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THE APPROXIMATION OF THE EXACT BOUNDARY CONDITIONS AT AN ARTIFICIAL BOUNDARY FOR LINEAR ELASTIC EQUATIONS AND ITS APPLICATION

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ABSTRACT. The exterior boundary value problems of linear elastic equations are considered. A sequence of approximations to the exact boundary conditions at an artificial boundary is given. Then the original problem is reduced to a boundary value problem on a bounded domain. Furthermore, a finite element approximation of this problem and optimal error estimates are obtained. Finally, a numerical example shows the effectiveness of this method.

1. INTRODUCTION

Many boundary value problems of partial differential equations arising in practical applications are given on unbounded domains, such as coupling of structures with foundation and fluid flow around obstacles. In finding the numerical solutions of these problems, it is often difficult to use the classical finite element or finite difference method. In engineering, the usual method is to introduce an artificial boundary and cut off the unbounded part of the domain and to set up an artificial boundary condition at the artificial boundary of the remaining bounded domain. For example, the Dirichlet condition and Neumann condition are often used for elliptic partial differential equations. In general, this artificial boundary condition at the artificial boundary is only a rough approximation of the exact boundary condition. Hence, the remaining bounded domain must be quite large when high accuracy is required. For such large domains, it is still difficult to compute the numerical solution.

In 1985, we found the exact boundary conditions at an artificial boundary for the Laplace equation as a model equation [7]. Moreover, a sequence of approximations to the exact boundary condition at the artificial boundary was given, and we reduced the original exterior problem to an equivalent (or approximate) boundary value problem on a bounded domain with integral boundary conditions. Then we solved the approximate boundary value problem on the bounded domain by a finite element method. An optimal error estimate of the finite element approximate solution was obtained and a numerical example showed the effectiveness of this method.

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Boundary value problems on unbounded domains have been studied for many years. For example, in 1982, Goldstein [3] studied Helmholtz-type equations. The problem was replaced by a boundary value problem on a fixed bounded domain. The behavior of the solution near infinity is incorporated in a nonlocal boundary condition. In 1984, Feng [4] studied asymptotic radiation conditions for the reduced wave equation; in 1986, Hagstrom and Keller [5] studied the exact boundary condition at an artificial boundary for partial differential equations in cylinders. Shortly thereafter, they used this technique to solve nonlinear problems of both elliptic and parabolic type [6]. This technique is a rather natural extension of related work on ordinary differential equations over infinite intervals by Keller [9], Jepson and Keller [8], and Lentini and Keller [11]. In 1988, Lenoir and Tounsi [10] studied the various convergence properties of the localized finite element method for the two-dimensional sea-keeping problem.

In this paper we show how this technique applies to the exterior problem for the linear elastic equations and obtain its finite element approximation on a bounded domain. An optimal error estimate of the finite element approximate solution is given; moreover, a numerical example shows this technique to be very effective.

2. The exact and approximate boundary conditions at an artificial boundary

Let Γ_i be a bounded, simply closed curve in R^2, and let Ω be the unbounded domain with boundary Γ_i. Consider the following exterior boundary value problem:

$$-\mu\Delta u - (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = f_1 \quad \text{in } \Omega, \tag{1}$$

$$-\mu\Delta v - (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = f_2 \quad \text{in } \Omega, \tag{2}$$

$$u = 0 \quad \text{on } \Gamma_i, \tag{3}$$

$$v = 0 \quad \text{on } \Gamma_i, \tag{4}$$

$$u,\, v \text{ are bounded when } r = (x^2+y^2)^{1/2} \to +\infty,$$

where (u, v) is the displacement, $\lambda, \mu > 0$ are the Lamé constants, and (f_1, f_2) is the density of the applied body force, the support of which is compact.

This problem is defined on an unbounded domain Ω. The usual method engineers use is to draw a circumference Γ_e with radius R. Then Ω is divided into two parts; the bounded part and the unbounded part are denoted by Ω_i and Ω_e (see Figure 1). Furthermore, suppose that the support of (f_1, f_2) is in Ω_i. If a certain boundary condition on the artificial boundary Γ_e is given, then we could solve the problem (1)–(4) on the bounded domain Ω_i. The goal of this section is to derive the exact and an approximate boundary condition for the solution of problem (1)–(4) on Γ_e.

We now consider the boundary value problem of linear elastic equations on

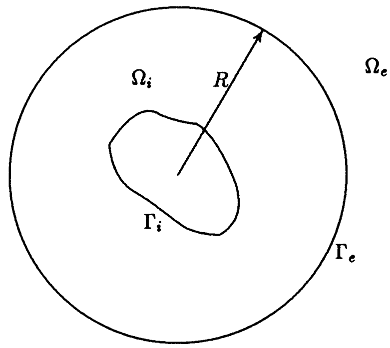

FIGURE 1

the unbounded domain Ω_e with boundary Γ_e:

$$-\mu\Delta u - (\lambda + u)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad \text{in } \Omega_e, \tag{5}$$

$$-\mu\Delta v - (\lambda + \mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad \text{in } \Omega_e, \tag{6}$$

$$u|_{\Gamma_e} = u(R, \theta), \tag{7}$$

$$v|_{\Gamma_e} = v(R, \theta), \tag{8}$$

$$u, v \text{ are bounded when } r \to +\infty.$$

We know that the problem (5)–(8) has a unique solution (u, v) if $(u(R, \theta), v(R, \theta))$ is given. This solution (u, v) can be found in [13, §83]. For our application, the solution (u, v) is given in the following form [7]:

$$u(r, \theta) = (r^2 - R^2)W_1 + G_1, \tag{9}$$

$$v(r, \theta) = (r^2 - R^2)W_2 + G_2, \tag{10}$$

where $x = r\cos\theta$ and $y = r\sin\theta$. Here G_1, G_2, W_1, and W_2 are harmonic functions, and

$$G_1(r, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)r^{-n}, \tag{11}$$

$$G_2(r, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos n\theta + d_n \sin n\theta)r^{-n}, \tag{12}$$

with

$$a_n = \frac{R^n}{\pi}\int_0^{2\pi} u(R, \theta)\cos n\theta \, d\theta, \qquad n = 0, 1, 2, \ldots, \tag{13}$$

$$b_n = \frac{R^n}{\pi}\int_0^{2\pi} u(R, \theta)\sin n\theta \, d\theta, \qquad n = 1, 2, \ldots, \tag{14}$$

$$c_n = \frac{R^n}{\pi}\int_0^{2\pi} v(R, \theta)\cos n\theta \, d\theta, \qquad n = 0, 1, 2, \ldots, \tag{15}$$

$$d_n = \frac{R^n}{\pi}\int_0^{2\pi} v(R, \theta)\sin n\theta \, d\theta, \qquad n = 1, 2, \ldots. \tag{16}$$

Furthermore, let

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \Theta := -\kappa p \quad \text{and} \quad \kappa = \frac{\mu}{\lambda + \mu}. \tag{17}$$

Then we know that p is a harmonic function, and

$$p(r, \theta) = \sum_{n=2}^{\infty} (p_n^1 \cos n\theta + p_n^2 \sin n\theta) r^{-n}, \tag{18}$$

with

$$(\tfrac{1}{2} + \kappa) p_n^1 = (n-1)(a_{n-1} - d_{n-1}), \tag{19}$$

$$(\tfrac{1}{2} + \kappa) p_n^2 = (n-1)(b_{n-1} + c_{n-1}), \tag{20}$$

and

$$W_1(r, \theta) = \frac{1}{4} \sum_{n=2}^{\infty} \{p_n^1 \cos(n+1)\theta + p_n^2 \sin(n+1)\theta\} r^{-n-1}, \tag{21}$$

$$W_2(r, \theta) = \frac{1}{4} \sum_{n=2}^{\infty} \{p_n^1 \sin(n+1)\theta - p_n^2 \cos(n+1)\theta\} r^{-n-1}. \tag{22}$$

Finally, a computation shows

$$x W_1 + y W_2 = \tfrac{1}{4} p, \tag{23}$$

$$\left(\frac{1}{2} + \kappa\right) p = -\left(\frac{\partial G_1}{\partial x} + \frac{\partial G_2}{\partial y}\right). \tag{24}$$

We now discuss the stress on the boundary Γ_e. From

$$X_x = \lambda\Theta + 2\mu \frac{\partial u}{\partial x}, \quad X_y = Y_x = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right), \quad Y_y = \lambda\Theta + 2\mu \frac{\partial v}{\partial y},$$

we obtain the vector components of stress acting on the boundary Γ_e:

$$X_n = (X_x \cos\theta + X_y \sin\theta)|_{\Gamma_e}, \qquad Y_n = (X_y \cos\theta + Y_y \sin\theta)|_{\Gamma_e}.$$

Furthermore, we get

$$\begin{aligned} X_n &= \left(\mu \frac{\partial u}{\partial r} - \lambda\kappa p \cos\theta + \mu \frac{\partial u}{\partial x} \cos\theta + \mu \frac{\partial v}{\partial x} \sin\theta\right)_{r=R} \\ &= \left\{\mu \frac{\partial u}{\partial r} - \lambda\kappa p \cos\theta + 2\mu x (W_1 \cos\theta + W_2 \sin\theta) \right. \\ &\qquad \left. + \mu \frac{\partial G_1}{\partial x} \cos\theta + \mu \frac{\partial G_2}{\partial x} \sin\theta\right\}_{r=R} \\ &= \mu \left(\frac{\partial u}{\partial r} - p\cos\theta + \frac{3\mu + \lambda}{2(\lambda + \mu)} p \cos\theta + \frac{\partial G_1}{\partial x} \cos\theta + \frac{\partial G_2}{\partial x} \sin\theta\right)_{r=R}. \end{aligned}$$

The last equality comes from (17) and (23).

A computation shows

$$\begin{aligned} \left(\frac{\partial u}{\partial r} - p\cos\theta\right)\bigg|_{r=R} &= 2R W_1(R, \theta) + \frac{\partial G_1}{\partial r}\bigg|_{r=R} - p(R, \theta)\cos\theta \\ &= \frac{2 + 2\kappa}{1 + 2\kappa} \frac{\partial G_1}{\partial r}\bigg|_{r=R} + \frac{1}{(1 + 2\kappa)R} \frac{\partial G_2}{\partial \theta}\bigg|_{r=R}. \end{aligned}$$

By equality (24) we have

$$\begin{aligned}\frac{3\mu+\lambda}{2(\lambda+\mu)}p\cos\theta &= \frac{3\mu+\lambda}{2(\lambda+\mu)}\frac{2}{1+2\kappa}\left(-\frac{\partial G_1}{\partial x}\bigg|_{r=R} - \frac{\partial G_2}{\partial y}\bigg|_{r=R}\right)\cos\theta \\ &= -\left(\frac{\partial G_1}{\partial x}+\frac{\partial G_2}{\partial y}\right)\bigg|_{r=R}\cos\theta.\end{aligned}$$

Hence, we get

$$X_n(u, v) = \mu\left(\frac{2+2\kappa}{1+2\kappa}\frac{\partial G_1}{\partial r}\bigg|_{r=R} - \frac{2\kappa}{(1+2\kappa)R}\frac{\partial G_2}{\partial \theta}\bigg|_{r=R}\right). \tag{25}$$

Similarly, we can get

$$Y_n(u, v) = \mu\left(\frac{2+2\kappa}{1+2\kappa}\frac{\partial G_2}{\partial r}\bigg|_{r=R} + \bigg|\frac{2\kappa}{(1+2\kappa)R}\frac{\partial G_1}{\partial \theta}\bigg|_{r=R}\right). \tag{26}$$

Substituting (11)–(16) into (25) and (26) and integrating by parts, we obtain the vector components of stress acting on the boundary Γ_e,

$$\begin{aligned}X_n(u, v) &= \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 u(R, \varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}\,d\varphi \\ &\qquad - \frac{2\kappa}{1+2\kappa}\frac{\mu}{R}\frac{\partial v(R, \theta)}{\partial\theta} \\ &\equiv T_1(u, v),\end{aligned} \tag{27}$$

$$\begin{aligned}Y_n(u, v) &= \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 v(R, \varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}\,d\varphi \\ &\qquad + \frac{2\kappa}{1+2\kappa}\frac{\mu}{R}\frac{\partial u(R, \theta)}{\partial\theta} \\ &\equiv T_2(u, v).\end{aligned} \tag{28}$$

The formula can also be rewritten in the following form:

$$\begin{aligned}X_n(u, v) = {}& \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 u(R, \varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}\,d\varphi \\ & - \frac{2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 v(R, \varphi)}{\partial\varphi^2}\frac{\sin n(\theta-\varphi)}{n}\,d\varphi,\end{aligned} \tag{29}$$

$$\begin{aligned}Y_n(u, v) = {}& \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 v(R, \varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}\,d\varphi \\ & + \frac{2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{\infty}\int_0^{2\pi}\frac{\partial^2 u(R, \varphi)}{\partial\varphi^2}\frac{\sin n(\theta-\varphi)}{n}\,d\varphi.\end{aligned} \tag{30}$$

We now get the exact boundary condition (27)–(28) (or (29)–(30)) at the artificial boundary Γ_e. Then the restriction of the solution (u, v) of problem

(1)–(4) to the bounded domain Ω_i is a solution of the following problem:

$$-\mu\Delta u - (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = f_1 \quad \text{in } \Omega_i, \tag{31}$$

$$-\mu\Delta v - (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = f_2 \quad \text{in } \Omega_i, \tag{32}$$

$$u = 0 \quad \text{on } \Gamma_i, \tag{33}$$

$$v = 0 \quad \text{on } \Gamma_i, \tag{34}$$

$$X_n = T_1(u, v) \quad \text{on } \Gamma_e, \tag{35}$$

$$Y_n = T_2(u, v) \quad \text{on } \Gamma_e. \tag{36}$$

This is a boundary value problem with global boundary condition on Γ_e. Let

$$\begin{aligned} T_1^N(u, v) = {} & \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2 u(R, \varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}\,d\varphi \\ & - \frac{2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2 v(R, \varphi)}{\partial\varphi^2}\frac{\sin n(\theta-\varphi)}{n}\,d\varphi, \\ T_2^N(u, v) = {} & \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2 v(R, \varphi)}{\partial\varphi^2}\frac{\cos n(\theta-\varphi)}{n}\,d\varphi \\ & + \frac{2\kappa}{1+2\kappa}\frac{\mu}{\pi R}\sum_{n=1}^{N}\int_0^{2\pi}\frac{\partial^2 u(R, \varphi)}{\partial\varphi^2}\frac{\sin n(\theta-\varphi)}{n}\,d\varphi \end{aligned}$$

and $T_1^0(u, v) = 0$, $T_2^0(u, v) = 0$. Then we get a sequence of approximate boundary condition on the artificial boundary Γ_e,

$$X_n = T_1^N(u, v) \quad \text{on } \Gamma_e, \tag{37}$$

$$Y_n = T_2^N(u, v) \quad \text{on } \Gamma_e, \tag{38}$$

for $N = 0, 1, 2, \ldots$. When $N = 0$, then (37)–(38) reduces to

$$X_n = 0 \quad \text{on } \Gamma_e, \qquad Y_n = 0 \quad \text{on } \Gamma_e,$$

which is often used in engineering.

By means of the approximate boundary condition (37)–(38), we reduce the original problem (1)–(4) to the following problem on the bounded domain Ω_i approximately for $N = 0, 1, 2, \ldots$:

$$-\mu\Delta u - (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = f_1 \quad \text{in } \Omega_i, \tag{39}$$

$$-\mu\Delta v - (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = f_2 \quad \text{in } \Omega_i, \tag{40}$$

$$u = 0 \quad \text{on } \Gamma_i, \tag{41}$$

$$v = 0 \quad \text{on } \Gamma_i, \tag{42}$$

$$X_n = T_1^N(u, v) \quad \text{on } \Gamma_e, \tag{43}$$

$$Y_n = T_2^N(u, v) \quad \text{on } \Gamma_e. \tag{44}$$

In the following section we will show that the boundary value problems (31)–(36) and (39)–(44) are well posed.

3. The solution of problems (31)–(36) and (39)–(44)

Let $H^m(\Omega_i)$ and $H^s(\Gamma_e)$ denote the usual Sobolev spaces on the domain Ω_i and the boundary Γ_e, with integer m and real s. Furthermore, let

$$H_*^1(\Omega_i) = \{v \in H^1(\Omega_i)\,;\, v|_{\Gamma_i} = 0\} \quad \text{with norm } \|v\|_{1,\Omega_i},$$

$$V = H_*^1(\Omega_i) \times H_*^1(\Omega_i) \quad \text{with norm } \|(u\,,v)\|_V^2 = \|u\|_{1,\Omega_i}^2 + \|v\|_{1,\Omega_i}^2.$$

Then the boundary value problem (31)–(36) is equivalent to the following variational problem:

Find $(u\,,v) \in V$ such that

$$A(u\,,v\,;\tilde u\,,\tilde v) + B(u\,,v\,;\tilde u\,,\tilde v) = \iint_{\Omega_i} (f_1\tilde u + f_2\tilde v)\,dx\,dy \quad \forall(\tilde u\,,\tilde v) \in V, \tag{45}$$

where

$$\begin{aligned} A(u\,,v\,;\tilde u\,,\tilde v) &= \iint_{\Omega_i}\left\{\lambda\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\left(\frac{\partial \tilde u}{\partial x}+\frac{\partial \tilde v}{\partial y}\right)+2\mu\left(\frac{\partial u}{\partial x}\frac{\partial \tilde u}{\partial x}+\frac{\partial v}{\partial y}\frac{\partial \tilde v}{\partial y}\right)\right.\\ &\qquad\left.+\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\left(\frac{\partial \tilde v}{\partial x}+\frac{\partial \tilde u}{\partial y}\right)\right\}dx\,dy \\ &\qquad \forall(u\,,v)\,,(\tilde u\,,\tilde v) \in V, \end{aligned}$$

$$\begin{aligned} B(u\,,v\,;\tilde u\,,\tilde v) &= \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{\partial u(R\,,\varphi)}{\partial\varphi}\frac{\partial \tilde u(R\,,\theta)}{\partial\theta}+\frac{\partial v(R\,,\varphi)}{\partial\varphi}\frac{\partial \tilde v(R\,,\theta)}{\partial\theta}\right\}\\ &\qquad\cdot\frac{\cos n(\theta-\varphi)}{n}\,d\varphi\,d\theta\\ &+\frac{2\kappa}{1+2\kappa}\frac{\mu}{\pi}\sum_{n=1}^{\infty}\int_0^{2\pi}\int_0^{2\pi}\left\{-\frac{\partial v(R\,,\varphi)}{\partial\varphi}\frac{\partial \tilde u(R\,,\theta)}{\partial\theta}+\frac{\partial u(R\,,\varphi)}{\partial\varphi}\frac{\partial \tilde v(R\,,\theta)}{\partial\theta}\right\}\\ &\qquad\cdot\frac{\sin n(\theta-\varphi)}{n}\,d\varphi\,d\theta \quad \forall(u\,,v)\,,(\tilde u\,,\tilde v)\in V. \end{aligned}$$

Furthermore, let

$$\begin{aligned} B_N(u\,,v\,;\tilde u\,,\tilde v) &= \frac{2+2\kappa}{1+2\kappa}\frac{\mu}{\pi}\sum_{n=1}^{N}\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{\partial u(R\,,\varphi)}{\partial\varphi}\frac{\partial \tilde u(R\,,\theta)}{\partial\theta}+\frac{\partial v(R\,,\varphi)}{\partial\varphi}\frac{\partial \tilde v(R\,,\theta)}{\partial\theta}\right\}\\ &\qquad\cdot\frac{\cos n(\theta-\varphi)}{n}\,d\varphi\,d\theta\\ &+\frac{2\kappa}{1+2\kappa}\frac{\mu}{\pi}\sum_{n=1}^{N}\int_0^{2\pi}\int_0^{2\pi}\left\{-\frac{\partial v(R\,,\varphi)}{\partial\varphi}\frac{\partial \tilde u(R\,,\theta)}{\partial\theta}+\frac{\partial u(R\,,\varphi)}{\partial\varphi}\frac{\partial \tilde v(R\,,\theta)}{\partial\theta}\right\}\\ &\qquad\cdot\frac{\sin n(\theta-\varphi)}{n}\,d\varphi\,d\theta \quad \forall(u\,,v)\,,(\tilde u\,,\tilde v)\in V. \end{aligned}$$

Then the boundary value problem (39)–(44) is equivalent to the following variational problem:

$$\text{(46)} \quad \begin{aligned} &\text{Find } (u_N, v_N) \in V \text{ such that} \\ &A(u_N, v_N; \tilde{u}, \tilde{v}) + B_N(u_N, v_N; \tilde{u}, \tilde{v}) \\ &\quad = \iint_{\Omega_i} (f_1\tilde{u} + f_2\tilde{v})\,dx\,dy \quad \forall(\tilde{u}, \tilde{v}) \in V. \end{aligned}$$

From Korn's inequality [12], we know that the following holds.

Lemma 1. *The bilinear form $A(u, v; \tilde{u}, \tilde{v})$ is symmetric, bounded, and coercive on $V \times V$. That is, there are two positive constants M_0 and β_0 such that*

$$|A(u, v; \tilde{u}, \tilde{v})| \le M_0 \|(u, v)\|_V \|(\tilde{u}, \tilde{v})\|_V \quad \forall(u, v), (\tilde{u}, \tilde{v}) \in V,$$
$$A(u, v; u, v) \ge \beta_0 \|(u, v)\|_V^2 \quad \forall(u, v) \in V.$$

For the bilinear forms $B(u, v; \tilde{u}, \tilde{v})$ and $B_N(u, v; \tilde{u}, \tilde{v})$, we have

Lemma 2. *The bilinear forms $B(u, v; \tilde{u}, \tilde{v})$ and $B_N(u, v; \tilde{u}, \tilde{v})$ are symmetric and bounded on $V \times V$, i.e., there is a constant $M_1 \ge 0$ such that*

$$\text{(47)} \quad |B(u, v; \tilde{u}, \tilde{v})| \le M_1 \|(u, v)\|_V \|(\tilde{u}, \tilde{v})\|_V \quad \forall(u, v), (\tilde{u}, \tilde{v}) \in V,$$
$$\text{(48)} \quad |B_N(u, v; \tilde{u}, \tilde{v})| \le M_1 \|(u, v)\|_V \|(\tilde{u}, \tilde{v})\|_V \quad \forall(u, v), (\tilde{u}, \tilde{v}) \in V.$$

Furthermore,

$$B(u, v; u, v) \ge 0 \quad \forall(u, v) \in V,$$
$$B_N(u, v; u, v) \ge 0 \quad \forall(u, v) \in V.$$

Proof. We recall an equivalent definition of Sobolev space $H^s(\Gamma_e)$ [14]:

$$u \in H^s(\Gamma_e) \Leftrightarrow u = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$
$$\text{and } \|u\|_{\Delta, s} = \left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(1 + n^2)^s(a_n^2 + b_n^2)\right)^{1/2} \le \infty,$$

where $\|u\|_{\Delta, s}$ is an equivalent norm in $H^s(\Gamma_e)$.

For any $(u, v), (\tilde{u}, \tilde{v}) \in V$ we know that $u|_{\Gamma_e}$, $v|_{\Gamma_e}$, $\tilde{u}|_{\Gamma_e}$, and $\tilde{v}|_{\Gamma_e}$ belong to the space $H^{1/2}(\Gamma_e)$ by the trace theorem. Assume

$$u(R, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta),$$
$$v(R, \theta) = \frac{c_0}{2} + \sum_{n=1}^{\infty}(c_n \cos n\theta + d_n \sin n\theta),$$
$$\tilde{u}(R, \theta) = \frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty}(\tilde{a}_n \cos n\theta + \tilde{b}_n \sin n\theta),$$
$$\tilde{v}(R, \theta) = \frac{\tilde{c}_0}{2} + \sum_{n=1}^{\infty}(\tilde{c}_n \cos n\theta + \tilde{d}_n \sin n\theta).$$

Then

$$\|u(R,\theta)\|_{\Delta,1/2} = \left(\frac{a_0}{2} + \sum_{n=1}^{\infty}(1+n^2)^{1/2}(a_n^2+b_n^2)\right)^{1/2} < \infty,$$

$$\|v(R,\theta)\|_{\Delta,1/2} = \left(\frac{c_0}{2} + \sum_{n=1}^{\infty}(1+n^2)^{1/2}(c_n^2+d_n^2)\right)^{1/2} < \infty,$$

$$\|\tilde{u}(R,\theta)\|_{\Delta,1/2} = \left(\frac{\tilde{a}_0}{2} + \sum_{n=1}^{\infty}(1+n^2)^{1/2}(\tilde{a}_n^2+\tilde{b}_n^2)\right)^{1/2} < \infty,$$

$$\|\tilde{v}(R,\theta)\|_{\Delta,1/2} = \left(\frac{\tilde{c}_0}{2} + \sum_{n=1}^{\infty}(1+n^2)^{1/2}(\tilde{c}_n^2+\tilde{d}_n^2)\right)^{1/2} < \infty.$$

A computation shows

$$\begin{aligned}
B(u,v;\tilde{u},\tilde{v}) &= \frac{2(1+\kappa)\pi\mu}{(1+2\kappa)}\sum_{n=1}^{\infty} n\{a_n\tilde{a}_n + b_n\tilde{b}_n + c_n\tilde{c}_n + d_n\tilde{d}_n\} \\
&\quad + \frac{2\kappa\pi\mu}{(1+2\kappa)}\sum_{n=1}^{\infty} n\{a_n\tilde{d}_n - b_n\tilde{c}_n - c_n\tilde{b}_n + d_n\tilde{a}_n\} \\
&= \frac{2\pi\mu}{(1+2\kappa)}\sum_{n=1}^{\infty} n\{(1+\kappa)(a_n\tilde{a}_n + b_n\tilde{b}_n + c_n\tilde{c}_n + d_n\tilde{d}_n) \\
&\qquad\qquad + \kappa(a_n\tilde{d}_n - b_n\tilde{c}_n - c_n\tilde{b}_n + d_n\tilde{a}_n)\} \\
&\le \frac{4(1+\kappa)\pi\mu}{(1+2\kappa)}\left\{\sum_{n=1}^{\infty} n(a_n^2+b_n^2+c_n^2+d_n^2)\right\}^{1/2} \\
&\qquad \cdot \left\{\sum_{n=1}^{\infty} n(\tilde{a}_n^2+\tilde{b}_n^2+\tilde{c}_n^2+\tilde{d}_n^2)\right\}^{1/2} \\
&\le \frac{4(1+\kappa)\pi\mu}{(1+2\kappa)}\|(u,v)\|_{\Delta,1/2}\|(\tilde{u},\tilde{v})\|_{\Delta,1/2} \\
&\le \frac{4(1+\kappa)\pi\mu c}{(1+2\kappa)}\|(u,v)\|_{1/2,\Gamma_e}\|(\tilde{u},\tilde{v})\|_{1/2,\Gamma_e},
\end{aligned}$$

where the last inequality is a consequence of the fact that $\|u\|_{\Delta,1/2}$ is equivalent to the norm $\|u\|_{1/2,\Gamma_e}$ in the Sobolev space $H^{1/2}(\Gamma_e)$, and c is a constant independent of N. By the trace theorem we obtain the inequality (47).

Furthermore,

$$\begin{aligned}
B(u,v;u,v) &= \frac{2\pi\mu}{(1+2\kappa)}\sum_{n=1}^{\infty} n\{a_n^2+b_n^2+c_n^2+d_n^2 \\
&\qquad + \kappa(b_n-c_n)^2 + \kappa(a_n+d_n)^2\} \ge 0 \quad \forall(u,v)\in V.
\end{aligned}$$

Similarly, for $B_N(u,v;\tilde{u},\tilde{v})$ we obtain

$$|B_N(u,v;\tilde{u},\tilde{v})| \le M_1\|(u,v)\|_V\|(\tilde{u},\tilde{v})\|_V \quad \forall(u,v),(\tilde{u},\tilde{v})\in V,$$
$$B_N(u,v;u,v) \ge 0 \quad \forall(u,v)\in V.$$

The proof of Lemma 2 is completed. □

On the other hand, we have

$$
\begin{aligned}
&|B(u, v; \tilde{u}, \tilde{v}) - B_N(u, v; \tilde{u}, \tilde{v})| \\
&= \left| \frac{2(1+\kappa)\pi\mu}{(1+2\kappa)} \sum_{n=N+1}^{\infty} n\{a_n\tilde{a}_n + b_n\tilde{b}_n + c_n\tilde{c}_n + d_n\tilde{d}_n\} \right. \\
&\qquad \left. + \frac{2\kappa\pi\mu}{(1+2\kappa)} \sum_{n=N+1}^{\infty} n\{a_n\tilde{d}_n - b_n\tilde{c}_n - c_n\tilde{b}_n + d_n\tilde{a}_n\} \right| \\
&\le \frac{4(1+\kappa)\pi\mu}{(1+2\kappa)} \left\{ \sum_{n=N+1}^{\infty} n(a_n^2 + b_n^2 + c_n^2 + d_n^2) \right\}^{1/2} \\
&\quad \cdot \left\{ \sum_{n=N+1}^{\infty} n(\tilde{a}_n^2 + \tilde{b}_n^2 + \tilde{c}_n^2 + \tilde{d}_n^2) \right\}^{1/2} \\
&\le \frac{4(1+\kappa)\pi\mu}{(1+2\kappa)N^{k-1}} \left\{ \sum_{n=N+1}^{\infty} (n^2)^{k-1/2}(a_n^2 + b_n^2 + c_n^2 + d_n^2) \right\}^{1/2} \\
&\quad \cdot \left\{ \sum_{n=N+1}^{\infty} n(\tilde{a}_n^2 + \tilde{b}_n^2 + \tilde{c}_n^2 + \tilde{d}_n^2) \right\}^{1/2} \\
&\le \frac{c}{N^{k-1}} \|(u, v)\|_{k-1/2,\Gamma_e} \|(\tilde{u}, \tilde{v})\|_{1/2,\Gamma_e}, \quad \forall k \ge 2.
\end{aligned}
$$

Hence we obtain the following error estimate:

Lemma 3. *The following error estimate holds*:

$$
(49)\quad |B(u, v; \tilde{u}, \tilde{v}) - B_N(u, v; \tilde{u}, \tilde{v})| \le \frac{c}{N^{k-1}} \|(u, v)\|_{k-1/2,\Gamma_e} \|(\tilde{u}, \tilde{v})\|_{1/2,\Gamma_e},
$$

with $k \ge 2$ *and* c *a constant independent of* N, (u, v), *and* $(\tilde{u}, \tilde{v})$. □

Theorem 1. *Suppose* $f_1, f_2 \in H^{-1}(\Omega_i)$; *then the variational problem* (45) *has a unique solution* $(u, v) \in V$ *and problem* (46) *has a unique solution* $(u_N, v_N) \in V$. *Furthermore, we have the following error estimate*:

$$
(50)\qquad \|(u - u_N, v - v_N)\|_V \le \frac{c}{\beta_0 N^{k-1}} \|(u, v)\|_{k-1/2,\Gamma_e}.
$$

Proof. By Lemmas 1 and 2, we know that $A(u, v; \tilde{u}, \tilde{v}) + B(u, v; \tilde{u}, \tilde{v})$ and $A(u, v; \tilde{u}, \tilde{v}) + B_N(u, v; \tilde{u}, \tilde{v})$ are two symmetric, bounded, and coercive bilinear functionals on $V \times V$. By Cauchy's inequality, $(f_1, f_2; \tilde{u}, \tilde{v})$ is a linear functional on V. From the Lax-Milgram theorem [2], we obtain that the problem (45) has a unique solution (u, v) which is the restriction to Ω_i of the solution (u, v) of the original problem (1)–(4), and the problem (46) has a unique solution (u_N, v_N).

Let $e_1 = u - u_N$ and $e_2 = v - v_N$; then (e_1, e_2) satisfies

$$
(51)\qquad \begin{aligned}
&A(e_1, e_2; \tilde{u}, \tilde{v}) + B_N(e_1, e_2; \tilde{u}, \tilde{v}) \\
&\quad = B_N(u, v; \tilde{u}, \tilde{v}) - B(u, v; \tilde{u}, \tilde{v}) \quad \forall (\tilde{u}, \tilde{v}) \in V.
\end{aligned}
$$

Taking $\tilde{u} = e_1$ and $\tilde{v} = e_2$ in (51), we get

$$\begin{aligned}\beta_0\|(e_1, e_1)\|_V^2 &\le A(e_1, e_2; e_1, e_2) \le |B_N(u, v; e_1, e_2) - B(u, v; e_1, e_2)| \\ &\le \frac{c}{N^{k-1}}\|(u, v)\|_{k-1/2, \Gamma_e}\|(e_1, e_2)\|_V.\end{aligned}$$

The last inequality comes from Lemma 3. The inequality (50) now follows immediately. □

4. The finite element approximation of problem (46)

For the sake of simplicity, let Γ_i be a polygonal line, and $\mathscr{T}_h$ be a triangulation of Ω_i satisfying

$$\Omega_i = \left(\bigcup_{K\in\mathscr{T}_h} K\right) \cap \left(\bigcup_{\widetilde{K}\in\mathscr{T}_h} \widetilde{K}\right),$$

where K is a triangle and $\widetilde{K}$ is a curved triangle with a curved side on Γ_e, and

$$\frac{h_k}{\rho_k} \le \sigma \quad \forall K, \widetilde{K} \in \mathscr{T}_h,$$

where $h_k =$ diameter of K or $\widetilde{K}$, $\rho_k =$ diameter of the inscribed circle of K or $\widetilde{K}$, and $h = \max_{K, \widetilde{K}\in\mathscr{T}_h} h_k$. Let

$$\begin{aligned}S_h(\Omega_i) &= \{v \in H_*^1(\Omega_i),\ v|_K\ (v|_{\widetilde{K}}) \text{ is a linear polynomial } \forall K(\widetilde{K}) \in \mathscr{T}_h\}, \\ V_h &= S_h(\Omega_i) \times S_h(\Omega_i).\end{aligned}$$

We know that the subspace V_h is a regular finite element space in the sense of Babuška and Aziz [1], which satisfies the following approximation property:

$$\inf_{(u_h, v_h)\in V_h} \|(u - u_h, v - v_h)\|_V \le ch\|(u, v)\|_{2,\Omega_i}. \tag{52}$$

We now consider the approximation problem of (46):

$$\begin{aligned}&\text{Find } (u_N^h, v_N^h) \in V_h \text{ such that} \\ &A(u_N^h, v_N^h; \tilde{u}, \tilde{v}) + B_N(u_N^h, v_N^h; \tilde{u}, \tilde{v}) \\ &\quad = \iint_{\Omega_i} (f_1\tilde{u} + f_2\tilde{v})\,dx\,dy \quad \forall(\tilde{u}, \tilde{v}) \in V_h.\end{aligned} \tag{53}$$

By the Lax-Milgram theorem we have

Theorem 2. *The variational problem* (53) *has a unique solution* $(u_N^h, v_N^h) \in V_h$.

Theorem 3. *Assume that* $u, v \in H^2(\Omega_i) \cap H^{k-1/2}(\Gamma_e)$, $k \ge 2$, *where* (u, v) *is the solution of problem* (45); *then the following error estimate holds*:

$$\|(u - u_N^h, v - v_N^h)\|_V \le c\left\{h\|(u, v)\|_{2,\Omega_i} + \frac{1}{N^{k-1}}\|(u, v)\|_{k-1/2,\Gamma_e}\right\},$$

where c *is a constant independent of* h *and* N.

Proof. From the equalities (45) and (53) we obtain

$$\begin{aligned}&A(u - u_N^h, v - v_N^h; \tilde{u}, \tilde{v}) + B_N(u - u_N^h, v - v_N^h; \tilde{u}, \tilde{v}) \\ &\quad = B_N(u, v; \tilde{u}, \tilde{v}) - B(u, v; \tilde{u}, \tilde{v}) \quad \forall(\tilde{u}, \tilde{v}) \in V_h.\end{aligned} \tag{54}$$

Then

$$\begin{aligned}
&\|(u_N^h-\tilde u, v_N^h-\tilde v)\|_V^2\\
&\quad\le \frac{1}{\beta_0}\{A(u_N^h-\tilde u, v_N^h-\tilde v; u_N^h-\tilde u, v_N^h-\tilde v)\\
&\qquad\quad + B_N(u_N^h-\tilde u, v_N^h-\tilde v; u_N^h-\tilde u, v_N^h-\tilde v)\}\\
&\quad= \frac{1}{\beta_0}\{A(u-\tilde u, v-\tilde v; u_N^h-\tilde u, v_N^h-\tilde v)\\
&\qquad\quad + B_N(u-\tilde u, v-\tilde v; u_N^h-\tilde u, v_N^h-\tilde v)\\
&\qquad\quad + B(u, v; u_N^h-\tilde u, v_N^h-\tilde v) - B_N(u, v; u_N^h-\tilde u, v_N^h-\tilde v)\}\\
&\quad\le \frac{(M_0+M_1)}{\beta_0}\|(u-\tilde u, v-\tilde v)\|_V\|(u_N^h-\tilde u, v_N^h-\tilde v)\|_V\\
&\qquad + \frac{c}{\beta_0 N^{k-1}}\|(u, v)\|_{k-1/2,\Gamma_e}\|(u_N^h-\tilde u, v_N^h-\tilde v)\|_V \quad \forall(\tilde u, \tilde v)\in V_h.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\|(u_N^h-\tilde u, v_N^h-\tilde v)\|_V \le{}& \frac{(M_0+M_1)}{\beta_0}\|(u-\tilde u, v-\tilde v)\|_V\\
&+ \frac{c}{\beta_0 N^{k-1}}\|(u, v)\|_{k-1/2,\Gamma_e} \quad \forall(\tilde u, \tilde v)\in V_h.
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}
\|(u-u_N^h, v-v_N^h)\|_V &\le \|(u-\tilde u, v-\tilde v)\|_V + \|(u_N^h-\tilde u, v_N^h-\tilde v)\|_V\\
&\le \left(\frac{(M_0+M_1)}{\beta_0}+1\right)\|(u-\tilde u, v-\tilde v)\|_V\\
&\quad + \frac{c}{\beta_0 N^{k-1}}\|(u, v)\|_{k-1/2,\Gamma_e}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
\|(u-u_N^h, v-v_N^h)\|_V \le{}& \frac{M_0+M_1+\beta_0}{\beta_0}\inf_{(\tilde u,\tilde v)\in V_h}\|(u-\tilde u, v-\tilde v)\|_V\\
&+ \frac{c}{\beta_0 N^{k-1}}\|(u, v)\|_{k-1/2,\Gamma_e}.
\end{aligned}$$

By inequality (52), the proof is completed. □

5. Numerical example

Suppose that the unbounded domain $\Omega = \{(x, y)\in\Omega,\ 1<|x| \text{ or } 1<|y|\}$ is the exterior domain of the square $[-1, 1]\times[-1, 1]$ with boundary Γ_i. Let

$$\begin{aligned}
u_1(x, y) = \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)}&\left\{\frac{1}{2}\log\frac{x^2+(y+0.5)^2}{x^2+(y-0.5)^2}\right.\\
&\left.+\frac{\lambda+\mu}{\lambda+3\mu}\left(\frac{x^2}{x^2+(y-0.5)^2}-\frac{x^2}{x^2+(y+0.5)^2}\right)\right\},\\
v_1(x, y) = \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)}&\left(\frac{x(y-0.5)}{x^2+(y-0.5)^2}-\frac{x(y+0.5)}{x^2+(y+0.5)^2}\right).
\end{aligned}$$

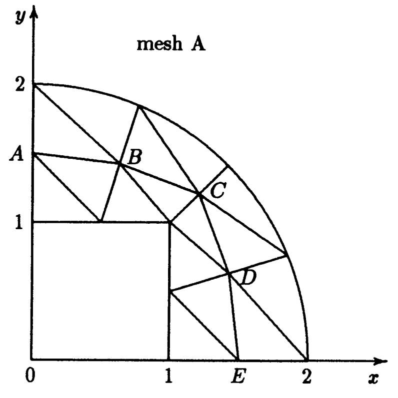

FIGURE 2

Then (u_1, v_1) is the unique solution of the following boundary value problem:

$$-\mu\Delta u - (\lambda+\mu)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0 \quad \text{in } \Omega,$$

$$-\mu\Delta v - (\lambda+\mu)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) = 0 \quad \text{in } \Omega,$$

$$u = u_1 \quad \text{on } \Gamma_i,$$

$$v = v_1 \quad \text{on } \Gamma_i,$$

$$u, v \text{ are bounded when } r \to +\infty.$$

We take Γ_e as a circumference with radius 2; then we consider the finite element approximation of (u, v) on the bounded domain $\Omega_i = \{(x, y) \in \Omega$ and $r < 2\}$.

Since u_1 and v_1 are symmetric about the x and y axes, respectively, and antisymmetric about the y and x axes, respectively, the domain of computation was taken to be the part lying in the first quadrant. The symmetric and antisymmetric boundary conditions were used along $x = 0$ and $y = 0$.

Three meshes were used in computation. Figure 2 shows the triangulation for mesh A. Mesh B was generated by dividing the triangles in mesh A into four small triangles, and mesh C was similarly generated. Linear finite element approximation was used in computation. Table 1 shows the maximum of the errors $u - u_N^h$ and $v - v_N^h$ over the mesh points when $N = 5$. Since the maximum norm of u is about 0.117, the maximum relative error for u is

TABLE 1. Maximum error for $N = 5$

mesh	A	B	C
h	0.36	0.18	0.09
$\max\lvert u_i - u_{N,i}^h\rvert$	0.370d-02	0.117d-02	0.294d-03
$\max\lvert v_i - v_{N,i}^h\rvert$	0.651d-02	0.252d-02	0.840d-03

TABLE 2. Maximum error for mesh A

N	0	1	3	5
$\max \lvert u_i - u^h_{N,i} \rvert$	0.433d-01	0.610d-02	0.358d-02	0.370d-02
$\max \lvert v_i - v^h_{N,i} \rvert$	0.721d-02	0.148d-01	0.658d-02	0.651d-02

TABLE 3. Maximum error for mesh B

N	0	1	3	5
$\max \lvert u_i - u^h_{N,i} \rvert$	0.412d-01	0.521d-02	0.116d-02	0.117d-02
$\max \lvert v_i - v^h_{N,i} \rvert$	0.493d-02	0.101d-01	0.231d-02	0.252d-02

TABLE 4. Maximum error for mesh C

N	0	1	3	5
$\max \lvert u_i - u^h_{N,i} \rvert$	0.409d-01	0.568d-02	0.433d-03	0.294d-03
$\max \lvert v_i - v^h_{N,i} \rvert$	0.408d-02	0.816d-02	0.624d-03	0.840d-03

about 3.2% for mesh A, 1% for mesh B, and 0.25% for mesh C. The maximum norm of v is about 0.555, hence the maximum relative error for v is about 11.7% for mesh A, 4.54% for mesh B, and 1.51% for mesh C. The convergence is fast; in fact, the rates are much higher than linear.

Table 2 shows the maximum of the errors $u - u^h_N$ and $v - v^h_N$ for mesh A when $N = 1, 3$, and 5; Tables 3–4 show the analogous results for meshes B and C. As we can see from the tables, for u, $N = 3$ is good enough for meshes A and B, since the meshes are too coarse and then the main errors are due to the coarse meshes. This becomes clear when the mesh is refined, $N = 5$ did improve the accuracy for mesh C. For v, the effects of N were not so significant as for u for meshes A and B; this is because on the boundary Γ_e, v is very close to zero, so even for $N = 0$, the error is already small. The effects of N showed up only for finer meshes, as is shown in Table 4 for mesh C.

Figures 3–5 show the results for u and v along some curves, where the interior points are the points along the curve $ABCDE$ shown in Figure 2, and the boundary points are the points along the boundary Γ_e, i.e., the circumference with radius 2. The effects of N are shown for meshes A–C; as shown in the figures, $N = 5$ gives good approximations, and therefore in the computation very few terms in the bilinear form $B_N(u, v; \tilde{u}, \tilde{v})$ are needed in order to get good accuracy.

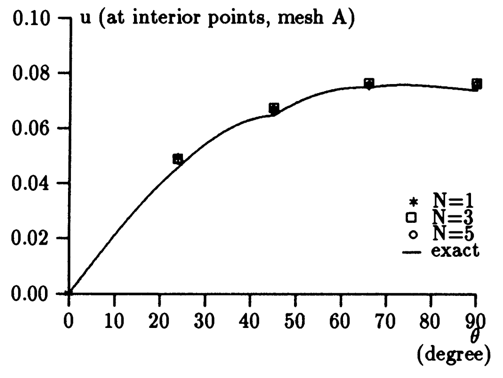

FIGURE 3

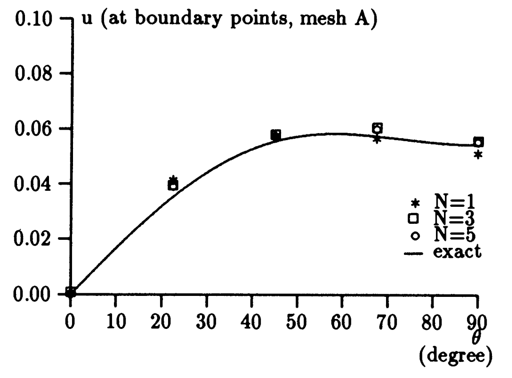

FIGURE 4

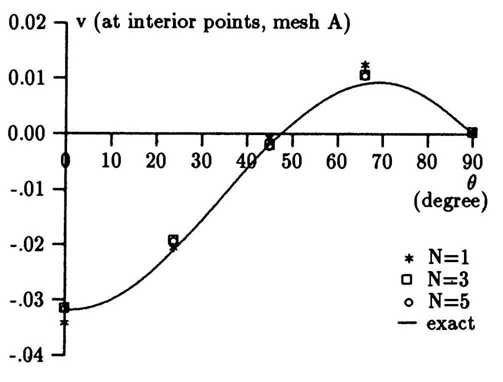

FIGURE 5

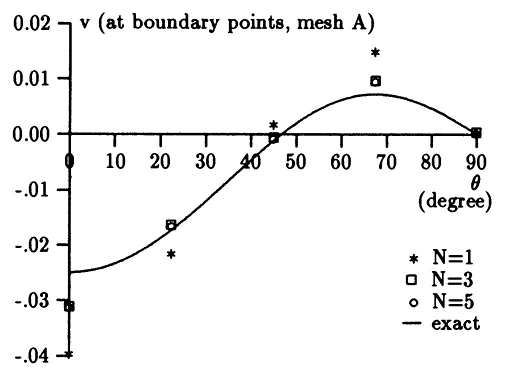

FIGURE 6

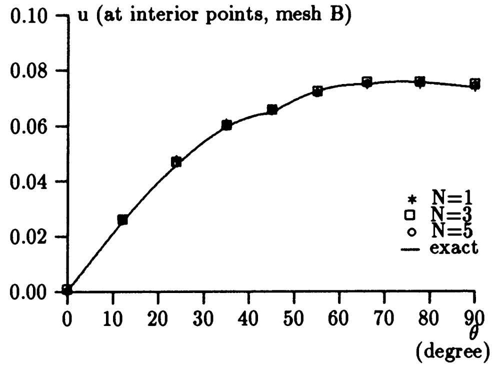

FIGURE 7

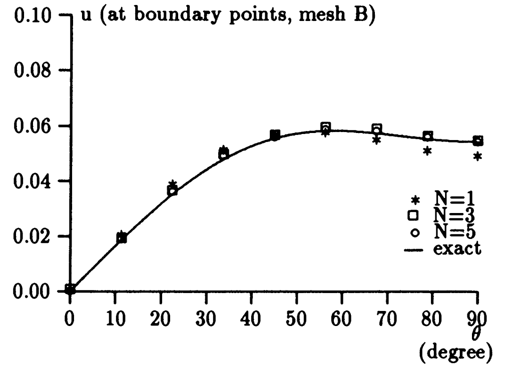

FIGURE 8

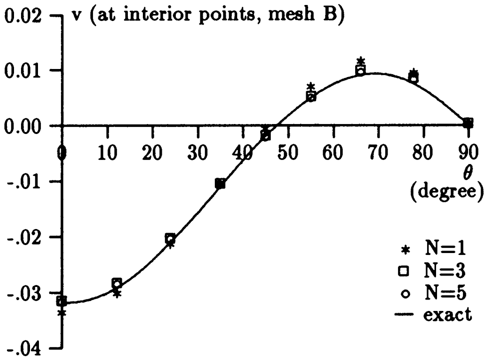

FIGURE 9

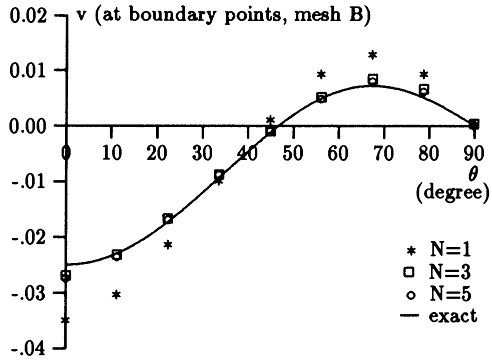

FIGURE 10

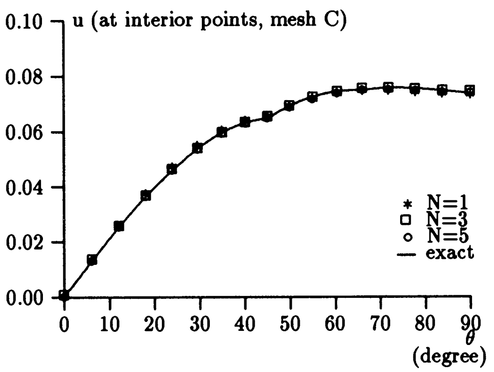

FIGURE 11

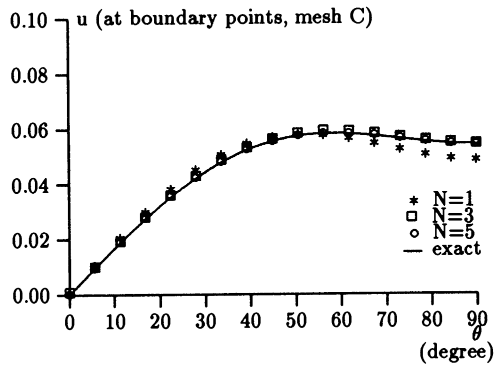

FIGURE 12

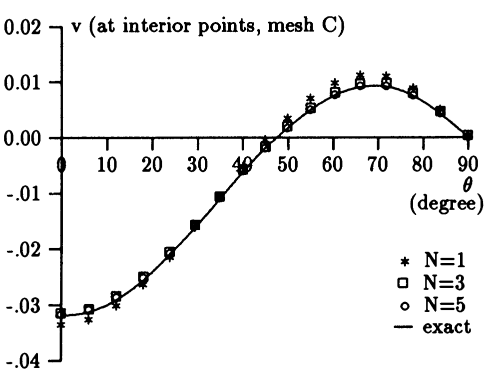

FIGURE 13

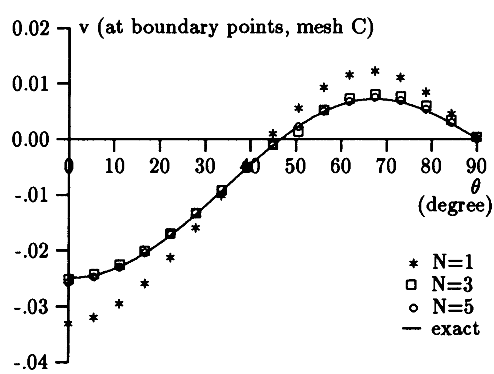

FIGURE 14

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