma+1}},$$

where $\gamma = v_\pi(f'_\theta(t_0))$. The Newton algorithm applied to $\overline{f}_\theta(T)$, with t_0 as the initial approximation, produces a zero $\overline{T}_0$ satisfying

$$T_0 \equiv \overline{T}_0 \pmod{\pi^{\rho M-\gamma}} \tag{8.1}$$

with $\rho = \min(e, c)$, *provided that* $\rho \le \gamma + 1$ *and* $\rho M > 2\gamma + 1$.

Proof. Apply Proposition 9 with $N = \rho M - \gamma - 1$ and $w = \rho$, observing that $a_j \equiv \overline{a}_j \pmod{\pi^{\rho M - \rho j}}$. The claimed congruence follows from (7.6). □

Remarks. The last two inequalities in Proposition 10 are fulfilled whenever c and M are chosen appropriately. Hence, there remains the question how to find a field E containing a zero T_0 and how to determine an initial approximation t_0. This will be discussed in the next section.

Once one zero T_0 of $f_\theta(T)$ has been computed, it may be better to deal with $f_\theta(T)/(T - T_0)$ when calculating further zeros. This is the case, in particular, if $T_0 = 0$, but sometimes also for other $T_0 \in \mathbb{Q}_p$. The above congruences are then to be modified slightly.

Our final task is to determine the zeros s_0 of $L_p(s, \theta\psi_n)$ which correspond to each T_0 lying in D_T^0 or on some C_n $(n \ge 1)$. By Propositions 2 and 3, and by (6.2), s_0 is obtained from

$$s_0 = \frac{\log(1 + T_0)}{\log(1 + dp)} = \frac{1}{p} \sum_{j=1}^{\infty} e_j T_0^j;$$

in the case $T_0 \in C_n$ one moreover has to check that $s_0 \in D_s$. Note that the field E containing T_0 also contains s_0.

For the computation in practice, let $\overline{e}_j$ be defined by $e_j \equiv \overline{e}_j \pmod{p^M}$, as in §6, and set

$$\overline{s}_0 = \frac{1}{p} \sum_{j=1}^{J} \overline{e}_j \overline{T}_0^j, \tag{8.2}$$

where $\overline{T}_0$ is given by Proposition 10.

Proposition 11. *If* J *is large enough and* $\rho M \ge \gamma + e$, *then*

$$s_0 \equiv \overline{s}_0 \pmod{\pi^{\rho M - \gamma - e}}.$$

The proof, which also implies an estimate for J, appears in §12 of the Supplements section.

9. Starting the Newton algorithm

We are left with the question of how to start the Newton algorithm leading to an approximation of a zero T_0 of $f_\theta(T)$.

Suppose that $v_p(T_0)$ is known. In fact, we assume that we know the Newton polygon of $f_\theta(T)$, that is, the p-adic values of sufficiently many of its first coefficients $a_0, a_1, \ldots$ (these values were computed in Program A).

More precisely, we are going to find the (minimal) extension E of $\mathbb{Q}_p$ containing T_0 and to determine $t_0 \in \pi\mathcal{O}_E$ so that $f_\theta(t_0) \equiv 0 \pmod{\pi^{2\gamma+1}}$, where $\gamma = v_\pi(f'(t_0)) = v_\pi(f'(T_0))$. Then, by Proposition 8, t_0 works as an initial approximation in the computation of T_0. In particular, $T_0 \equiv t_0 \pmod{\pi^{\gamma+1}}$, so that one can proceed in the way described in §8 (Proposition 10).

The above conditions for t_0 include the implicit assumption that the zero T_0 be simple. This was satisfied in all examples computed by us. We point out that Ferrero and Greenberg [4] have proved that $s_0 = 0$ is at most a simple zero of $L_p(s, \chi)$. From this it follows that $T_0 = 0$ never occurs as a multiple zero of $f_\theta(T)$. Whether the same holds true for nonzero T_0 is not known.

In the following, we set

$$a_j = \sum_{k=0}^{\infty} a_{jk} p^k \qquad \text{with } 0 \leq a_{jk} < p \quad (j = 0, 1, \dots)$$

and restrict ourselves to two basic types of Newton polygon met in the computations. Our discussion not only covers the majority of all examples but also presents the essential features of the required techniques in most, though not all, of the remaining cases. The treatment of these will be completed in Part II.

The *first type* consists of polygons determined by the conditions $v_p(a_0) = 1$, λ prime to p. Hence, there is but one (finite) nonzero slope, and this equals $-1/\lambda$. For any zero T_0 one has $v_p(T_0) = 1/\lambda$, so that $E/\mathbb{Q}_p$ is fully ramified of degree λ. Set $q = (\lambda, p-1)$. Since the ramification is tame, we know (e.g., [5, Ch. II, §16]) that E is one of the q (nonconjugate) fields

$$\mathbb{Q}_p(\pi), \quad \pi^\lambda = rp \quad \text{with} \quad r \equiv g^b \pmod{p}, \quad 0 \leq b \leq q-1,$$

where g is a fixed primitive root of p. We also have

$$\gamma = v_\pi(f'(T_0)) = v_\pi(\lambda a_\lambda T_0^{\lambda-1}) = \lambda - 1.$$

To determine t_0, assume that $\lambda > 1$, the case $\lambda = 1$ being trivial ($t_0 = 0$). Write

$$T_0 \equiv x_1 \pi \pmod{\pi^2}, \qquad 0 < x_1 < p.$$

By Krasner's lemma, this fixes $E = \mathbb{Q}_p(T_0)$ uniquely (up to conjugates), since $v_p(\pi - \alpha\pi) = 1/\lambda$ whenever $\alpha^\lambda = 1$. Let us make this result explicit as follows. From $f_\theta(T_0) \equiv 0 \pmod{\pi^{\lambda+1}}$ we have that $a_{01} + a_{\lambda 0} r x_1^\lambda \equiv 0 \pmod{p}$, or

$$r x_1^\lambda \equiv -a_{01}/a_{\lambda 0} \pmod{p}.$$

This yields a unique solution for $r \bmod p$, and q solutions for $x_1 \bmod p$, corresponding to q zeros $T_0 \in E$ (note that there are λ/q extensions conjugate to E).

If $\lambda > 2$, we go on by setting

$$T_0 \equiv x_1 \pi + \cdots + x_{\lambda-1} \pi^{\lambda-1} \pmod{\pi^\lambda}.$$

Observe that

$$T_0^\lambda \equiv x_1^\lambda \pi^\lambda + c_1 \pi^{\lambda+1} + \cdots + c_{\lambda-2} \pi^{2\lambda-2} \pmod{\pi^{2\lambda-1}},$$

where the coefficients c_j are of the form $\lambda x_1^{\lambda-1} x_{j+1}$ plus terms depending only on $x_1, \dots, x_j$. The congruence $f_\theta(T_0) \equiv 0 \pmod{\pi^{2\gamma+1}}$ is easily seen to be equivalent to

$$A_1 + A_2 \pi + \cdots + A_{\lambda-2} \pi^{\lambda-3} \equiv 0 \pmod{\pi^{\lambda-2}}$$

with each $A_j = A_j(x_1, \dots, x_{j+1}) \in \mathbb{Z}$, hence equivalent to $A_1 \equiv \cdots \equiv A_{\lambda-2} \equiv 0 \pmod{p}$. Since

$$A_j \equiv \lambda a_{\lambda 0} r x_1^{\lambda-1} x_{j+1} + A_j'(x_1, \dots, x_j) \pmod{p},$$

where A_j' does not depend on x_{j+1}, we find that $x_{j+1} \bmod p$ becomes uniquely determined once x_1 is fixed ($j = 1, \dots, \lambda-2$). As final result we get that $t_0 = x_1 \pi + \cdots + x_{\lambda-1} \pi^{\lambda-1}$ satisfies the desired conditions.

The *second type* we consider is the case in which $v_p(a_0) = 2$ and $\lambda = 2$. Then, for both zeros $T_0 = T_0'$ and T_0'', say, $v_p(T_0) = 1$, and so E is one of the fields $\mathbb{Q}_p$, $\mathbb{Q}_p(\sqrt{g})$, $\mathbb{Q}_p(\sqrt{p})$, $\mathbb{Q}_p(\sqrt{gp})$. In any case, we may write

$$T_0 \equiv xp \pmod{p^{3/2}} \quad \text{with } x = u + v\sqrt{g} \neq 0,$$

where $0 \le u < p$, $0 \le v < p$. Now the congruence $f_\theta(T_0) \equiv 0 \pmod{p^{5/2}}$ becomes

$$a_{02} + a_{11}x + a_{20}x^2 \equiv 0 \pmod{p},$$

and this gives us two different solutions $x \in \mathbb{Z}$ or $x \in \mathbb{Z}[\sqrt{g}]$ according to whether the discriminant $D = a_{11}^2 - 4a_{02}a_{20}$ is a quadratic residue or nonresidue mod p, respectively, and a double root $x \in \mathbb{Z}$ if $D \equiv 0 \pmod{p}$.

In the former case ($D \not\equiv 0$) we have by Krasner's lemma $E = \mathbb{Q}_p$ for $x \in \mathbb{Z}$, while $E = \mathbb{Q}_p(\sqrt{g})$ for $x \in \mathbb{Z}[\sqrt{g}]$. Moreover, since $a_{11} + 2a_{20}x \not\equiv 0 \pmod{p}$, we see that $\gamma = v_p(f'(T_0)) = 1$. In this case, $f_\theta(xp) \equiv 0 \pmod{p^3}$, which shows that $t_0 = xp$ works as an initial approximation.

Suppose that $D \equiv 0 \pmod{p}$; then $v_p(T_0' - T_0'') \ge 3/2$. We will postpone the cases with $v_p(T_0' - T_0'') \ge 2$ to Part II and let here this distance be $3/2$. This means that the zeros are of the form $T_0 \equiv xp \pm yp\sqrt{rp} \pmod{p^2}$, where $r = 1$ or g and $0 < x < p$, $0 < y < p$. In particular, $E = Q_p(\pi)$ with $\pi = \sqrt{rp}$. Using the congruence $a_{11} + 2a_{20}x \equiv 0 \pmod{p}$, we find that

$$f_\theta'(T_0) \equiv \pm 2a_{20}yp\pi \pmod{\pi^4},$$

and so $\gamma = 3$. Hence, to obtain r and t_0 we must consider the congruence $f_\theta(T_0) \equiv 0 \pmod{\pi^7}$. This is equivalent to

$$a_{20}ry^2 \equiv -b - a_{03} - a_{12}x - a_{21}x^2 - a_{30}x^3 \pmod{p},$$

where $b \equiv (a_{02} + a_{11}x + a_{20}x^2)/p \pmod{p}$. Upon solving for r and y, we are done: take $t_0 = xp + yp\pi$.

10. Numerical results

A table including the main results from Program A appears in the Supplements section of this issue. This table gives a first approximation for $f_\theta(T)$, where θ runs through all the characters treated in this program (about 1150 in number).

A sample of the results from Program B is exhibited in Tables I–V of the Supplements. The present section contains a description of these tables together with some further examples.

Each item in the tables is headed by a triple (p, Δ, t) identifying the character $\theta = \theta_m\omega^t$. Here, Δ denotes the discriminant of $\mathbb{Q}(\sqrt{m})$, hence $\Delta = m$ or $4m$ and $|\Delta|$ equals d, the conductor of θ_m.

Of the two columns below the triple, the first (left) lists the first coefficients of $L_p(s, \theta) = \sum_{i=0}^{\infty} u_i s^i$ and the second (right) those of $f_\theta(T) = \sum_{j=0}^{\infty} a_j T^j$. For example, the first item of Table I (with $p = 3$ and $\theta = \theta_{-47}\omega$) gives the following data about the coefficients:

$$\begin{array}{ll} u_0 = 0 & a_0 = 0 \\ u_1 = 0.0020\cdots & a_1 = 0.0100\cdots \\ u_2 = 0.0120\cdots & a_2 = 1.1222\cdots \\ u_3 = 0.0012\cdots & a_3 = 1.002\cdots \\ u_4 = 0.0010\cdots & a_4 = 2.00\cdots \\ u_5 = 0.0000\cdots & a_5 = 1.1\cdots \\ u_6 = 0.0001\cdots & a_6 = (1.)\cdots \end{array}$$

All the nonzero p-adic numbers are written in the "decimal" form:

$$\sum_{\nu=-a}^{b} x_\nu p^\nu = x_{-a}x_{-a+1}\cdots x_0.x_1x_2\cdots x_b \qquad (0 \le x_\nu < p,\ a \ge 0,\ b \ge 0).$$

For $p = 11$ the digit $x_\nu = 10$ is denoted by α.

The coefficient columns are followed by the pairs T_k, s_k, for $k = 1, \dots, \lambda$, where T_k is a zero of $f_\theta(T)$ and $s_k = \log(1+T_k)/\log(1+dp)$. Observe that $v_p(s_k) = v_p(T_k) - 1$ if $T_k \neq 0$ and $v_p(T_k) > 1/(p-1)$. When s_k is not a zero of $L_p(s, \theta)$, it is denoted by s_k^*; this occurs only in Tables IV and V. In case T_k generates a proper extension E of $\mathbb{Q}_p$, it is preceded by a "standard" generator of E. If $E/\mathbb{Q}_p$ is ramified, this generator, a prime element of $\mathcal{O}_E$, is denoted by π (or π', π_k), otherwise by ξ.

Each nonzero value of u_i, a_j, T_k, s_k is an approximation. When we write 0 for a value of these numbers, it is exact. The approximations are truncated, according to the above propositions, to contain only correct digits. To save space, we in many cases list the coefficients u_i and a_j with an accuracy that is lower than actually computed. However, when tabulating u_i, we follow the principle that all the coefficients which are nonzero to the displayed accuracy are included (cf. (5.3) and (5.4)). Thus, in the above example,

$$u_i = 0.0000\cdots \qquad \text{for all } i \ge 7.$$

We now comment on the tables separately and offer some additional examples.

Table I provides examples in which $\lambda = 2$ and $a_0 = 0$. This is an easy case: apart from the trivial zero $T_1 = 0$, $f_\theta(T)$ has a unique zero $T_2 \in \mathbb{Q}_p$ and this can be computed by replacing $f_\theta(T)$ with $f_\theta(T)/T$ and choosing 0 as an initial approximation. In the examples of this type, it is not rare that $v_p(T_2) > 1$, while the previously known examples (with $\lambda = 1$) always are about zeros with p-ordinal 1. The maximum values we found for $v_p(T_2)$ occurred in the following cases:

EXAMPLE 1.

$$\begin{aligned}(3, -971, 1)\colon T_1 &= 0, \qquad s_1 = 0, \\ T_2 &= 0.000011, \qquad s_2 = 0.00021.\end{aligned}$$

EXAMPLE 2.

$$\begin{aligned}(3, -2351, 1)\colon T_1 &= 0, \qquad s_1 = 0, \\ T_2 &= 0.0000021, \qquad s_2 = 0.000012.\end{aligned}$$

In Table II we have $\lambda = 2$ and $a_0 \neq 0$. Here we give only examples with $v_p(T_k) = 1/2$ or 1; the cases with higher $v_p(T_k)$ will be discussed in Part II.

Note that if $p = 3$ and $v_3(T_k) = 1/2$, s_k is not a zero of $L_3(s, \theta)$. These cases are postponed to Part II as well.

The next example, being of the type of Table II, corrects some erroneous data in [13].

EXAMPLE 3.

$$\begin{aligned}(5, 109, 0)\colon T_{1,2} &= 0.4122 \pm 2.1134\sqrt{2 \cdot 5},\\ s_{1,2} &= 0.0322 \pm 33.211\sqrt{2 \cdot 5}.\end{aligned}$$

(With the definition of $f_\theta(T)$ used in [13] one has $T_{1,2} = 0.4434 \pm 2.3244\sqrt{2 \cdot 5}$.)

The examples in Table IIa, with $\lambda = 3$ and $a_0 = 0$, are analogous to those in Table II: exclude the trivial zero $T_1 = 0$ and consider the power series $f_\theta(T)/T$.

Table III is about the case $\lambda = 3$, $a_0 \neq 0$. Except for one example, the zeros lie in ramified cubic extensions. The number of such extensions (not counting conjugates) is one for $p = 5$, and three for $p = 7$. For $p = 3$, the ramification is wild. This case will be studied in Part II, but below we give just one example completing a discussion begun in [13].

EXAMPLE 4.

$$\begin{aligned}(3, 281, 0)\colon &\pi_k \text{ zero of } X^3 - 6X - 3 \qquad (k = 1, 2, 3),\\ &T_k \equiv \pi_k + \pi_k^3 \pmod{\pi_k^5},\\ &\text{no zero } s_k \text{ corresponding to } T_k.\end{aligned}$$

The next example, still about $p = 3$, falls into the same category ($\lambda = 3$, $a_0 \neq 0$) but is of another kind: $f_\theta(T)$ has a linear factor. This example was also computed in [13] (with some errors in the last digits).

EXAMPLE 5.

$$\begin{aligned}(3, 733, 0)\colon T_1 &= 0.12001, \qquad s_1 = 1.1220,\\ \pi &= \sqrt{-3},\\ T_{2,3} &= 0.20120 \pm 1.12000\pi,\\ s^*_{2,3} &= 1.2022 \pm 2.1010\pi\end{aligned}$$

(s_2^* and s_3^* are zeros of $L_3(s, \theta\psi_1)$; see Part II).

Table IIIa is again a companion to the preceding table, with $\lambda = 4$ and $a_0 = 0$.

Table IV lists examples in the case $\lambda = 4$, $a_0 \neq 0$. In all examples, $v_p(T_k) = 1/4$. Thus, we know about s_k that it is a zero of $L_p(s, \theta)$ for $p = 7$ but no zero of any $L_p(s, \theta\psi_n)$ for $p = 3$. For $p = 5$, the answer depends on $\mathbb{Q}_5(\pi)$; in our examples $\mathbb{Q}_5(\pi)$ never happens to be $\mathbb{Q}_5(\sqrt{-5}) = \mathbb{Q}_5(\zeta_5)$, and so there is no s_k corresponding to T_k (see Proposition 5).

Finally, Table V contains three examples with $\lambda = 5$ and one with $\lambda = 6$. (In the last case, λ was not computed in [3].) Again, most of the zeros T_k lie in quartic extensions, and the s_k obtained from these are no zeros of $L_p(s, \theta)$, not even zeros of $L_p(s, \theta\psi_n)$.

The previously computed tables [3, 8] list 12 further examples with $\lambda \geq 6$. We computed $v_p(T_k)$ for all zeros T_k in these examples. There are three cases (with $p = 3$) in which some of the zeros lie in wildly ramified sextic extensions, and we will return to them in Part II. In the remaining cases one can determine the extension $E = \mathbb{Q}_p(\pi)$, up to conjugates, from $v_p(T_k)$; they are as follows:

p	Δ	t	λ	π	Remarks
3	−239, −4088, −1427,−17252, −36140	1	6	$\sqrt[5]{3}$	$T_1 = 0$
3	−21592	1	7	$\sqrt[7]{3}$	–
3	−11156, −30584	1	8	$\sqrt[7]{3}$	$T_1 = 0$
7	−1371	1	7	$\sqrt[5]{7}$	$T_1 = 0$, $T_2 \in \mathbb{Q}_7$

On the basis of the numerical data computed so far it seems natural to expect that the zeros T_i and s_i are distributed randomly as regards their p-adic value (within the prescribed limits) and their inclusion in various extensions of $\mathbb{Q}_p$. Also the p-adic expansions of the zeros fail to show any regularity.

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