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Supplement to
MULTIPLICITIES OF DIHEDRAL DISCRIMINANTS

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4. NUMERICAL EXAMPLES

Generally, $p = 3$ in all examples. The information about the 14 totally real cubic discriminants with multiplicities 4, 6 has been taken from [10]. Data for the 69 complex cubic discriminants with multiplicities 6, 9 has kindly been made available by the authors of [5] in personal communication.

A common feature of all examples is that the 3-rank of the ring class group mod f of the quadratic field k is always $\rho + t + w - \delta(f) = 3$, whence the sum of all partial multiplicities equals $\frac{1}{2}(3^{\rho+t+w-\delta} - 1) = 13$.

Both parts ($w \leq 1$ and $w = 2$) begin with the unconstrained cases and are arranged according to increasing quadratic 3-class rank ρ of k. Correspondingly, the values of $t + w$ (resp. t) and of u decrease.

For any integer $n \geq 1$, denote by $\mathcal{O}_n$ the suborder $\mathbb{Z} \oplus \mathbb{Z}n\frac{1}{2}(d_k + \omega)$ of the maximal order $\mathcal{O}_k$, where $\omega = \sqrt{d_k}$. Then the unit group $U(\mathcal{O}_n)$ is exactly the intersection $U_k \cap \mathbb{Q}^\times(n) \cdot k_n^\times$.

In the totally real case, the fundamental unit of k is denoted by η. If $\rho = 0$, then the condition $\eta \in \mathcal{O}_n$ is certainly sufficient for $\delta(n) = 0$.

Part 1. Applications of Corollary 3.2 to conductors with $w \leq 1$.

a) A totally real cubic field with

$$\rho = 0,\ t = 2,\ w = 1,\ \delta_{\max} = 0,\ u = 3 = t + w,$$
$$f = q_1 \cdot q_2 \cdot q_3 \text{ with } q_1 = 2,\ q_2 = 7,\ q_3 = 3.$$

Here, we have $I_{k,3}(f) = U_k \cdot k^\times(f)^3$. There are no constraints from the fundamental unit η of k, since $\delta(q_1) = 0$, $\delta(q_2) = 0$ and $\delta(q_3) = 0$. The quadratic discriminant d_k is simultaneously congruent $-3 (\bmod 9)$ and congruent $5 (\bmod 8)$, since $2 \mid f$, $\left(\frac{d_k}{2}\right) \equiv 2 \equiv -1 (\bmod 3)$.

No.	d_L	d_k	η	
1	8 250 228	4677	$2\,796\,250\,463 + 40\,887\,672\omega$	$\in \mathcal{O}_2 \cap \mathcal{O}_7 \cap \mathcal{O}_3$

The partial multiplicities $m_3(d_k, f')$ of divisors f' of the conductor f can be determined by means of Corollary 3.2.

f'	1	q_1	q_2	q_3	q_1q_2	q_1q_3	q_2q_3	$q_1q_2q_3$
$m_3(d_k, f')$	0	1	1	1	2	2	2	4

b) 14 complex cubic fields with

$$\rho = 1,\ t + w = 2,\ \delta_{\max} = 0,\ u = 2 = t + w,$$
$$f = q_1 \cdot q_2 \text{ with } q_1 = 2.$$

In this case, $I_{k,3}(f)/k^\times(f)^3 = \langle\alpha\rangle$. There do not arise restrictions from the single generating principal ideal cube α of k, i. e., $\alpha \in \mathbb{Q}^\times(f) \cdot k_f^\times \cdot k^\times(f)^3$, since $\delta(q_1) = 0$ and $\delta(q_2) = 0$. All quadratic discriminants d_k are congruent $5 (\bmod 8)$, since $2 \mid f$. The class numbers of all cubic fields L are divisible by 9.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

No.	d_L	d_k		q_2	w
1	−225 612	−6267	$\equiv -3(9)$	3	1
2	−439 628	−2243	$\equiv +1(3)$	7	0
3	−483 948	−13443	$\equiv +3(9)$	3	1
4	−593 676	−16491	$\equiv -3(9)$	3	1
5	−649 612	−307	$\equiv -1(3)$	23	0
6	−729 708	−3723	$\equiv +3(9)$	7	0
7	−762 156	−21171	$\equiv -3(9)$	3	1
8	−772 300	−7723	$\equiv -1(3)$	5	0
9	−789 868	−547	$\equiv -1(3)$	19	0
10	−803 628	−22323	$\equiv -3(9)$	3	1
11	−812 300	−8123	$\equiv +1(3)$	5	0
12	−966 700	−9667	$\equiv -1(3)$	5	0
13	−968 300	−9683	$\equiv +1(3)$	5	0
14	−990 700	−9907	$\equiv -1(3)$	5	0

f'	1	q_1	q_2	q_1q_2
$m_3(d_k, f')$	1	3	3	6

c) 2 totally real cubic fields with

$$\rho = 1,\ t = 1,\ w = 1,\ \delta_{\max} = 0,\ u = 2 = t + w,$$
$$f = q_1 \cdot q_2 \text{ with } q_1 = 2,\ q_2 = 3.$$

Here, $I_{k,3}(f)/U_k \cdot k^\times(f)^3 = \langle\alpha\rangle$. Constraints neither arise from the fundamental unit η of k nor from the single generating principal ideal cube α of k, because $\delta(q_1) = 0$, $\delta(q_2) = 0$, whence $\alpha, \eta \in \mathbb{Q}^\times(f) \cdot k_f^\times \cdot k^\times(f)^3$. The quadratic discriminants d_k are congruent 5(mod 8), since $2 \mid f$.

No.	d_L	d_k	
2	3 054 132	84837	$\equiv +3(9)$
3	4 735 476	131541	$\equiv -3(9)$

f'	1	q_1	q_2	q_1q_2
$m_3(d_k, f')$	1	3	3	6

d) 8 complex cubic fields with

$$\rho = 2,\ t + w = 1,\ \delta_{\max} = 0,\ u = 1 = t + w,$$
$$f = q_1.$$

In this case, $I_{k,3}(f)/k^\times(f)^3 = \langle\alpha, \beta\rangle$ and there are no restrictions from the two generating principal ideal cubes α, β of k, because $\delta(q_1) = 0$, and $\alpha, \beta \in \mathbb{Q}^\times(f) \cdot k_f^\times \cdot k^\times(f)^3$. All quadratic discriminants d_k are congruent 5(mod 8). The class numbers of all cubic fields L are divisible by 9.

No.	d_L	d_k		q_1	w
15	−301 675	−12067	$\equiv -1(3)$	5	0
16	−414 508	−103627	$\equiv -1(3)$	2	0
17	−429 676	−107419	$\equiv -1(3)$	2	0
18	−706 540	−176635	$\equiv -1(3)$	2	0
19	−738 572	−184643	$\equiv +1(3)$	2	0
20	−795 180	−198795	$\equiv -3(9)$	2	0
21	−821 452	−205363	$\equiv -1(3)$	2	0
22	−864 243	−96027	$\equiv +3(9)$	3	1

f'	1	q_1
$m_3(d_k, f')$	4	9

e) 5 complex cubic fields with

$$\rho = 1,\ t + w = 3,\ \delta_{\max} = 1,\ u = 1 < t + w,$$
$$f = q_1 \cdot q_2 \cdot q_3 \text{ with } q_1 = 2.$$

Restrictions arise from the single generating principal ideal cube α of k, where $I_{k,3}(f)/k^\times(f)^3 = \langle\alpha\rangle$, but $\alpha \notin \mathbb{Q}^\times(f) \cdot k_f^\times \cdot k^\times(f)^3$. All quadratic discriminants d_k are congruent 5(mod 8), since $2 \mid f$. The class numbers of all cubic fields L are divisible by 9, and two of the six fields with discriminant $d_L = -936\,648$ even have 3-class rank 3.

No.	d_L	d_k		q_2	q_3	w	$\delta(q_1)$	$\delta(q_2)$	$\delta(q_3)$
23	−406 700	−83	$\equiv 1(3)$	5	7	0	1	0	1
24	−597 996	−339	$\equiv 3(9)$	7	3	1	0	1	1
25	−672 300	−83	$\equiv 1(3)$	5	3^2	1	1	0	1
26	−866 700	−107	$\equiv 1(3)$	5	3^2	1	1	1	0
27	−936 684	−59	$\equiv 1(3)$	7	3^2	1	1	0	1

No.	$f' =$	1	q_1	q_2	q_3	q_1q_2	q_1q_3	q_2q_3	$q_1q_2q_3$
23	$m_3(d_k, f') =$	1	0	3	0	0	3	0	6
24		1	3	0	0	0	0	3	6
25		1	0	3	0	0	3	0	6
26		1	0	0	3	3	0	0	6
27		1	0	3	0	0	3	0	6

f) A single complex cubic field with

$$\rho = 2,\ t = 1,\ w = 1,\ \delta_{\max} = 1,\ u = 0 < t + w,$$
$$f = q_1 \cdot q_2 \text{ with } q_1 = 2,\ q_2 = 3.$$

Restrictions arise from the two generating principal ideal cubes α, β of k, where $I_{k,3}(f)/k^\times(f)^3 = \langle\alpha, \beta\rangle$ and $\alpha \notin \mathbb{Q}^\times(f) \cdot k_f^\times \cdot k^\times(f)^3$. However, the other ideal cube is not independent from α, i. e., $\beta \in \langle\alpha\rangle \cdot \mathbb{Q}^\times(f) \cdot k_f^\times \cdot k^\times(f)^3$. Here, we have $\delta(q_1) = 1$ and $\delta(q_2) = 1$. The quadratic discriminant d_k is simultaneously congruent $-3 \pmod 9$ and congruent $5 \pmod 8$, since $2 \mid f$. The class numbers of all cubic fields L are divisible by 9.

No.	d_L	d_k
28	−941 004	−26139

f'	1	q_1	q_2	$q_1 q_2$
$m_3(d_k, f')$	4	0	0	9

Part 2. Applications of Corollary 3.3 to conductors with $w = 2$.

All quadratic discriminants d_k are congruent $-3 \pmod 9$, since $w = 2$.

a) 30 complex cubic fields with

$$\rho = 0,\ t = 1,\ w = 2,\ \delta_{\max} = 0,\ u = 1 = t,$$
$$f = 3^2 \cdot q_1.$$

There are no possible constraints, since $I_{k,3}(f) = k^\times(f)^3$. Therefore, $\delta(3^2) = 0$ and $\delta(q_1) = 0$ In general, the 3-class groups of all cubic fields L are cyclic.

No.	d_L	d_k	q_1	No.	d_L	d_k	q_1
29	−70 956	−219	2	44	−537 516	−1659	2
30	−94 284	−291	2	45	−560 844	−1731	2
31	−140 940	−435	2	46	−584 172	−1803	2
32	−187 596	−579	2	47	−630 828	−1947	2
33	−210 924	−651	2	48	−654 156	−2019	2
34	−234 252	−723	2	49	−700 812	−2163	2
35	−257 580	−795	2	50	−747 468	−2307	2
36	−304 236	−939	2	51	−753 300	−372	5
37	−370 575	−183	5	52	−770 796	−2379	2
38	−374 220	−1155	2	53	−794 124	−2451	2
39	−397 548	−1227	2	54	−826 200	−408	5
40	−420 876	−1299	2	55	−864 108	−2667	2
41	−440 559	−111	7	56	−887 436	−2739	2
42	−461 700	−228	5	57	−910 764	−2811	2
43	−467 532	−1443	2	58	−912 951	−39	17

The partial multiplicities $m_3(d_k, f')$ of divisors f' of the conductor f can be determined with the aid of Corollary 3.3.

f'	1	q_1	3	$3q_1$	3^2	$3^2 q_1$
$m_3(d_k, f')$	0	1	1	2	3	6

b) A totally real cubic field with

$$\rho = 0,\ t = 1,\ w = 2,\ \delta_{\max} = 0,\ u = 1 = t,$$
$$f = 3^2 \cdot q_1 \text{ with } q_1 = 2$$

Here, we have $I_{k,3}(f) = U_k \cdot k^\times(f)^3$ without constraints from the fundamental unit η of k, i. e., $\delta(3^2) = 0$ and $\delta(q_1) = 0$. The quadratic discriminant d_k is congruent $5 \pmod 8$, since $2 \mid f$

No.	d_L	d_k
4	9 796 788	30237

f'	1	q_1	3	$3q_1$	3^2	$3^2 q_1$
$m_3(d_k, f')$	0	1	1	2	3	6

c) 2 complex cubic fields with

$$\rho = 1,\ t = 0,\ w = 2,\ \delta_{\max} = 0,\ u = 0 = t,$$
$$f = 3^2.$$

In this case, $I_{k,3}(f)/k^\times(f)^3 = \langle \alpha \rangle$ without constraints from the single generating principal ideal cube α of k; $\alpha \in \mathbb{Q}^\times(f) \cdot k_f^\times\ k^\times(f)^3$, since $\delta(3^2) = 0$ In general, the 3-class groups of all cubic fields L are cyclic.

No.	d_L	d_k	No.	d_L	d_k
59	−274 347	−3387	60	−659 259	−8139

f'	1	3	3^2
$m_3(d_k, f')$	1	3	9

d) 9 totally real cubic fields with

$$\rho = 0,\ t = 2,\ w = 2,\ \delta_{\max} = 1,\ u < t,$$
$$f = 3^2 \cdot q_1\ \ q_2 \text{ with } q_1 = 2.$$

Here, $I_{k,3}(f) = U_k \cdot k^\times(f)^3$, and constraints arise from the fundamental unit η of k. All quadratic discriminants d_k are congruent $5 \pmod 8$, since $2 \mid f$. Except for the discriminant $d_k = 717$ with $\delta(3^2) = 0$, all examples belong to the case $\delta(3) = 1$.

No.	d_L	d_k	q_2	η		$\delta(q_1)$	$\delta(q_2)$	u
5	1 725 300	213	5	$(73 + 5\omega)/2$	$\in \mathcal{O}_5$	1	0	1
6	2 238 516	141	7	$95 + 8\omega$	$\in \mathcal{O}_2$	0	1	1
7	2 891 700	357	5	$(19 + \omega)/2$		1	1	0
8	4 641 300	573	5	$383 + 16\omega$	$\in \mathcal{O}_2$	0	1	1
9	5 807 700	717	5	$(241 + 9\omega)/2$	$\in \mathcal{O}_9$	1	1	0
10	6 810 804	429	7	$(145 + 7\omega)/2$	$\in \mathcal{O}_7$	1	0	1
11	7 557 300	933	5	$75263 + 2464\omega$	$\in \mathcal{O}_2$	0	1	1
12	7 953 876	501	7	$(28225 + 1261\omega)/2$		1	1	0
13	8 723 700	1077	5	$(361 + 11\omega)/2$		1	1	0

It is illuminating to compare the various multiplicities $m_3(d_k, f')$ of divisors f' of the conductor f.

No.	1	q_1	q_2	q_1q_2	3	$3q_1$	$3q_2$	$3q_1q_2$	3^2	3^2q_1	3^2q_2	$3^2q_1q_2$
5	0	0	1	0	0	1	0	2	1	2	2	4
6	0	1	0	0	0	0	1	2	1	2	2	4
7	0	0	0	1	0	1	1	1	1	2	2	4
8	0	1	0	0	0	0	1	2	1	2	2	4
9	0	0	0	1	1	0	0	2	3	0	0	6
10	0	0	1	0	0	1	0	2	1	2	2	4
11	0	1	0	0	0	0	1	2	1	2	2	4
12	0	0	0	1	0	1	1	1	1	2	2	4
13	0	0	0	1	0	1	1	1	1	2	2	4

e) 9 complex cubic fields with

$$\rho = 1,\ t = 1,\ w = 2,\ \delta_{\max} = 1,\ u \le t,$$
$$f = 3^2 \cdot q_1$$

Restrictions arise from the single generating principal ideal cube α of k, where $I_{k,3}(f)/k^\times(f)^3 = \langle\alpha\rangle$ but $\alpha \notin \mathbb{Q}^\times(f) \cdot k_f^\times \cdot k^\times(f)^3$. Here, always $\delta(3^2) = 1$. In general, the 3-class groups of all cubic fields L are cyclic, but the class numbers of the six fields with discriminant $d_L = -724\,140$ are divisible by 9

No.	d_L	d_k	q_1	u	$\delta(q_1)$	$\delta(3)$
61	−327 564	−1011	2	0	1	1
62	−444 204	−1371	2	0	1	1
63	−490 860	−1515	2	0	1	1
64	−662 175	−327	5	0	1	1
65	−677 484	−2091	2	0	1	0
66	−724 140	−2235	2	1	0	1
67	−840 780	−2595	2	0	1	1
68	−957 420	−2955	2	0	1	0
69	−980 748	−3027	2	0	1	1

No.	$f' =$	1	q_1	3	$3q_1$	3^2	3^2q_1
61	$m_3(d_k, f') =$	1	0	0	3	3	6
62		1	0	0	3	3	6
63		1	0	0	3	3	6
64		1	0	0	3	3	6
65		1	0	3	0	0	9
66		1	3	0	0	3	6
67		1	0	0	3	3	6
68		1	0	3	0	0	9
69		1	0	0	3	3	6

f) Finally, a single totally real cubic field with

$$\rho = 1,\ t = 1,\ w = 2,\ \delta_{\max} = 1,\ u = 0 < t,$$
$$f = 3^2 \cdot q_1 \text{ with } q_1 = 2.$$

Here, we have a single generating principal ideal cube α of k beside the fundamental unit η of k, $I_{k,3}(f)/U_k \cdot k^\times(f)^3 = \langle\alpha\rangle$. $\delta(3) = 1$ and $\delta(q_1) = 1$, because constraints arise from the fundamental unit η of k, but $\alpha \in U_k \quad \mathbb{Q}^\times(f) \quad k_f^\times \cdot k^\times(f)^3$ is not independent. The quadratic discriminant d_k is congruent 5(mod 8), since $2 \mid f$.

No.	d_L	d_k	η
14	6 367 572	19653	$(8\,968\,321 + 63\,973\omega)/2$

f'	1	q_1	3	$3q_1$	3^2	3^2q_1
$m_3(d_k, f')$	1	0	0	3	3	6