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FINITE ELEMENT APPROXIMATION TO INITIAL-BOUNDARY VALUE PROBLEMS OF THE SEMICONDUCTOR DEVICE EQUATIONS WITH MAGNETIC INFLUENCE

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ABSTRACT. We shall consider Zlámal's approach to the nonstationary equations of the semiconductor device theory under magnetic fields, with mixed boundary conditions. Owing to the reduced smoothness of the electric potential ψ and carrier densities n and p caused by considering the mixed boundary conditions, we must use a nonstandard analysis for this procedure. Existence as well as uniqueness of the approximate solution is proved. The convergence rates obtained in this paper are slower than those previously obtained for pure Dirichlet or Neumann boundary conditions.

1. INTRODUCTION

We shall consider a system of three quasilinear partial differential equations in a bounded polygonal domain $\Omega \in \mathbf{R}^2$, which form a basic model of the transient behavior of a semiconductor device in a magnetic field (cf. Allegretto, Mun, Nathan, and Baltes [1], and Wang [30]):

$$(1.1)\qquad \begin{aligned} &\text{(a)}\ -\Delta\psi = \frac{q}{\varepsilon}(p-n+N),\\ &\text{(b)}\ \frac{\partial n}{\partial t} - \nabla\cdot[e^{a\psi}A_1\nabla(e^{-a\psi}n)] + R_n(n,p) = 0,\\ &\text{(c)}\ \frac{\partial p}{\partial t} - \nabla\cdot[e^{-a\psi}A_2\nabla(e^{a\psi}p)] + R_p(n,p) = 0. \end{aligned}$$

The unknowns are the electrostatic potential ψ and the electron and hole densities n and p, while q and ε are constants (q is the electron charge, ε is the permittivity). The function N is the total electric active net impurity density or "doping", and a is a positive constant; $R_n(n,p)$ and $R_p(n,p)$ are the recombination rates. The matrices

$$(1.2)\qquad A_i = \begin{pmatrix} a_{11}^{(i)} & a_{12}^{(i)} \\ a_{21}^{(i)} & a_{22}^{(i)} \end{pmatrix} = \frac{1}{1+\beta^2}\begin{pmatrix} 1 & (-1)^i\beta \\ (-1)^{i-1}\beta & 1 \end{pmatrix}, \qquad i = 1, 2,$$

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are positive definite nonsymmetric. The reason for the asymmetry of A_i $(i = 1, 2)$ is the presence of a magnetic field $\vec{B} = (0, 0, \beta)$, where $\beta = \beta(x)$ is Lipschitz continuous. Indeed, if $\vec{B} = 0$, then A_i $(i = 1, 2)$ are positive definite symmetric. Since A_i $(i = 1, 2)$ are positive definite and bounded, there are two positive constants a_0 and M such that, for any $u = (u_1, u_2)$, $v = (v_1, v_2) \in \mathbf{R}^2$,

$$a_0|u|^2 \le (A_i u, u), \qquad i = 1, 2, \tag{1.3}$$

$$|(A_i u, v)| \le M|u||v|, \qquad i = 1, 2, \tag{1.4}$$

with $|u|^2 = \sum_{i=1}^2 u_i^2$.

We simplify the system, taking $a = q/\varepsilon = 1$ and $R_n(n, p) = R_p(n, p) = R(n, p)$ (assumed Lipschitz continuous). These simplifications are not essential, neither for the construction of the approximate solution nor for the results of this paper. We can write (1.1) in the form

$$\begin{aligned} &\text{(a)}\ -\Delta\psi = p - nN, \\ &\text{(b)}\ \frac{\partial n}{\partial t} - \nabla\cdot[A_1(\nabla n - n\nabla\psi)] + R(n, p) = 0, \\ &\text{(c)}\ \frac{\partial p}{\partial t} - \nabla\cdot[A_2(\nabla p + p\nabla\psi)] + R(n, p) = 0. \end{aligned} \tag{1.5}$$

For simplicity, consider the PDE system (1.5) with homogeneous mixed boundary conditions:

$$\{\psi, n, p\} = \{0, 0, 0\} \quad \text{on } \partial\Omega_D \times I, \tag{1.6}$$

$$\left\{\frac{\partial\psi}{\partial\nu}, J_n\cdot\nu, J_p\cdot\nu\right\} = \{0, 0, 0\} \quad \text{on } \partial\Omega_N \times I. \tag{1.7}$$

Here, the boundary $\partial\Omega$ of Ω has been decomposed into the union $\partial\Omega_D \cup \partial\Omega_N$, where $\partial\Omega_D$ is of positive measure in $\partial\Omega$, $I = [0, T]$, ν is the outward unit normal vector on $\partial\Omega$, $J_n = A_1(\nabla n - n\nabla\psi)$, and $J_p = A_2(\nabla p + p\nabla\psi)$.

In addition, we have the initial condition

$$n = n^0(x), \quad p = p^0(x) \quad \text{in } \Omega. \tag{1.8}$$

Remark 1.1. For nonhomogeneous mixed boundary conditions with smooth data, the problems can be homogenized by Banasiak and Roach's trace theorem in [2].

There is much work concerning the basic semiconductor device equations with no magnetic fields. For stationary problems, Mock [23, 25] showed the existence and uniqueness of a solution subject to the mixed boundary conditions (with $R = 0$). A very similar existence proof was given by Bank, Jerome, and Rose [4], and effective numerical algorithms were also presented in their paper. Later, Jerome [17] proved the existence for a more general stationary problem. A singular perturbation analysis for the problems was given by Markowich [19, 20], Markowich and Ringhofer [21], and Selberherr and Ringhofer [27]. Finite difference or finite element methods are discussed in Markowich [20] and in references therein. Recently, Ringhofer and Schmeiser [26] analyzed an iterative method and its convergence. For nonstationary problems, Mock [24] was the first to prove a global existence and uniqueness result, and a more general type of the boundary conditions was discussed in Gajewski [13] and Gajewski and Gröger [14]. With regard to numerical treatments, Zlámal [33] has proposed two fully discrete finite element schemes (one is nonlinear, the other is partly linear) and discussed the existence (for both schemes) and uniqueness (for the second scheme) of the approximations. Stability, uniqueness, and convergence (for the first scheme) of the approximations have been investigated under stronger assumptions in Zlámal [34]. The mixed finite element-characteristic procedure for the one-dimensional Dirichlet problem was introduced by Douglas, Gamba, and Squeff [7], and Gamba and Squeff [15], Douglas and Yuan [8], and Douglas, Yuan, and Li [9, 10] have discussed, respectively, the finite difference-characteristic finite difference procedure, the mixed finite element-characteristic finite difference procedure, and the mixed finite element-characteristic finite element procedure for the two-dimensional Dirichlet problem and the Neumann problem, and have given the convergence analyses under the assumption of a smooth solution.

However, to our knowledge, there is not much work on problems in the presence of magnetic fields. A finite element analysis for stationary problems was given by Allegretto, Mun, Nathan, and Baltes [1], but there was no theoretical analysis of approximation in their paper. Recently, the author [32] presented and analyzed the problem (1.1) by the finite difference-characteristic finite difference procedure, considering nonhomogeneous Dirichlet boundary conditions.

Unfortunately, it is well known (see, e.g., [2, 28]) that in general the solutions of mixed boundary value problems for elliptic equations are not smooth, no matter how smooth the data may be, and moreover, the loss of smoothness occurs in the vicinity of $\overline{\partial\Omega_D}\cap\overline{\partial\Omega_N}$. Hence, the solutions of (1.5)–(1.8) are certainly not smooth. Similar to the idea of Ewing and Wheeler [12], we shall use in this paper a nonstandard analysis for Zlámal's approach. Since the resulting functions are considerably less smooth than previously assumed, the convergence rates obtained in this paper are slower than those previously obtained. Recently, Markowich and Zlámal [22] have generalized Zlámal's approach to mixed boundary value problems of second-order elliptic equations.

The paper contains two additional sections. In §2, terminology is developed, a variational form of the problem (1.5)–(1.8), basic regularity and boundedness assumptions are presented, and the continuous-time Zlámal's approach to (1.5)–(1.8) is defined. In §3, existence, uniqueness, and a priori error estimates for this approach are obtained. Throughout, the symbols C and δ will denote, respectively, a generic constant and a generic small positive constant.

2. Preliminaries and description of approximations

Let $(u, v) = \int_\Omega uv\,dx$ and $\|u\|^2 = (u, u)$ be the standard L^2 inner product and norm. Let $W_q^k(\Omega)$ be the Sobolev space on Ω with norm

$$\|u\|_{W_q^k} = \left(\sum_{|\alpha|\le k}\left\|\frac{\partial^\alpha u}{\partial x^\alpha}\right\|_{L^q(\Omega)}^q\right)^{1/q}, \tag{2.1}$$

with the usual modification for $q = \infty$. If $U = (u_1, u_2)$, write $\|U\|_{W_q^k}$ in place of $(\|u_1\|_{W_q^k}^q + \|u_2\|_{W_q^k}^q)^{1/q}$. When $q = 2$, denote $\|u\|_{W_q^k} = \|u\|_{H^k} = \|u\|_k$.

If $k=0$, $\|u\|_0=\|u\|$. We also denote by $H^{1+\sigma}(\Omega)$ (σ is a real number with $0<\sigma<1$) the noninteger Sobolev space on Ω with norm (see, e.g., Girault and Raviart [16])

$$\begin{aligned}\|u\|_{1+\sigma} &= \|u\|_{H^{1+\sigma}(\Omega)} \\ &= \left\{\|u\|_1^2+\sum_{|\alpha|=1}\int_\Omega\int_\Omega \frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{2(1+\sigma)}}\,dx\,dy\right\}^{1/2},\end{aligned} \tag{2.2}$$

where $|x|$ denotes the Euclidean norm of $\mathbf{R}^2$. Let $I=[0,T]$, and

$$V=\{v\in H^1(\Omega)\,;\, v|_{\partial\Omega_D}=0\}. \tag{2.3}$$

Multiplying (1.5) by a function $v\in V$, integrating over Ω, and using Green's theorem and (1.7), we have

$$\begin{aligned}&\text{(a)}\ d(\psi,v)=(p-n+N,v), && v\in V,\\ &\text{(b)}\ \left(\frac{\partial n}{\partial t},v\right)+\nu(\psi;n,v)+(R(n,p),v)=0, && v\in V,\\ &\text{(c)}\ \left(\frac{\partial p}{\partial t},v\right)+\pi(\psi;p,v)+(R(n,p),v)=0, && v\in V,\end{aligned} \tag{2.4}$$

where

$$\begin{aligned}d(\psi,v)&=\int_\Omega \nabla\psi\cdot\nabla v\,dx,\\ \nu(\psi;n,v)&=\int_\Omega A_1(\nabla n-n\nabla\psi)\cdot\nabla v\,dx,\\ \pi(\psi;p,v)&=\int_\Omega A_2(\nabla p+p\nabla\psi)\cdot\nabla v\,dx.\end{aligned} \tag{2.5}$$

We are looking for $\{\psi,n,p\}\colon I\to V\times V\times V$.

Let $\{\psi,n,p\}$, the solution of (1.5)–(1.8), satisfy the following regularity assumptions:

$$\begin{aligned}&\text{(a)}\ \|\psi\|_{L^\infty(H^{1+\sigma})}+\|n\|_{L^2(H^{1+\sigma})}+\|p\|_{L^2(H^{1+\sigma})}\le C,\\ &\text{(b)}\ \|n\|_{L^\infty(L^\infty)}+\|p\|_{L^\infty(L^\infty)}\le C,\\ &\text{(c)}\ \left\|\frac{\partial n}{\partial t}\right\|_{L^2(L^2)}+\left\|\frac{\partial p}{\partial t}\right\|_{L^2(L^2)}\le C,\end{aligned} \tag{2.6}$$

where $0<\sigma<1$ and C are fixed constants, $\|u\|_{L^q(X)}=\|u\|_{L^q(I;X)}$, $q=2,\infty$, and X is a Sobolev space on Ω. In view of [1, 2, 4, 13, 14, 28], the above assumptions are reasonable.

In this paper we restrict our attention to continuous-time Zlámal's approach (nonlinear scheme) to (1.5)–(1.8). We consider a family $\{T_h\}$ of triangulations of $\overline{\Omega}$. Let K denote an element of T_h, $h_K=\operatorname{diam}(K)$, and $h=\max_{K\in T_h}h_K$. As in Zlámal [34], we assume that the family $\{T_h\}$ satisfies the minimum angle condition and is of acute type. Thus, if J is the Jacobian matrix of the linear mapping which maps a given triangle K on the reference triangle $\widehat{K}$, then

$$Ch^2 \leq |\det J| \leq C^{-1}h^2 \quad \forall K \in \bigcup_h T_h.$$

We shall use the above implicitly in some places.

With each partition from $\{T_h\}$ we associate the finite-dimensional space $V_h = \{v_h \in C(\overline{\Omega})\,;\ v_h$ is a linear polynomial on each $K \in T_h\,,\ v_h|_{\partial\Omega_D} = 0\}$.

We use the same idea as in Mock [25] and Zlámal [33, 34]: the quantities J_n, J_p, and $\|\nabla\psi\|$ are approximated by constants on each element K. Let ψ_h, n_h, p_h, and v_h belong to V_h. The discrete analogs of the forms $\nu(\psi\,; n\,, v)$ and $\pi(\psi\,; p\,, v)$ are

$$\nu_h(\psi_h\,; n_h\,, v_h) = \sum_{K\in T_h}\sum_{r=j,k,m} v_{hr}\left\{\int_K A_1^K (J^T)^{-1} D^K J^T \nabla n_h \cdot \nabla v^r\,dx - \int_K n_{hr} A_1^K \nabla\psi_h \cdot \nabla v^r\,dx\right\}, \tag{2.7}$$

$$\pi_h(\psi_h\,; p_h\,, v_h) = \sum_{K\in T_h}\sum_{r=j,k,m} v_{hr}\left\{\int_K A_2^K (J^T)^{-1} B^K J^T \nabla p_h \cdot \nabla v^r\,dx + \int_K p_{hr} A_2^K \nabla\psi_h \cdot \nabla v^r\,dx\right\}, \tag{2.8}$$

where

$$A_i^K = \begin{pmatrix} a^K & (-1)^i b^K \\ (-1)^{i-1} b^K & a^K \end{pmatrix}, \qquad i = 1\,, 2\,,$$

$a^K = 1/(1+(\beta^K)^2)$, $b^K = \beta^K/(1+(\beta^K)^2)$, $\beta^K = \beta(x^K)$, and x^K is the center of gravity of the element K. Here, J is the Jacobian matrix of the mapping which maps K on $\widehat{K}$ in such a way that the node x^r is mapped on the vertex $(0\,, 0)$ in the reference plane (see Zlámal [33]), v_{hr} is the value $v_h(x^r)$, v^r is the basis function associated with the node x^r, and B^K, D^K are the matrices

$$\begin{aligned} B^K &= \operatorname{diag}(B(\psi_{h1} - \psi_{h2})\,, B(\psi_{h1} - \psi_{h3}))\,, \\ D^K &= \operatorname{diag}(D(\psi_{h1} - \psi_{h2})\,, D(\psi_{h1} - \psi_{h3})). \end{aligned} \tag{2.9}$$

Here, $B(\zeta) = \zeta(e^\zeta - 1)^{-1}$ and $D(\zeta) = e^\zeta B(\zeta) = B(-\zeta)$, $-\infty < \zeta < \infty$. Furthermore, ψ_{h1}, ψ_{h2}, ψ_{h3} are the local notations of the values of ψ_h at the vertices x^j, x^k, x^m such that $\psi_{h1} = \psi_{hr}$, $r = j\,, k\,, m$.

Remark 2.1. For A_i^K, $i = 1\,, 2$, the inequalities (1.3) and (1.4) are still valid.

The $L^2(\Omega)$-scalar product $(\cdot\,, \cdot)$ will be approximated by $(\cdot\,, \cdot)_h$ defined in Zlámal [33] ($(u\,, v)_h = \sum_{j=1}^q m_j u_j v_j$, $m_j > 0$, q is the number of all nodes not lying on $\partial\Omega_D$). Let f_I denote the interpolate of a given function f.

Now we can introduce the continuous-time approximation of $\{\psi\,, n\,, p\}$ as follows: let $\{\psi_h\,, n_h\,, p_h\}\colon I \to V_h \times V_h \times V_h$ be defined by

$$(2.10)\quad \begin{aligned} &\text{(a) } d(\psi_h, v_h) = (p_h - n_h + N_I, v_h)_h, \qquad v_h \in V_h, \\ &\text{(b) } \left(\frac{\partial n_h}{\partial t}, v_h\right)_h + \nu_h(\psi_h; n_h, v_h) + (R(n_h, p_h), v_h)_h = 0, \qquad v_h \in V_h, \\ &\text{(c) } \left(\frac{\partial p_h}{\partial t}, v_h\right)_h + \pi_h(\psi_h; p_h, v_h) + (R(n_h, p_h), v_h)_h = 0, \qquad v_h \in V_h, \\ &\text{(d) } n_h(0) = n_I^0, \qquad p_h(0) = p_I^0. \end{aligned}$$

The main results of this paper are the existence, uniqueness, and a priori error estimates for the approximation $\{\psi_h, n_h, p_h\}$. These will be developed in the next section.

3. Existence, uniqueness, and a priori error estimates

Similar to the idea introduced by Ewing and Wheeler [12] for miscible displacement problems, we first define the L^2 projection $\{\bar{n}, \bar{p}\}$ of $\{n, p\}$ into $V_h \times V_h$ by

$$(3.1)\quad \begin{aligned} &\text{(a) } (n - \bar{n}, v_h) = 0, \qquad v_h \in V_h, \text{ or} \\ &\text{(b) } \left(\frac{\partial n}{\partial t} - \frac{\partial \bar{n}}{\partial t}, v_h\right) = 0, \qquad v_h \in V_h; \\ &\text{(c) } (p - \bar{p}, v_h) = 0, \qquad v_h \in V_h, \text{ or} \\ &\text{(d) } \left(\frac{\partial p}{\partial t} - \frac{\partial \bar{p}}{\partial t}, v_h\right) = 0, \qquad v_h \in V_h. \end{aligned}$$

We are led to use the L^2 projection of $\{n, p\}$ into $V_h \times V_h$ instead of the now more standard H^1 projection, owing to smoothness restrictions on n and p. Since we assume that $\frac{\partial n}{\partial t}$ and $\frac{\partial p}{\partial t}$ are only in $L^2(I; L^2)$, we are not able to treat terms like $\frac{\partial}{\partial t}(n - \bar{n})$ and $\frac{\partial}{\partial t}(p - \bar{p})$ in the usual fashion. Thus, we have used $\{\bar{n}, \bar{p}\}$ in (3.1b, d) to remove this problem. Using the theory of interpolation spaces, we obtain (see, e.g., Ewing and Wheeler [12])

Lemma 3.1. *There exists a positive constant C such that, for each $t \in I$,*

$$(3.2)\quad \begin{aligned} &\text{(a) } \|n - \bar{n}\| + h\|n - \bar{n}\|_1 \le C\|n\|_{s_1} h^{s_1}, \qquad 1 \le s_1 \le 1 + \sigma, \\ &\text{(b) } \|p - \bar{p}\| + h\|p - \bar{p}\|_1 \le C\|p\|_{s_1} h^{s_1}, \qquad 1 \le s_1 \le 1 + \sigma, \\ &\text{(c) } \|n - \bar{n}\|_{L^\infty} \le C\|n\|_{W_\infty^{s_2}} h^{s_2}, \qquad 0 \le s_2 \le 1, \\ &\text{(d) } \|p - \bar{p}\|_{L^\infty} \le C\|p\|_{W_\infty^{s_2}} h^{s_2}, \qquad 0 \le s_2 \le 1, \\ &\text{(e) } \|v - v_I\| + h\|v - v_I\|_1 \le C\|v\|_{s_3} h^{s_3} \quad \forall v \in H^{s_3}(\Omega), \ 1 \le s_3 \le 1 + \sigma. \end{aligned}$$

Assuming that the family $\{V_h\}$ satisfies the following inverse inequalities (see Ciarlet [5] and Thomée [29]), we also have

Lemma 3.2. *There exists a positive constant C such that, for any $v_h \in V_h$,*

$$(3.3)\quad \begin{aligned} &\text{(a) } \|v_h\|_{L^q} \le Ch^{2/q-1}\|v_h\|, \qquad 2 \le q \le \infty, \\ &\text{(b) } \|\nabla v_h\|_{L^q} \le Ch^{2/q-1}\|\nabla v_h\|, \qquad 2 \le q \le \infty, \\ &\text{(c) } \|v_h\|_1 \le Ch^{-1}\|v_h\|, \\ &\text{(d) } \|v_h\|_{L^\infty} \le C|\log h|^{1/2}\|\nabla v_h\|. \end{aligned}$$

Finally, we need (see Ciarlet [5])

Lemma 3.3. *There exists a positive constant* C *such that, for all real* q *with* $1 \le q \le \infty$,

$$(3.4)\qquad \begin{aligned} &\text{(a) } \|\hat{v}\|_{L^q(\widehat{K})} \le C h_K^{-2/q} \|v\|_{L^q(K)} \quad \forall v \in L^q(K), \\ &\text{(b) } \|\nabla\hat{v}\|_{L^q(\widehat{K})} \le C h_K^{1-2/q} \|\nabla v\|_{L^q(K)} \quad \forall v \in W_q^1(K), \end{aligned}$$

where $\hat{v}(\zeta) = v(x(\zeta))$, $x(\zeta) = \sum_{r=j,k,m} x^r v^r(\zeta)$.

We shall prove the main results of this paper similarly as was done by the author in [31]. By elementary, but tedious computations, we can write the form ν_h as follows:

$$(3.5)\qquad \nu_h(\psi_h; n_h, v_h) = a_1(\psi_h; n_h, v_h) - a_2(\psi_h; n_h, v_h) - c(\psi_h; n_h, v_h),$$

$$(3.6)\qquad \begin{aligned} a_1(\psi_h; n_h, v_h) = \sum_{K\in T_h} a^K\{&\alpha_j^K b_{mk}(n_{hm} - n_{hk})(v_{hm} - v_{hk}) \\ &+ \alpha_k^K b_{jm}(n_{hj} - n_{hm})(v_{hj} - v_{hm}) \\ &+ \alpha_m^K b_{kj}(n_{hk} - n_{hj})(v_{hk} - v_{hj})\}, \end{aligned}$$

$$(3.7)\qquad \begin{aligned} a_2(\psi_h; n_h, v_h) = \frac{1}{2}\sum_{K\in T_h} b^K\{&b_{mk}(n_{hm} - n_{hk})(v_{hm} + v_{hk}) \\ &+ b_{jm}(n_{hj} - n_{hm})(v_{hj} + v_{hm}) \\ &+ b_{kj}(n_{hk} - n_{hj})(v_{hk} + v_{hj})\}, \end{aligned}$$

$$(3.8)\qquad c(\psi_h; n_h, v_h) = c_1(\psi_h; n_h, v_h) - c_2(\psi_h; n_h, v_h),$$

(3.9)
$$\begin{aligned} c_1(\psi_h; n_h, v_h) = \frac{1}{2}\sum_{K\in T_h} a^K\{&\alpha_j^K(\psi_{hm} - \psi_{hk})(n_{hm} + n_{hk})(v_{hm} - v_{hk}) \\ &+ \alpha_k^K(\psi_{hj} - \psi_{hm})(n_{hj} + n_{hm})(v_{hj} - v_{hm}) \\ &+ \alpha_m^K(\psi_{hk} - \psi_{hj})(n_{hk} + n_{hj})(v_{hk} - v_{hj})\}, \end{aligned}$$

$$(3.10)\qquad \begin{aligned} c_2(\psi_h; n_h, v_h) = \frac{1}{4}\sum_{K\in T_h} b^K\{&(\psi_{hm} - \psi_{hk})(n_{hm} + n_{hk})(v_{hm} + v_{hk}) \\ &+ (\psi_{hj} - \psi_{hm})(n_{hj} + n_{hm})(v_{hj} + v_{hm}) \\ &+ (\psi_{hk} - \psi_{hj})(n_{hk} - n_{hj})(v_{hk} + v_{hj})\}. \end{aligned}$$

Here, $\alpha_r^K = \frac{1}{2}\cot\theta_r$, $r = j, k, m$, where θ_r denotes the measure of the angle of K lying at the vertex x^r, and $b_{rs} = b_{sr} = \frac{1}{2}(B(\psi_{hr} - \psi_{hs}) + B(\psi_{hs} - \psi_{hr}))$. From the acuteness and the minimum angle condition it follows immediately that

$$(3.11)\qquad 0 \le \alpha_r^K \le C, \qquad r = j, k, m, \ \forall K \in \bigcup_h T_h.$$

It is furthermore known that (see Zlámal [34]) $b_{rs} \geq 1$. Therefore,

$$a_1(\psi_h; v_h, v_h) \geq a_1(0; v_h, v_h) \geq a_0\|\nabla v_h\|^2 \quad \forall v_h \in V_h. \tag{3.12}$$

In a similar way we can derive

$$\pi_h(\psi_h; p_h, v_h) = a_1(\psi_h; p_h, v_h) + a_2(\psi_h; p_h, v_h) - \bar{c}(\psi_h; p_h, v_h), \tag{3.13}$$

$$\bar{c}(\psi_h; p_h, v_h) = c_1(\psi_h; p_h, v_h) + c_2(\psi_h; p_h, v_h). \tag{3.14}$$

Theorem 3.1. *The problem* (2.10) *has a unique local solution* $\{\psi_h, n_h, p_h\}$, *i.e., there exists a positive constant* T^* (*defined by* (3.28)) *such that the problem* (2.10) *is uniquely solvable for* $t \in [0, T^*]$.

Proof. (I) *Existence.* Let $\{n_h, p_h\} \in V_h \times V_h$ be fixed. Then there exists a unique $\psi_h \in V_h$ such that

$$d(\psi_h, v_h) = (p_h - n_h + N_I, v_h)_h, \qquad v_h \in V_h. \tag{3.15}$$

Let $v_h = \psi_h$ in (3.15), and note that

$$\|v_h\| \leq \|v_h\|_h \leq C\|v_h\| \quad \forall v_h \in V_h. \tag{3.16}$$

We obtain

$$\|\nabla \psi_h\| \leq C\{\|n_h\| + \|p_h\| + 1\}. \tag{3.17}$$

We denote by $\mathbb{P}$ the mapping from V_h into itself, assigning $u_h = \{n_h, p_h\}$ to $w_h = \{w_h^{(1)}, w_h^{(2)}\}$ such that

$$\text{(3.18)}\quad \begin{aligned} &\text{(a)}\ \left(\frac{\partial w_h^{(1)}}{\partial t}, v_h\right)_h + a_1(\psi_h; w_h^{(1)}, v_h) \\ &\qquad = a_2(\psi_h; n_h, v_h) + c(\psi_h; n_h, v_h) - (R(n_h, p_h), v_h)_h, \qquad v_h \in V_h, \\ &\text{(b)}\ \left(\frac{\partial w_h^{(2)}}{\partial t}, v_h\right)_h + a_1(\psi_h; w_h^{(2)}, v_h) \\ &\qquad = -a_2(\psi_h; p_h, v_h) + \bar{c}(\psi_h; p_h, v_h) - (R(n_h, p_h), v_h)_h, \qquad v_h \in V_h, \\ &\text{(c)}\ w_h^{(1)}(0) = n_I^0, \qquad w_h^{(2)}(0) = p_I^0. \end{aligned}$$

It is easy to see that the problem (3.18) is uniquely solvable. We shall prove the solvability of (2.10) by showing that $\mathbb{P}$ has a fixed point. Let $v_h = w_h^{(1)}$ in (3.18a), integrate over τ in $[0, t]$, and note (3.12) and (3.16); the left-hand side of the resulting equation is then bounded below by

$$\tfrac{1}{2}\{\|w_h^{(1)}(t)\|^2 - \|w_h^{(1)}(0)\|^2\} + a_0 \int_0^t \|\nabla w_h^{(1)}(\tau)\|^2 \, d\tau. \tag{3.19}$$

Next, we consider bounds for the terms on the right-hand side. Since (see Zlámal [34])

$$|\chi'(\zeta)| \le \tfrac{1}{2} \quad \forall \zeta \in (-\infty, \infty), \tag{3.20}$$

where $\chi(\zeta) = \frac{1}{2}[B(\zeta) + B(-\zeta)] = \frac{1}{2}\zeta(e^{\zeta}+1)/(e^{\zeta}-1)$, and $\chi(0) = 1$, it follows from (3.7), (3.4), (3.17), and (3.3) that

$$\begin{aligned}
&\left|\int_0^t a_2(\psi_h; n_h, w_h^{(1)})\, d\tau\right| \\
&\quad \le C\int_0^t \sum_{K\in T_h}\left\{\left(1+\frac{1}{2}|\psi_{hm}-\psi_{hk}|\right)|n_{hm}-n_{hk}||w_{hm}^{(1)}+w_{hk}^{(1)}| + \cdots\right\} d\tau \\
&\quad \le C\int_0^t \sum_{K\in T_h}\{\|\nabla\hat{\psi}_h\|_{L^2(\widehat{K})}\|\nabla\hat{n}_h\|_{L^2(\widehat{K})}\|\hat{w}_h^{(1)}\|_{L^\infty(\widehat{K})} \\
&\qquad\qquad + \|\nabla\hat{n}_h\|_{L^2(\widehat{K})}\|\hat{w}_h^{(1)}\|_{L^2(\widehat{K})}\}\, d\tau \\
&\quad \le C\int_0^t \sum_{K\in T_h}\{\|\nabla\psi_h\|_{L^2(K)}\|\nabla n_h\|_{L^2(K)}\|w_h^{(1)}\|_{L^\infty(K)} \\
&\qquad\qquad + h_K^{-1}\|\nabla n_h\|_{L^2(K)}\|w_h^{(1)}\|_{L^2(K)}\}\, d\tau \\
&\quad \le C\int_0^t \{\|\nabla\psi_h\|\,\|\nabla n_h\|\,\|w_h^{(1)}\|_{L^\infty} + h^{-1}\|\nabla n_h\|\,\|w_h^{(1)}\|\}\, d\tau \\
&\quad \le C\int_0^t \{h^{-1}|\log h|^{1/2}(\|n_h\|+\|p_h\|+1)\|n_h\|\,\|\nabla w_h^{(1)}\| \\
&\qquad\qquad + h^{-2}\|n_h\|\,\|w_h^{(1)}\|\}\, d\tau \\
&\quad \le C(h)\int_0^t \{(\|n_h\|^2+\|p_h\|^2)^2+1\}\, d\tau + \frac{1}{5}a_0\int_0^t \|\nabla w_h^{(1)}(\tau)\|^2\, d\tau \\
&\quad \le C(h)\int_0^t \{\|u_h(\tau)\|^4+1\}\, d\tau + \frac{1}{5}a_0\int_0^t \|\nabla w_h^{(1)}(\tau)\|^2\, d\tau.
\end{aligned} \tag{3.21}$$

By (3.9), (3.3), and (3.17),

$$\begin{aligned}
&\left|\int_0^t c_1(\psi_h; n_h, w_h^{(1)})\, d\tau\right| \\
&\quad \le \frac{1}{2}\int_0^t \sum_{K\in T_h} a^K\{\alpha_j^K|\psi_{hm}-\psi_{hk}||n_{hm}+n_{hk}||w_{hm}^{(1)}-w_{hk}^{(1)}| + \cdots\}\, d\tau \\
&\quad \le C\int_0^t \|n_h\|_{L^\infty}\|\nabla\psi_h\|\,\|\nabla w_h^{(1)}\|\, d\tau \\
&\quad \le Ch^{-1}\int_0^t \|n_h\|(\|n_h\|+\|p_h\|+1)\|\nabla w_h^{(1)}\|\, d\tau \\
&\quad \le C(h)\int_0^t \{\|u_h(\tau)\|^4+1\}\, d\tau + \frac{1}{5}a_0\int_0^t \|\nabla w_h^{(1)}(\tau)\|^2\, d\tau.
\end{aligned} \tag{3.22}$$

From (3.10), (3.4), (3.3), and (3.17), we get

$$\begin{aligned}
&\left|\int_0^t c_2(\psi_h; n_h, w_h^{(1)})\,d\tau\right| \\
&\quad\le \frac{1}{4}\int_0^t \sum_{K\in T_h} b^K\{|\psi_{hm}-\psi_{hk}||n_{hm}+n_{hk}||w_{hm}^{(1)}+w_{hk}^{(1)}|+\cdots\}\,d\tau \\
&\quad\le C\int_0^t \sum_{K\in T_h} \|\nabla\hat{\psi}_h\|_{L^2(\widehat{K})}\|\hat{n}_h\|_{L^2(\widehat{K})}\|\hat{w}_h^{(1)}\|_{L^\infty(\widehat{K})}\,d\tau \\
&\quad\le Ch^{-1}\int_0^t \sum_{K\in T_h} \|\nabla\psi_h\|_{L^2(K)}\|n_h\|_{L^2(K)}\|w_h^{(1)}\|_{L^\infty(K)}\,d\tau \\
&\quad\le Ch^{-1}|\log h|^{1/2}\int_0^t \|\nabla\psi_h\|\,\|n_h\|\,\|\nabla w_h^{(1)}\|\,d\tau \\
&\quad\le C(h)\int_0^t\{\|u_h(\tau)\|^4+1\}\,d\tau+\frac{1}{5}a_0\int_0^t\|\nabla w_h^{(1)}(\tau)\|^2\,d\tau.
\end{aligned} \tag{3.23}$$

Noting that $R(n, p)$ is Lipschitz continuous, and using (3.16), we obtain

$$\begin{aligned}
&\left|\int_0^t (R(n_h, p_h), w_h^{(1)})_h\,d\tau\right| \\
&\quad\le C\int_0^t\{\|n_h\|+\|p_h\|+1\}\|w_h^{(1)}\|\,d\tau \\
&\quad\le C\int_0^t\{\|n_h\|^2+\|p_h\|^2+1\}\,d\tau+\frac{1}{5}a_0\int_0^t\|\nabla w_h^{(1)}(\tau)\|^2\,d\tau.
\end{aligned} \tag{3.24}$$

Thus, by (3.18c) and the fact that

$$\|w_h^{(1)}(0)\| \le C\|n^0\|,$$

we have

$$\begin{aligned}
\|w_h^{(1)}(t)\|^2 &\le \|w_h^{(1)}(0)\|^2 + C(h)\int_0^t\{\|u_h(\tau)\|^4+1\}\,d\tau \\
&\le C\|n^0\|^2 + C(h)\int_0^t\{\|u_h(\tau)\|^4+1\}\,d\tau \\
&\le C_1(h) + C_1(h)\int_0^t\{\|u_h(\tau)\|^4+1\}\,d\tau.
\end{aligned} \tag{3.25}$$

Similarly,

$$\|w_h^{(2)}(t)\|^2 \le C_2(h) + C_2(h)\int_0^t\{\|u_h(\tau)\|^4+1\}\,d\tau. \tag{3.26}$$

By (3.25) and (3.26),

$$\|w_h(t)\|^2 \le C^*(h) + C^*(h)\int_0^t\{\|u_h(\tau)\|^4+1\}\,d\tau, \tag{3.27}$$

where $C^*(h)$ depends on h, $\|n^0\|$, and $\|p^0\|$.

Let

$$T^* = \frac{1}{1+4(C^*(h))^2} > 0 \tag{3.28}$$

and

$$\mathscr{Q} = \{u_h \in V_h \times V_h \,;\, \|u_h\|^2 \le 2C^*(h)\}. \tag{3.29}$$

Then, by (3.27), for $0 \le t \le T^*$, we can prove that $\mathbb{P}(\mathscr{Q}) \subset \mathscr{Q}$. By usual arguments one can show that $\mathbb{P}$ is continuous. Therefore, the Brouwer Fixed Point Theorem yields the existence of a fixed point of $\mathbb{P}$.

(II) *Uniqueness.* Let $\{\psi_h, n_h, p_h\}$ and $\{\bar\psi_h, \bar n_h, \bar p_h\}$ be solutions of (2.10) for $t \in [0, T^*]$, and let $\tilde\psi = \psi_h - \bar\psi_h$, $\tilde n = n_h - \bar n_h$, and $\tilde p = p_h - \bar p_h$. Then

$$\|\nabla\tilde\psi\| \le C\{\|\tilde n\| + \|\tilde p\|\} \tag{3.30}$$

and

$$\begin{aligned}\left(\frac{\partial \tilde n}{\partial t}, v_h\right)_h &+ a_1(\psi_h\,;\tilde n, v_h)\\ &= a_1(\bar\psi_h\,;\bar n_h, v_h) - a_1(\psi_h\,;\bar n_h, v_h)\\ &\quad + a_2(\bar\psi_h\,;\bar n_h, v_h) - a_2(\psi_h\,;n_h, v_h) + c(\bar\psi_h\,;\bar n_h, v_h)\\ &\quad - c(\psi_h\,;n_h, v_h) - (R(n_h, p_h) - R(\bar n_h, \bar p_h), v_h)_h, \qquad v_h \in V_h.\end{aligned} \tag{3.31}$$

Let $v_h = \tilde n$ in (3.31); then, by (3.12),

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\tilde n\|_h^2 + a_0\|\nabla\tilde n\|^2 &\le |a_1(\bar\psi_h\,;\bar n_h, \tilde n) - a_1(\psi_h\,;\bar n_h, \tilde n)|\\ &\quad + |a_2(\bar\psi_h\,;\bar n_h, \tilde n) - a_2(\psi_h\,;n_h, \tilde n)|\\ &\quad + |c(\bar\psi_h\,;\bar n_h, \tilde n) - c(\psi_h\,;n_h, \tilde n)|\\ &\quad + |(R(n_h, p_h) - R(\bar n_h, \bar p_h), \tilde n)_h|\\ &= I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{3.32}$$

We integrate (3.32) over τ in $I_t = [0, t]$ for $t \in I^* = [0, T^*]$ and note that $\tilde n(0) = 0$ and (3.16); then the left-hand side of the resulting equation is bounded below by

$$\frac{1}{2}\|\tilde n(t)\|^2 + a_0\int_0^t \|\nabla\tilde n(\tau)\|^2\,d\tau. \tag{3.33}$$

On the right-hand side of the resulting equation, we have, by (3.6), (3.20), (3.3), (3.30), and $\|\bar n_h\|_{L^\infty(I^*;L^2)} \le C(h)$, that

$$\begin{aligned}\int_0^t I_1\,d\tau &\le \int_0^t \sum_{K\in T_h} a^K\{\alpha_j^K|b_{mk} - \bar b_{mk}||\bar n_{hm} - \bar n_{hk}||\tilde n_m - \tilde n_k| + \cdots\}\,d\tau\\ &\le C\int_0^t \sum_{K\in T_h}\{|\tilde\psi_m - \tilde\psi_k||\bar n_{hm} - \bar n_{nk}||\tilde n_m - \tilde n_k| + \cdots\}\,d\tau\\ &\le C\int_0^t \|\nabla\tilde\psi\|\,\|\bar n_h\|_{L^\infty}\|\nabla\tilde n\|\,d\tau\\ &\le Ch^{-1}\int_0^t \|\nabla\tilde\psi\|\,\|\bar n_h\|\,\|\nabla\tilde n\|\,d\tau\\ &\le C(h)\int_0^t\{\|\tilde n\|^2 + \|\tilde p\|^2\}\,d\tau + \frac{a_0}{5}\int_0^t \|\nabla\tilde n\|^2\,d\tau.\end{aligned} \tag{3.34}$$

Noting (3.7), (3.20), (3.3), (3.30), and

$$\|\nabla\bar\psi_h\|_{L^\infty(I^*;L^2)} + \|n_h\|_{L^\infty(I^*;L^2)} \le C(h), \tag{3.35}$$

we get

$$
\begin{aligned}
\int_0^t I_2\,d\tau \le & \int_0^t \{|a_2(\bar\psi_h\,;\,\bar n_h\,,\,\tilde n) - a_2(\bar\psi_h\,;\,n_h\,,\,\tilde n)| \\
& \qquad + |a_2(\bar\psi_h\,;\,n_h\,,\,\tilde n) - a_2(\psi_h\,;\,n_h\,,\,\tilde n)|\}\,d\tau \\
\le & \, C\int_0^t \sum_{K\in T_h} \{[|\bar\psi_{hm} - \bar\psi_{hk}||\tilde n_m - \tilde n_k||\tilde n_m + \tilde n_k| + \cdots] \\
& \qquad + [|\tilde n_m - \tilde n_k||\tilde n_m + \tilde n_k| + \cdots] \\
& \qquad + |\tilde\psi_{hm} - \tilde\psi_{hk}||n_{hm} - n_{hk}||\tilde n_m + \tilde n_k| + \cdots]\}\,d\tau \\
\le & \, C\int_0^t \{\|\nabla\bar\psi_h\|\,\|\nabla\tilde n\|\,\|\tilde n\|_{L^\infty} + h^{-1}\|\tilde n\|\,\|\nabla\tilde n\| \\
& \qquad + \|\nabla\tilde\psi\|\,\|\nabla n_h\|\,\|\tilde n\|_{L^\infty}\}\,d\tau \\
\le & \, Ch^{-1}\int_0^t \{\|\nabla\bar\psi_h\|\,\|\nabla\tilde n\|\,\|\tilde n\| + \|\tilde n\|\,\|\nabla\tilde n\| \\
& \qquad + |\log h|^{1/2}\|\nabla\tilde\psi\|\,\|n_h\|\,\|\nabla\tilde n\|\}\,d\tau \\
\le & \, C(h)\int_0^t \{\|\tilde n\|^2 + \|\tilde p\|^2\}\,d\tau + \frac{a_0}{5}\int_0^t \|\nabla\tilde n\|^2\,d\tau.
\end{aligned} \tag{3.36}
$$

Breaking I_3 into two parts, we have

$$
\begin{aligned}
I_3 &\le |c(\bar\psi_h\,;\,\bar n_h\,,\,\tilde n) - c(\bar\psi_h\,;\,n_h\,,\,\tilde n)| + |c(\bar\psi_h\,;\,n_h\,,\,\tilde n) - c(\psi_h\,;\,n_h\,,\,\tilde n)| \\
&= |c(\bar\psi_h\,;\,\tilde n\,,\,\tilde n)| + |c(\tilde\psi\,;\,n_h\,,\,\tilde n)| = I_{31} + I_{32}.
\end{aligned} \tag{3.37}
$$

From (3.8)–(3.10), (3.4), (3.3), and (3.35), we obtain

$$
\begin{aligned}
\int_0^t I_{31}\,d\tau \le & \, C\int_0^t \sum_{K\in T_h} \{[|\bar\psi_{hm} - \bar\psi_{hk}||\tilde n_m + \tilde n_k||\tilde n_m - \tilde n_k| + \cdots] \\
& \qquad + [|\bar\psi_{hm} - \bar\psi_{hk}||\tilde n_m + \tilde n_k|^2 + \cdots]\}\,d\tau \\
\le & \, C\int_0^t \{\|\nabla\bar\psi_h\|\,\|\nabla\tilde n\|\,\|\tilde n\|_{L^\infty} + h^{-1}\|\nabla\bar\psi_h\|\,\|\tilde n\|\,\|\tilde n\|_{L^\infty}\}\,d\tau \\
\le & \, C(h)\int_0^t \|\nabla\bar\psi_h\|\,\|\nabla\tilde n\|\,\|\tilde n\|\,d\tau \\
\le & \, C(h)\int_0^t \|\tilde n\|^2\,d\tau + \frac{a_0}{5}\int_0^t \|\nabla\tilde n\|^2\,d\tau.
\end{aligned} \tag{3.38}
$$

By noting (3.30), we get

$$
\begin{aligned}
\int_0^t I_{32}\,d\tau \le & \, C\int_0^t \sum_{K\in T_h} \{[|\tilde\psi_m - \tilde\psi_k||n_{hm} + n_{hk}||\tilde n_m - \tilde n_k| + \cdots] \\
& \qquad + [|\tilde\psi_m - \tilde\psi_k||n_{hm} + n_{hk}||\tilde n_m + \tilde n_k| + \cdots]\}\,d\tau \\
\le & \, C\int_0^t \{\|\nabla\tilde\psi\|\,\|\nabla\tilde n\|\,\|n_h\|_{L^\infty} + h^{-1}\|\nabla\tilde\psi\|\,\|\tilde n\|\,\|n_h\|_{L^\infty}\}\,d\tau \\
\le & \, C(h)\int_0^t \|\nabla\tilde\psi\|\,\|\nabla\tilde n\|\,\|n_h\|\,d\tau \\
\le & \, C(h)\int_0^t \{\|\tilde n\|^2 + \|\tilde p\|^2\}\,d\tau + \frac{a_0}{5}\int_0^t \|\nabla\tilde n\|^2\,d\tau.
\end{aligned} \tag{3.39}
$$

Noting that $R(n, p)$ is Lipschitz continuous, and using (3.16), we easily see that

$$\int_0^t I_4\,d\tau \leq C\int_0^t \{\|\tilde{n}\|^2 + \|\tilde{p}\|^2\}\,d\tau. \tag{3.40}$$

From (3.32)–(3.34) and (3.36)–(3.40), we have, for each $t \in I^*$,

$$\|\tilde{n}(t)\|^2 \leq C(h)\int_0^t \{\|\tilde{n}\|^2 + \|\tilde{p}\|^2\}\,d\tau. \tag{3.41}$$

Similarly,

$$\|\tilde{p}(t)\|^2 \leq C(h)\int_0^t \{\|\tilde{n}\|^2 + \|\tilde{p}\|^2\}\,d\tau. \tag{3.42}$$

By (3.41) and (3.42),

$$\|\tilde{n}(t)\|^2 + \|\tilde{p}(t)\|^2 \leq C(h)\int_0^t \{\|\tilde{n}\|^2 + \|\tilde{p}\|^2\}\,d\tau.$$

Gronwall's Lemma and (3.30) now complete the proof. □

Let $e_\psi = \psi - \psi_h$, $e_n = n - n_h = n - \bar{n} + \bar{n} - n_h = \eta_n + \xi_n$, and $e_p = p - p_h = p - \bar{p} + \bar{p} - p_h = \eta_p + \xi_p$. Then we have

Theorem 3.2. *With* T^* *defined by* (3.28), *there exists a positive constant* C *such that, for* h *sufficiently small*

$$\begin{aligned}\|\xi_n\|_{L^\infty(I^*;L^2)} &+ \|\xi_p\|_{L^\infty(I^*;L^2)} \\ &+ \|\nabla\xi_n\|_{L^2(I^*;L^2)} + \|\nabla\xi_p\|_{L^2(I^*;L^2)} \leq Ch^\sigma.\end{aligned} \tag{3.43}$$

Proof. Subtract (2.10a) from (2.4a) to obtain

$$\begin{aligned} d(e_\psi, v_h) &= (p - n + N, v_h) - (p_h - n_h + N_I, v_h)_h \\ &= (e_p - e_n + N - N_I, v_h) + (p_h - n_h + N_I, v_h) \\ &\quad - (p_h - n_h + N_I, v_h)_h, \qquad v_h \in V_h. \end{aligned}$$

From Lemma 4.3 in Zlámal [34], (2.6), and (3.2), we have, for each $v_h \in V_h$,

$$\begin{aligned} d(e_\psi, v_h) &\leq \{\|e_n\| + \|e_p\| + \|N - N_I\|\}\|v_h\| + Ch\{\|n_h\| + \|p_h\| + \|N_I\|\}\|\nabla v_h\| \\ &\leq \{\|e_n\| + \|e_p\| + \|N - N_I\|\}\|v_h\| \\ &\quad + Ch\{1 + \|e_n\| + \|e_p\| + \|N - N_I\|\}\|\nabla v_h\| \\ &\leq \{\|\xi_n\| + \|\xi_p\| + \|\eta_n\| + \|\eta_p\| + \|N - N_I\|\}\|v_h\| \\ &\quad + Ch\{1 + \|\xi_n\| + \|\xi_p\| + \|\eta_n\| + \|\eta_p\| + \|N - N_I\|\}\|\nabla v_h\| \\ &\leq \{\|\xi_n\| + \|\xi_p\| + h\}\|v_h\| + Ch\{1 + \|\xi_n\| + \|\xi_p\| + h\}\|\nabla v_h\|. \end{aligned}$$

Thus, by (3.2e),

$$\begin{aligned} \|\nabla e_\psi\|^2 = d(e_\psi, e_\psi) &= d(e_\psi, \psi - \psi_I) + d(e_\psi, \psi_I - \psi_h) \\ &\leq \tfrac{1}{2}\|\nabla e_\psi\|^2 + C\{\|\nabla(\psi - \psi_I)\|^2 + \|\xi_n\|^2 + \|\xi_p\|^2 + h^2\} \\ &\leq \tfrac{1}{2}\|\nabla e_\psi\|^2 + C\{\|\xi_n\|^2 + \|\xi_p\|^2 + h^{2\sigma}\}. \end{aligned}$$

Therefore,

$$\|\nabla e_\psi\|^2 \le C\{\|\xi_n\|^2 + \|\xi_p\|^2 + h^{2\sigma}\}. \tag{3.44}$$

To estimate ξ_n, subtract (2.10b) from (2.4b). Letting

$$A_1^K(n, v) = (A_1^K \nabla n, \nabla v), \tag{3.45}$$

$$\nu^K(\psi; n, v) = A_1^K(n, v) - (nA_1^K \nabla\psi, \nabla v), \tag{3.46}$$

and noting (2.5), (3.1), (3.5), and $a_1(0; \bar{n}, v_h) - a_2(0; \bar{n}, v_h) = A_1^K(\bar{n}, v_h)$ (obtained by (2.7), (3.5)–(3.10), (3.45)), we have, for each $v_h \in V_h$,

$$\begin{aligned} &\left(\frac{d\xi_n}{\partial t}, v_h\right)_h + a_1(\psi_h; \xi_h, v_h) \\ &\quad = \left(\frac{\partial \bar{n}}{\partial t}, v_h\right)_h - \left(\frac{\partial \bar{n}}{\partial t}, v_h\right) + \nu^K(\psi; n, v_h) - \nu(\psi; n, v_h) \\ &\qquad + a_1(\psi_h; \bar{n}, v_h) - a_1(0; \bar{n}, v_h) - \{a_2(\psi_h; n_h, v_h) - a_2(0; \bar{n}, v_h)\} \\ &\qquad + (nA_1^K\nabla\psi, \nabla v_h) - (n_hA_1^K\nabla\psi_h, \nabla v_h) \\ &\qquad + (n_hA_1^K\nabla\psi_h, \nabla v_h) - c(\psi_h; n_h, v_h) \\ &\qquad - A_1^K(\eta_n, v_h) + (R(n_h, p_h), v_h)_h - (R(n, p), v_h). \end{aligned} \tag{3.47}$$

Let $v_h = \xi_n$ in (3.47), and note that

$$a_2(0; \xi_n, \xi_n) = \frac{1}{2}\sum_{K\in T_h} b^K[(\xi_{nm}^2 - \xi_{nk}^2) + (\xi_{nj}^2 - \xi_{nm}^2) + (\xi_{nk}^2 - \xi_{nj}^2)] = 0,$$

so, $a_2(0; \bar{n}, \xi_n) = a_2(0; n_h, \xi_n)$; then by (3.12), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\xi_n\|_h^2 + a_0\|\nabla\xi_n\|^2 \\ &\quad \le \left|\left(\frac{\partial \bar{n}}{\partial t}, \xi_n\right)_h - \left(\frac{\partial \bar{n}}{\partial t}, \xi_n\right)\right| + |\nu^K(\psi; n, \xi_n) - \nu(\psi; n, \xi_n)| \\ &\qquad + |a_1(\psi_h; \bar{n}, \xi_n) - a_1(0; \bar{n}, \xi_n)| + |a_2(\psi_h; n_h, \xi_n) - a_2(0; n_h, \xi_n)| \\ &\qquad + |(nA_1^K\nabla\psi, \nabla\xi_n) - (n_hA_1^K\nabla\psi_h, \nabla\xi_n)| \\ &\qquad + |(n_hA_1^K\nabla\psi_h, \nabla\xi_n) - c(\psi_h; n_h, \xi_n)| \\ &\qquad + |A_1^K(\eta_n, \xi_n)| + |(R(n_h, p_h), \xi_n)_h - (R(n, p), \xi_n)| = \sum_{i=1}^{8} F_i. \end{aligned} \tag{3.48}$$

Integrating (3.48) termwise over τ in $I_t = [0, t]$ for $t \in I^*$, we find that the left-hand side of the resulting equation is bounded below by

$$\tfrac{1}{2}\{\|\xi_n(t)\|_h^2 - \|\xi_n(0)\|_h^2\} + a_0\|\nabla\xi_n\|_{L^2(I_t; L^2)}^2. \tag{3.49}$$

Next, we consider bounds for the terms on the right-hand side of the resulting equation. Using Lemma 4.3 in Zlámal [34], (3.1b) and (2.6c), we have

$$\int_0^t F_1\,d\tau \le Ch\int_0^t \left\|\frac{\partial \bar n}{\partial t}\right\| \|\nabla\xi_n\|\,d\tau \le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} + Ch^2. \tag{3.50}$$

By (2.5), (3.45), and (3.46), and noting that $\beta(x)$ is Lipschitz continuous, and (2.6), we get

$$\begin{aligned}\int_0^t F_2\,d\tau &\le \int_0^t \left\{\sum_{K\in T_h}\int_K |A_1 - A_1^K||\nabla n - n\nabla\psi||\nabla\xi_n|\,dx\right\}d\tau\\ &\le Ch\int_0^t \sum_{K\in T_h}\|\nabla n - n\nabla\psi\|_{L^2(K)}\|\nabla\xi_n\|_{L^2(K)}\,d\tau\\ &\le Ch\int_0^t \|\nabla n - n\nabla\psi\|\,\|\nabla\xi_n\|\,d\tau\\ &\le Ch\int_0^t \{\|\nabla n\| + \|\nabla\psi\|\,\|n\|_{L^\infty}\}\|\nabla\xi_n\|\,d\tau\\ &\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} + Ch^2.\end{aligned} \tag{3.51}$$

From (3.6), (3.20), (3.4), (3.3), the Sobolev Imbedding Theorem, (3.2), (3.44), and (2.6), we deduce

$$\begin{aligned}\int_0^t F_3\,d\tau &\le \int_0^t \sum_{K\in T_h} a^K\{\alpha_j^K|b_{mk}-1||\bar n_m - \bar n_k||\xi_{nm}-\xi_{nk}| + \cdots\}\,d\tau\\ &\le C\int_0^t \sum_{K\in T_h}\{|\psi_{hm}-\psi_{hk}|[|n_m - n_k| + |\eta_{nm}-\eta_{nk}|]\\ &\qquad\qquad \times|\xi_{nm}-\xi_{nk}| + \cdots\}\,d\tau\\ &\le C\int_0^t \sum_{K\in T_h}\{\|\nabla\hat\psi_h\|_{L^2(\widehat K)}\|\nabla\hat n_I\|_{L^{2/(1-\sigma)}(\widehat K)}\|\nabla\hat\xi_n\|_{L^{2/\sigma}(\widehat K)}\\ &\qquad + \|\nabla\hat\psi_h\|_{L^2(\widehat K)}\|\nabla(\hat\eta_n)_I\|_{L^2(\widehat K)}\|\nabla\hat\xi_n\|_{L^\infty(\widehat K)}\}\,d\tau\\ &\le C\int_0^t \sum_{K\in T_h} h_K\{\|\nabla\psi_h\|_{L^2(K)}\|\nabla n_I\|_{L^{2/(1-\sigma)}(K)}\|\nabla\xi_n\|_{L^{2/\sigma}(K)}\\ &\qquad + \|\nabla\psi_h\|_{L^2(K)}\|\nabla(\eta_n)_I\|_{L^2(K)}\|\nabla\xi_n\|_{L^\infty(K)}\}\,d\tau\\ &\le Ch\int_0^t \|\nabla\psi_h\|\{\|\nabla n\|_{L^{2/(1-\sigma)}}\|\nabla\xi_n\|_{L^{2/\sigma}} + \|\nabla\eta_n\|\,\|\nabla\xi_n\|_{L^\infty}\}\,d\tau\\ &\le C\int_0^t \{\|\nabla\psi\| + \|\nabla e_\psi\|\}\{h^\sigma\|\nabla n\|_{L^{2/(1-\sigma)}} + \|\nabla\eta_n\|\}\|\nabla\xi_n\|\,d\tau\\ &\le Ch^\sigma\int_0^t \{\|\psi\|_{1+\sigma} + \|\nabla e_\psi\|\}\|n\|_{1+\sigma}\|\nabla\xi_n\|\,d\tau\\ &\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} + Ch^{2\sigma}\{\|\xi_n\|^2_{L^\infty(I_t;L^2)} + \|\xi_p\|^2_{L^\infty(I_t;L^2)} + 1\}.\end{aligned} \tag{3.52}$$

Breaking F_4 into pieces, we have

$$
\begin{aligned}
F_4 &\le \frac{1}{2}\sum_{K\in T_h} b^K\{|b_{mk}-1||n_{hm}-n_{hk}||\xi_{nm}+\xi_{nk}|+\cdots\} \\
&\le C\sum_{K\in T_h}\{|\psi_{hm}-\psi_{hk}||n_{hm}-n_{hk}||\xi_{nm}+\xi_{nk}|+\cdots\} \\
&\le C\sum_{K\in T_h}\{[|\psi_m-\psi_k||n_m-n_k||\xi_{nm}+\xi_{nk}|+\cdots] \\
&\qquad +[|\psi_m-\psi_k||\xi_{nm}-\xi_{nk}||\xi_{nm}+\xi_{nk}|+\cdots] \\
&\qquad +[|\psi_m-\psi_k||\eta_{nm}-\eta_{nk}||\xi_{nm}+\xi_{nk}|+\cdots] \\
&\qquad +[|e_{\psi m}-e_{\psi k}||n_m-n_k||\xi_{nm}+\xi_{nk}|+\cdots] \\
&\qquad +[|e_{\psi m}-e_{\psi k}||\xi_{nm}-\xi_{nk}||\xi_{nm}+\xi_{nk}|+\cdots] \\
&\qquad +[|e_{\psi m}-e_{\psi k}||\eta_{nm}-\eta_{nk}||\xi_{nm}+\xi_{nk}|+\cdots]\} \\
&= \sum_{i=1}^{6} F_{4i}.
\end{aligned}
\tag{3.53}
$$

By (3.4), the Sobolev Imbedding Theorem, (3.3), and (2.6), we get

$$
\begin{aligned}
\int_0^t F_{41}\,d\tau &\le C\int_0^t \sum_{K\in T_h}\{|\psi_m-\psi_k||n_m-n_k||\xi_{nm}+\xi_{nk}|+\cdots\}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h}\|\nabla\hat\psi_I\|_{L^{2/(1-\sigma)}(\widehat K)}\|\nabla\hat n_I\|_{L^{2/(1-\sigma)}(\widehat K)}\|\hat\xi_n\|_{L^{1/\sigma}(\widehat K)}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h}\|\nabla\hat\psi_I\|_{L^{2/(1-\sigma)}(\widehat K)}\|\nabla\hat n_I\|_{L^{2/(1-\sigma)}(\widehat K)}\|\hat\xi_n\|_{L^{\infty}(\widehat K)}\,d\tau \\
&\le Ch^{2\sigma}\int_0^t \sum_{K\in T_h}\|\nabla\psi_I\|_{L^{2/(1-\sigma)}(K)}\|\nabla n_I\|_{L^{2/(1-\sigma)}(K)}\|\xi_n\|_{L^{\infty}(K)}\,d\tau \\
&\le Ch^{2\sigma}\int_0^t \sum_{K\in T_h}\|\psi\|_{H^{1+\sigma}(K)}\|n\|_{H^{1+\sigma}(K)}\|\xi_n\|_{L^{\infty}(K)}\,d\tau \\
&\le Ch^{2\sigma}\int_0^t \|\psi\|_{1+\sigma}\|n\|_{1+\sigma}\|\xi_n\|_{L^\infty}\,d\tau \\
&\le Ch^{2\sigma}|\log h|^{1/2}\int_0^t \|\psi\|_{1+\sigma}\|n\|_{1+\sigma}\|\nabla\xi_n\|\,d\tau \\
&\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} + Ch^{4\sigma}|\log h|.
\end{aligned}
\tag{3.54}
$$

Using the Gagliardo-Nirenberg Inequality

$$\|v\|_{L^q} \le C(q)\|v\|^{2/q}\|\nabla v\|^{(q-2)/q}, \qquad v \in V, \; q > 2, \tag{3.55}$$

and Young's Inequality

$$ab \le C(\delta, q)a^q + \delta b^{q/(1-q)}, \qquad a, b \ge 0, \; q > 1, \tag{3.56}$$

we have (let $q = 2/\sigma$)

$$\begin{aligned}
\int_0^t F_{42}\,d\tau &\le C\int_0^t \sum_{K\in T_h} \{|\psi_m - \psi_k||\xi_{nm} - \xi_{nk}||\xi_{nm} + \xi_{nk}| + \cdots\}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h} \|\nabla\hat{\psi}_I\|_{L^{2/(1-\sigma)}(\widehat{K})}\|\nabla\hat{\xi}_n\|_{L^2(\widehat{K})}\|\hat{\xi}_n\|_{L^{2/\sigma}(\widehat{K})}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h} \|\nabla\psi_I\|_{L^{2/(1-\sigma)}(K)}\|\nabla\xi_n\|_{L^2(K)}\|\xi_n\|_{L^{2/\sigma}(K)}\,d\tau \\
&\le C\int_0^t \|\nabla\psi\|_{L^{2/(1-\sigma)}}\|\nabla\xi_n\|\,\|\xi_n\|_{L^{2/\sigma}}\,d\tau \\
&\le C\int_0^t \|\psi\|_{1+\sigma}\|\xi_n\|^{\sigma}\|\nabla\xi_n\|^{2-\sigma}\,d\tau \\
&\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;\,L^2)} + C\|\xi_n\|^2_{L^2(I_t;\,L^2)},
\end{aligned} \tag{3.57}$$

and by (3.4), (3.2), and (3.3),

$$\begin{aligned}
\int_0^t F_{43}\,d\tau &\le C\int_0^t \sum_{K\in T_h} \{|\psi_m - \psi_k||\eta_{nm} - \eta_{nk}||\xi_{nm} + \xi_{nk}| + \cdots\}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h} \|\nabla\hat{\psi}_I\|_{L^{2/(1-\sigma)}(\widehat{K})}\|\nabla(\hat{\eta}_n)_I\|_{L^2(\widehat{K})}\|\hat{\xi}_n\|_{L^{2/\sigma}(\widehat{K})}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h} \|\nabla\psi_I\|_{L^{2/(1-\sigma)}(K)}\|\nabla(\eta_n)_I\|_{L^2(K)}\|\xi_n\|_{L^{2/\sigma}(K)}\,d\tau \\
&\le C\int_0^t \|\nabla\psi\|_{L^{2/(1-\sigma)}}\|\nabla\eta_n\|\,\|\xi_n\|_{L^{2/\sigma}}\,d\tau \\
&\le C\int_0^t \|\psi\|_{1+\sigma}\|\nabla\eta_n\|\,\|\nabla\xi_n\|\,d\tau \\
&\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;\,L^2)} + Ch^{2\sigma}.
\end{aligned} \tag{3.58}$$

From (3.4), the Sobolev Imbedding Theorem, (3.3), (2.6), and (3.44), we get

$$
\begin{aligned}
\int_0^t F_{44}\,d\tau &\le C\int_0^t \sum_{K\in T_h}\{|e_{\psi m}-e_{\psi k}||n_m-n_k||\xi_{nm}+\xi_{nk}|+\cdots\}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h}\|\nabla(\hat e_\psi)_I\|_{L^2(\widehat K)}\|\nabla \hat n_I\|_{L^{2/(1-\sigma)}(\widehat K)}\|\hat\xi_n\|_{L^{2/\sigma}(\widehat K)}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h}\|\nabla(\hat e_\psi)_I\|_{L^2(\widehat K)}\|\nabla \hat n_I\|_{L^{2/(1-\sigma)}(\widehat K)}\|\hat\xi_n\|_{L^{\infty}(\widehat K)}\,d\tau \\
&\le Ch^\sigma\int_0^t \sum_{K\in T_h}\|\nabla(e_\psi)_I\|_{L^2(K)}\|\nabla n_I\|_{L^{2/(1-\sigma)}(K)}\|\xi_n\|_{L^{\infty}(K)}\,d\tau \qquad (3.59)\\
&\le Ch^\sigma\int_0^t \sum_{K\in T_h}\|\nabla e_\psi\|_{L^2(K)}\|n\|_{H^{1+\sigma}(K)}\|\xi_n\|_{L^{\infty}(K)}\,d\tau \\
&\le Ch^\sigma|\log h|^{1/2}\int_0^t \|\nabla e_\psi\|\,\|n\|_{1+\sigma}\|\nabla\xi_n\|\,d\tau \\
&\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} \\
&\quad + Ch^{2\sigma}|\log h|\{\|\xi_n\|^2_{L^\infty(I_t;L^2)}+\|\xi_p\|^2_{L^\infty(I_t;L^2)}+h^{2\sigma}\},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t F_{45}\,d\tau &\le C\int_0^t \sum_{K\in T_h}\{|e_{\psi m}-e_{\psi k}||\xi_{nm}-\xi_{nk}||\xi_{nm}+\xi_{nk}|+\cdots\}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h}\|\nabla(\hat e_\psi)_I\|_{L^2(\widehat K)}\|\nabla\hat\xi_n\|_{L^2(\widehat K)}\|\hat\xi_n\|_{L^{\infty}(\widehat K)}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h}\|\nabla(e_\psi)_I\|_{L^2(K)}\|\nabla\xi_n\|_{L^2(K)}\|\xi_n\|_{L^{\infty}(K)}\,d\tau \qquad (3.60)\\
&\le C\int_0^t \|\nabla e_\psi\|\,\|\nabla\xi_n\|\,\|\xi_n\|_{L^\infty}\,d\tau \\
&\le C|\log h|^{1/2}\{\|\xi_n\|_{L^\infty(I_t;L^2)}+\|\xi_p\|_{L^\infty(I_t;L^2)}+h^\sigma\}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)},
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t F_{46}\,d\tau &\le C\int_0^t \sum_{K\in T_h}\{|e_{\psi m}-e_{\psi k}||\eta_{nm}-\eta_{nk}||\xi_{nm}+\xi_{nk}|+\cdots\}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h}\|\nabla(\hat e_\psi)_I\|_{L^2(\widehat K)}\|\nabla(\hat\eta_n)_I\|_{L^2(\widehat K)}\|\hat\xi_n\|_{L^{\infty}(\widehat K)}\,d\tau \\
&\le C\int_0^t \sum_{K\in T_h}\|\nabla(e_\psi)_I\|_{L^2(K)}\|\nabla(\eta_n)_I\|_{L^2(K)}\|\xi_n\|_{L^{\infty}(K)}\,d\tau \qquad (3.61)\\
&\le C\int_0^t \|\nabla e_\psi\|\,\|\nabla\eta_n\|\,\|\xi_n\|_{L^\infty}\,d\tau \\
&\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} \\
&\quad + Ch^{2\sigma}|\log h|\{\|\xi_n\|^2_{L^\infty(I_t;L^2)}+\|\xi_p\|^2_{L^\infty(I_t;L^2)}+h^{2\sigma}\}.
\end{aligned}
$$

In order to estimate F_5, we shall break it into pieces as follows:

$$
\begin{aligned}
F_5 &= |(A_1^K e_n \nabla\psi, \nabla\xi_n)| + |(A_1^K \bar{n} \nabla e_\psi, \nabla\xi_n)| + |(A_1^K \xi_n \nabla e_\psi, \nabla\xi_n)| \\
&= |(A_1^K \xi_n \nabla\psi, \nabla\xi_n)| + |(A_1^K \eta_n \nabla\psi, \nabla\xi_n)| \\
&\quad + |(A_1^K \bar{n} \nabla e_\psi, \nabla\xi_n)| + |(A_1^K \xi_n \nabla e_\psi, \nabla\xi_n)| \\
&= F_{51} + F_{52} + F_{53} + F_{54}.
\end{aligned} \tag{3.62}
$$

From Remark 2.1, (2.6), and using the Sobolev Imbedding Theorem, the Gagliardo-Nirenberg Inequality (3.55), and Young's Inequality (3.56), we have

$$
\begin{aligned}
\int_0^t F_{51}\, d\tau &\le M \int_0^t \|\xi_n\|_{L^{2/\sigma}} \|\nabla\psi\|_{L^{2/(1-\sigma)}} \|\nabla\xi_n\|_{L^2}\, d\tau \\
&\le C\|\psi\|_{L^\infty(H^{1+\sigma})} \int_0^t \|\xi_n\|^\sigma \|\nabla\xi_n\|^{2-\sigma}\, d\tau \\
&\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} + C\|\xi_n\|^2_{L^2(I_t;L^2)}.
\end{aligned} \tag{3.63}
$$

Furthermore, by (3.2),

$$
\begin{aligned}
\int_0^t F_{52}\, d\tau &\le M \int_0^t \|\eta_n\|_{L^{2/\sigma}} \|\nabla\psi\|_{L^{2/(1-\sigma)}} \|\nabla\xi_n\|_{L^2}\, d\tau \\
&\le C\|\psi\|_{L^\infty(H^{1+\sigma})} \int_0^t \|\nabla\eta_n\|\, \|\nabla\xi_n\|\, d\tau \\
&\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} + Ch^{2\sigma},
\end{aligned} \tag{3.64}
$$

by (3.44),

$$
\begin{aligned}
\int_0^t F_{53}\, d\tau &\le M \int_0^t \|\bar{n}\|_{L^\infty} \|\nabla e_\psi\|\, \|\nabla\xi_n\|\, d\tau \\
&\le C\|n\|_{L^\infty(L^\infty)} \int_0^t \|\nabla e_\psi\|\, \|\nabla\xi_n\|\, d\tau \\
&\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} + C\|\nabla e_\psi\|^2_{L^2(I_t;L^2)} \\
&\le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)} + C\{\|\xi_n\|^2_{L^2(I_t;L^2)} + \|\xi_p\|^2_{L^2(I_t;L^2)} + h^{2\sigma}\},
\end{aligned} \tag{3.65}
$$

and by (3.3),

$$
\begin{aligned}
\int_0^t F_{54}\, d\tau &\le M \int_0^t \|\xi_n\|_{L^\infty} \|\nabla e_\psi\|\, \|\nabla\xi_n\|\, d\tau \\
&\le C|\log h|^{1/2}\{\|\xi_n\|_{L^\infty(I_t;L^2)} \\
&\qquad + \|\xi_p\|_{L^\infty(I_t;L^2)} + h^\sigma\}\|\nabla\xi_n\|^2_{L^2(I_t;L^2)}.
\end{aligned} \tag{3.66}
$$

Since

$$
A_i^K(\psi_n, \xi_h) = \sum_{K\in T_h} \int_K A_1^K \nabla\psi_h \cdot \nabla\xi_h\, dx = \sum_{K\in T_h} \operatorname{area}(K) A_1^K \nabla\psi_h \cdot \nabla\xi_h
$$

and

$$
\begin{aligned}
A_1^K(\psi_h, \xi_h) = \nu_h(0;\psi_h, \xi_h) = &\sum_{K\in T_h} a^K[\alpha_j^K(\psi_{hm} - \psi_{hk})(\xi_{nm} - \xi_{nk}) + \cdots] \\
&- \frac{1}{2}\sum_{K\in T_h} b^K[(\psi_{hm} - \psi_{hk})(\xi_{nm} + \xi_{nk}) + \cdots],
\end{aligned}
$$

we get

$$\operatorname{area}(K)A_1^K\nabla\psi_h\cdot\nabla\xi_h = a^K[\alpha_j^K(\psi_{hm}-\psi_{hk})(\xi_{nm}-\xi_{nk})+\cdots] - \frac{1}{2}b^K[(\psi_{hm}-\psi_{hk})(\xi_{nm}+\xi_{nk})+\cdots]$$

and

$$\begin{aligned}
F_6 &= \left| c(\psi_h\,;n_h\,,\xi_n) - \sum_{K\in T_h} 2\operatorname{area}(K)A_1^K\nabla\psi_h\cdot\nabla\xi_h\int_{\widehat{K}}\hat{n}_h\,d\zeta\right| \\
&= \left|\sum_{K\in T_h} a^K\left\{\alpha_j^K(\psi_{hm}-\psi_{hk})(\xi_{nm}-\xi_{nk})\left[\frac{1}{2}(n_{hm}+n_{hk})-2\int_{\widehat{K}}\hat{n}_h\,d\zeta\right]+\cdots\right\}\right. \\
&\qquad -\frac{1}{2}\sum_{K\in T_h} b^K\left\{(\psi_{hm}-\psi_{hk})(\xi_{nm}+\xi_{nk})\right. \\
&\qquad\qquad \left.\left.\times\left[\frac{1}{2}(n_{hm}+n_{hk})-2\int_{\widehat{K}}\hat{n}_h\,d\zeta\right]+\cdots\right\}\right| \\
&= \frac{1}{6}\left|\sum_{K\in T_h} a^K\{\alpha_j^K(\psi_{hm}-\psi_{hk})(\xi_{nm}-\xi_{nk})[(n_{hm}-n_{hj})+(n_{hk}-n_{hj})]+\cdots\}\right. \\
&\qquad \left.-\frac{1}{2}\sum_{K\in T_h} b^K\{(\psi_{hm}-\psi_{hk})(\xi_{nm}+\xi_{nk})[(n_{hm}-n_{hj})+(n_{hk}-n_{hj})]+\cdots\}\right| \\
&\le C\sum_{K\in T_h}\{|\psi_{hm}-\psi_{hk}|[|\xi_{nm}-\xi_{nk}|+|\xi_{nm}+\xi_{nk}|] \\
&\qquad\qquad \times|(n_{hm}-n_{hj})+(n_{hk}-n_{hj})|+\cdots\}.
\end{aligned}$$

Similarly to the treatment of F_4, we can show that

$$\begin{aligned}
\int_0^t F_6\,d\tau \le{}& \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t\,;L^2)} + C\|\xi_n\|^2_{L^2(I_t\,;L^2)} \\
&+ Ch^{2\sigma}|\log h|\{\|\xi_n\|^2_{L^\infty(I_t\,;L^2)}+\|\xi_p\|^2_{L^\infty(I_t\,;L^2)}+h^{2\sigma}\} \\
&+ C|\log h|^{1/2}\{\|\xi_n\|_{L^\infty(I_t\,;L^2)}+\|\xi_p\|_{L^\infty(I_t\,;L^2)}+h^\sigma\}\|\nabla\xi_n\|^2_{L^2(I_t\,;L^2)}.
\end{aligned}\tag{3.67}$$

By Remark 2.1 and (3.2),

$$\int_0^t F_7\,d\tau \le M\int_0^t\|\nabla\eta_n\|\,\|\nabla\xi_n\|\,d\tau \le \frac{a_0}{16}\|\nabla\xi_n\|^2_{L^2(I_t\,;L^2)} + Ch^{2\sigma}.\tag{3.68}$$

Using Theorem 4.1.5 in Ciarlet [5], we have

$$\begin{aligned}
|(R(n\,,p)\,,\xi_n)_h &- (R(n\,,p)\,,\xi_n)| \\
&\le C\sum_{K\in T_h} h_K(\operatorname{area}(K))^{\sigma/2}\|R(n\,,p)\|_{W^1_{2/(1-\sigma)}(K)}\|\nabla\xi_n\|_{L^2(K)} \\
&\le Ch(\operatorname{area}(\Omega))^{\sigma/2}\|R(n\,,p)\|_{W^1_{2/(1-\sigma)}}\|\nabla\xi_n\| \\
&\le Ch\{\|n\|_{1+\sigma}+\|p\|_{1+\sigma}+1\}\|\nabla\xi_n\|,
\end{aligned}$$

where, in the last inequality, we have made use of (3.13) in Zlámal [33] and the Sobolev Imbedding Theorem. Hence, by (3.16), (3.2), and (2.6),

$$
\begin{aligned}
\int_0^t F_8\, d\tau \le \int_0^t \{ &|(R(n, p), \xi_n)_h - (R(n, p), \xi_n)| \\
&+ |(R(n_h, p_h), \xi_n)_h - (R(n, p), \xi_n)_h| \}\, d\tau \\
\le \frac{a_0}{16} &\|\nabla \xi_n\|^2_{L^2(I_t; L^2)} + C\{\|\xi_n\|^2_{L^2(I_t; L^2)} + \|\xi_p\|^2_{L^2(I_t; L^2)} + h^{2\sigma}\}.
\end{aligned} \tag{3.69}
$$

Combining (3.48)–(3.54), (3.57)–(3.69), and noting (3.16), $\|\xi_n(0)\| \le \|n^0 - n_I^0\| + \|n^0 - \bar{n}^0\| \le Ch^\sigma$, and $C|\log h|^{1/2} h^\sigma \le a_0/16$ for h sufficiently small, we have

$$
\begin{aligned}
&\frac{1}{2}\|\xi_n(t)\|^2 + \frac{a_0}{16}\|\nabla \xi_n\|^2_{L^2(I_t; L^2)} \\
&\quad \le C|\log h|^{1/2}\{\|\xi_n\|_{L^\infty(I_t; L^2)} + \|\xi_p\|_{L^\infty(I_t; L^2)}\}\|\nabla \xi_n\|^2_{L^2(I_t; L^2)} \\
&\quad\quad + C\{\|\xi_n\|^2_{L^2(I_t; L^2)} + \|\xi_p\|^2_{L^2(I_t; L^2)} + h^{2\sigma}\} \\
&\quad\quad + Ch^{2\sigma}|\log h|\{\|\xi_n\|^2_{L^\infty(I_t; L^2)} + \|\xi_p\|^2_{L^\infty(I_t; L^2)}\}.
\end{aligned} \tag{3.70}
$$

Similarly, we have

$$
\begin{aligned}
&\frac{1}{2}\|\xi_p(t)\|^2 + \frac{a_0}{16}\|\nabla \xi_p\|^2_{L^2(I_t; L^2)} \\
&\quad \le C|\log h|^{1/2}\{\|\xi_n\|_{L^\infty(I_t; L^2)} + \|\xi_p\|_{L^\infty(I_t; L^2)}\}\|\nabla \xi_p\|^2_{L^2(I_t; L^2)} \\
&\quad\quad + C\{\|\xi_n\|^2_{L^2(I_t; L^2)} + \|\xi_p\|^2_{L^2(I_t; L^2)} + h^{2\sigma}\} \\
&\quad\quad + Ch^{2\sigma}|\log h|\{\|\xi_n\|^2_{L^\infty(I_t; L^2)} + \|\xi_p\|^2_{L^\infty(I_t; L^2)}\}.
\end{aligned} \tag{3.71}
$$

Let $\xi = \{\xi_n, \xi_p\}$; then

$$
\begin{aligned}
&\frac{1}{2}\|\xi(t)\|^2 + \frac{a_0}{16}\|\nabla \xi\|^2_{L^2(I_t; L^2)} \\
&\quad \le C|\log h|^{1/2}\|\xi\|_{L^\infty(I_t; L^2)}\|\nabla \xi\|^2_{L^2(I_t; L^2)} \\
&\quad\quad + C\{\|\xi\|^2_{L^2(I_t; L^2)} + h^{2\sigma}\} + Ch^{2\sigma}|\log h|\|\xi\|^2_{L^\infty(I_t; L^2)}.
\end{aligned} \tag{3.72}
$$

As in [6, 11, 31], let us make the induction hypothesis that

$$
C|\log h|^{1/2}\|\xi\|_{L^\infty(I^*; L^2)} \le \frac{a_0}{32}. \tag{3.73}
$$

Obviously, (3.73) holds for $t = 0$. Thus, (3.73) will hold for $t \le t^*$ for some $t^* > 0$. We shall show for h sufficiently small that $t^* = T^*$ and that (3.43) holds.

It follows from (3.72), (3.73), and Gronwall's Lemma that

$$
\|\xi\|^2_{L^\infty(I^*; L^2)} + \|\nabla \xi\|^2_{L^2(I^*; L^2)} \le Ch^{2\sigma}, \tag{3.74}
$$

where C is independent of T^*. Note that (3.74) implies that the induction hypothesis (3.73) holds for small h, so that the entire argument is validated. □

From Theorem 3.2 and (3.2), one easily obtains the following corollary.

Corollary 3.1. *Let*

$$\widetilde{M} = \{\|n\|_{L^\infty(L^2)} + \|p\|_{L^\infty(L^2)}\} + 1\,; \tag{3.75}$$

then

$$\|n_h\|_{L^\infty(I^*;L^2)} + \|p_h\|_{L^\infty(I^*;L^2)} \le \widetilde{M}.$$

If we substitute $\widetilde{M}$ defined by (3.75) for $\|n^0\|_{L^2}$ and $\|p^0\|_{L^2}$ on which $C^*(h)$ (in (3.27)) depends, then T^* defined by (3.28) depends only on h. Thus, for h fixed, T^* is a fixed constant, and we can show that the problem (2.10) has a unique global solution by extending gradually the local solution defined by Theorem 3.1. Therefore, Theorem 3.2 holds for I instead of I^*. Noting (3.2) and (3.44), we have the following main results of this paper.

Theorem 3.3. *Problem* (2.10) *is uniquely solvable. Let* $\{\psi, n, p\}$ *satisfy* (1.5)–(1.8) *and* $\{\psi_h, n_h, p_h\}$ *satisfy* (2.10). *If the regularity assumptions* (2.6) *hold, then there exists a positive constant* C *such that, for* h *sufficiently small,*

$$\|n - n_h\|_{L^\infty(L^2)} + \|p - p_h\|_{L^\infty(L^2)} + \|\nabla(\psi - \psi_h)\|_{L^\infty(L^2)} + \|\nabla(n - n_h)\|_{L^2(L^2)} + \|\nabla(p - p_h)\|_{L^2(L^2)} \le Ch^\sigma.$$

Corollary 3.2. *Theorem* 3.3 *holds for the case when* $\vec{B} = 0$, *considered by Zlámal* [33, 34].

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