ON BESSEL FUNCTIONS AND RATE OF CONVERGENCE
OF ZEROS OF LOMMEL POLYNOMIALS

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ABSTRACT. In this note, we solve an open problem of Flajolet and Schott concerning the rate of convergence of the zeros of Lommel polynomials to the zeros of a Bessel function (considered as a function of the order). The average case analysis of dynamic data structures was the initial motivation for this investigation. The Maple program, whose results are reported at the end, illustrates that our result is quite good.

1. INTRODUCTION

Coulomb [1] first investigated the zeros of the Bessel functions $J_\nu(x)$, with x fixed. He observed that the zeros converge to the negative integers as $\nu \to -\infty$. More recently, Flajolet and Schott [2] gave a quantitative version of Coulomb's result and solved some combinatorial problems related to nonoverlapping partitions. They mentioned that the dynamic analysis of data structures apparently required investigating the speed of convergence of zeros of Lommel polynomials to zeros of Bessel functions. This provided the initial motivation for this investigation. It is to be noted that Lommel polynomials arise via the convergents of the continued fraction expansion converging to the Bessel function. It is likely that our technique may prove useful in similar cases for other classes of functions.

2. LOMMEL POLYNOMIALS AND BESSEL FUNCTIONS

The reader is referred to Watson's treatise [3] for the basic theory of Bessel functions. Let us just recall that the Bessel function J_ν is defined by

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{n! \Gamma(n + \nu + 1)}$$

and satisfies the fundamental recurrence

$$2\nu x^{-1} J_\nu(x) = J_{\nu+1}(x) + J_{\nu-1}(x).$$

In the following, the argument of the Bessel function is fixed and ν is variable,

and will be denoted by x. Consider the (modified) Lommel polynomials

$$H_m(x) = \sum_{n=0}^{[m/2]} \binom{m-n}{n} (-1)^n (x+n)_{m-2n}$$

with $(x)_a = x(x+1)\cdots(x+a-1)$.

Note. Throughout the following, x is any complex number that is not an integer.

We have, in the notation of [3, p. 294ff]

(1) $$H_m(x) \equiv R_{m,x}(2).$$

We let

(2) $$J(x) \equiv J_x(2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(x+1+n)}.$$

For $m \to \infty$, recalling that x is nonintegral,

(3) $$J(x+m) = \frac{1}{\Gamma(x+m+1)}\left[1 + O\left(\frac{1}{m}\right)\right]$$

and

(4) $$J(-x-m) = \frac{1}{\Gamma(-x-m+1)}\left[1 + O\left(\frac{1}{m}\right)\right],$$

where the O-terms are uniform (in x) on compact sets. The principal formula required is the following, which may be found in [3, p. 295]:

(*) $$H_m(x) = \frac{\pi}{\sin \pi x}[J(x+m)J(1-x) + (-1)^m J(-x-m)J(x-1)].$$

Now let μ be a positive integer. Define the domain $\Omega_\mu = \{x: 0 < |x+\mu| < 1\}$. The first lemma is related to Hurwitz' result for Lommel polynomials [3, p. 302].

Lemma 1. *On Ω_μ, $H_m(x)/\Gamma(x+m)$ as $m \to \infty$ converges to $J(x-1)$ uniformly on compact subsets ('normally').*

Proof. By (*),

(5)
$$\frac{H_m(x)}{\Gamma(x+m)} = O\left(\frac{1}{\Gamma(x+m)\Gamma(x+m+1)}\right)$$
$$+ \frac{\pi}{\sin \pi x} \frac{(-1)^m J(-x-m)J(x-1)}{\Gamma(x+m)}.$$

The first term is negligible. We have to show that the coefficient of $J(x-1)$ in the second term converges normally to 1. In fact,

(6)
$$\frac{\pi}{\sin \pi x} \frac{(-1)^m J(-x-m)}{\Gamma(x+m)} = \frac{\pi}{\sin \pi(x+\mu)} \frac{(-1)^{m+\mu}}{\Gamma(x+\mu)\Gamma(1-x-\mu)}$$
$$\times \frac{\Gamma(x+\mu)}{\Gamma(x+m)} \frac{\Gamma(1-x-\mu)}{\Gamma(1-x-m)}\left[1 + O\left(\frac{1}{m}\right)\right],$$

where we note that the O-terms are uniform on compact subsets of Ω_μ. By the reflection formula for the gamma function, the first two factors reduce, and thus the expression is asymptotic to

$$(-1)^{m+\mu} \frac{\Gamma(x+\mu)\Gamma(1-x-\mu)}{\Gamma(x+m)\Gamma(1-x-m)} = 1 . \quad \square$$

Recall that the zeros of $J(x)$ are very close to negative integers as $x \to -\infty$ [1].

Lemma 2. *Let* $J(\zeta - 1) = 0$. *Let* ζ_m *be the zero of* $H_m(x)$ *closest to* ζ. *Then, as* $m \to \infty$, *we have the asymptotic relation*

$$(7) \qquad\qquad |\zeta_m - \zeta| \sim |J'(\zeta - 1)|^{-1} \left| \frac{H_m(\zeta)}{\Gamma(\zeta + m)} \right| .$$

Proof. (The idea of the proof is similar to that of Newton's method for solving equations.) Let $f_m(x) = H_m(x)/\Gamma(x + m)$. So f_m and H_m have the same roots near ζ. By the mean value theorem, there exists ξ_m, e.g., on the segment between ζ and ζ_m, such that

$$(8) \qquad\qquad f_m(\zeta) - f_m(\zeta_m) = (\zeta - \zeta_m) f_m'(\xi_m) .$$

Thus, since $f_m(\zeta_m) = 0$, as $m \to \infty$,

$$(9) \qquad\qquad |\zeta - \zeta_m| = \left| \frac{f_m(\zeta)}{f_m'(\xi_m)} \right| \sim |f_m'(\zeta)|^{-1} |f_m(\zeta)| ,$$

where, since $\{f_m(x)\}$ converges normally to $J(x - 1)$ (by Cauchy's formula), the derivatives are equicontinuous. For large m, we can replace $f_m'(\zeta)$ by $J'(\zeta - 1)$. $\square$

Finally we have:

Theorem. *Let* ζ *denote a zero of* $J(x - 1)$. *Let* ζ_m *denote the zero of* $H_m(x)$ *nearest to* ζ. *Then, as* $m \to \infty$,

$$(10) \qquad |\zeta_m - \zeta| \sim \left| \frac{\pi}{\sin \pi \zeta} \frac{J(1 - \zeta)}{J'(\zeta - 1)} \frac{1}{\Gamma(\zeta + m)\Gamma(\zeta + m + 1)} \right| .$$

Proof. We apply $(*)$ at $x = \zeta$, so that the $J(x - 1)$ term vanishes. Thus,

$$(11) \qquad\qquad \frac{H_m(\zeta)}{\Gamma(\zeta + m)} = \frac{\pi}{\sin \pi \zeta} J(1 - \zeta) \frac{J(\zeta + m)}{\Gamma(\zeta + m)} .$$

Combining (3) with Lemma 2 yields the result. $\square$

Remark. Relation (10) may be written in the form

$$(12) \qquad\qquad |\zeta_m - \zeta| \sim \left| \frac{J(1 - \zeta)}{J'(\zeta - 1)} \frac{\Gamma(-\zeta - m)}{\Gamma(\zeta + m)} \right|$$

via the reflection formula for the gamma function.

Some numerical results illustrating the theorem are given in Table 1.

Table 1

m	$\|\xi^L - \xi_1\|$ Theory	Practice	$\|\xi^L - \xi_5\|$ Theory	Practice	$\|\xi^L - \xi_8\|$ Theory	Practice
4	1×10^{-3}	9×10^{-4}				
5	1×10^{-5}	4×10^{-5}				
6	1×10^{-6}	1×10^{-6}				
7	3×10^{-8}	2×10^{-8}	8×10^{-2}	4×10^{-2}		
8	4×10^{-10}	4×10^{-10}	7×10^{-3}	4×10^{-3}		
9			3×10^{-4}	2×10^{-4}		
10			1×10^{-5}	8×10^{-6}	8×10^{-2}	4×10^{-2}
11			3×10^{-7}	2×10^{-7}	7×10^{-3}	4×10^{-3}
12			5×10^{-9}	4×10^{-9}	3×10^{-4}	2×10^{-4}
13					1×10^{-5}	8×10^{-6}
14					3×10^{-7}	2×10^{-7}
15					5×10^{-9}	

ξ_i denotes a zero of $J(\nu) = J_\nu(2)$,

ξ^L denotes a zero of a Lommel polynomial,

We have (approximately) $\xi_1 = 0.2538058170$, $\xi_5 = -4.9997743198$,

$\xi_8 = -7.9999999961$.

Bibliography

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