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NU-CONFIGURATIONS IN TILING THE SQUARE

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ABSTRACT. In order to tile the unit square with rational triangles, at least four triangles are needed. There are four candidate configurations: one is conjectured not to exist; two others are dealt with elsewhere; the fourth is the "nu-configuration," corresponding to rational points on a quartic surface in affine 3-space. This surface is examined via a pencil of elliptic curves. One rank-3 curve is treated in detail, and rational points are given on 772 curves of the pencil. Within the range of the search there are roughly equal numbers of odd and even rank and those of rank 2 or more seem to be at least 0.45 times as numerous as those of rank 0. Symmetrical solutions correspond to rational points on a curve of rank 1, which exhibits an almost periodic behavior.

1. INTRODUCTION

The problem of tiling an integer-sided square with integer-sided triangles is considered in [6]. This is clearly equivalent to the problem of tiling the unit square with rational-sided ("rational") triangles. A dissection into two such triangles is trivially impossible, and it is known (see [6] for a proof and references) that a dissection into three rational triangles is also impossible.

Accordingly, the first interesting case is a dissection into four rational triangles. Figure 1 shows the four candidates for such a configuration, named respectively chi-, delta-, kappa-, and nu-configurations. A rational chi-configuration is conjectured to be impossible. A detailed discussion of how to obtain "all" delta- and kappa-configurations is given in [1]. Here we give a parallel discussion of nu-configurations.

In Figure $1(\nu)$, take OB, OA as axes and U the point $(1, 1)$. If P is the point $(X, 1)$ and Q the point $(X+Y, 0)$, then the rationality of the lengths OP, PQ, QU immediately implies that

$$X^2+1=\square,\quad Y^2+1=\square,\quad (1-X-Y)^2+1=\square. \tag{1}$$

We call a rational number, x, satisfying $x^2+1=\square$, a *rectangular number.* Then it is clear from (1) that nu-configurations are characterized by solutions of the equation

$$X+Y+Z=1 \tag{2}$$

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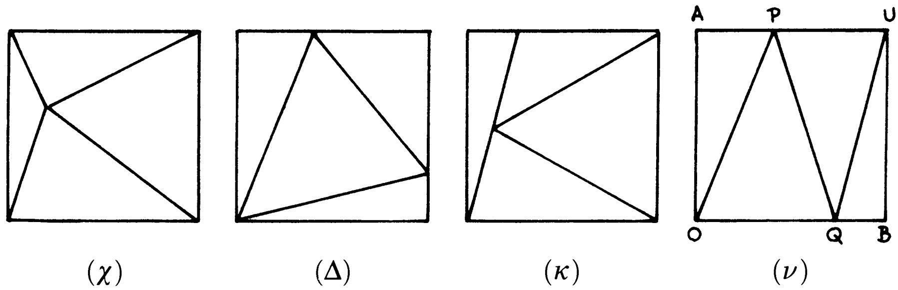

FIGURE 1. Four candidates for a tiling of the square with four rational triangles

in which X, Y, Z are rectangular numbers. We investigate (2) in the manner of Bremner and Guy [1].

2. THE PARAMETERS (r, s), (u, v), (m, n)

In (2), there is no loss of generality in putting

$$X = \frac{r^2 - s^2}{2rs}, \quad Y = \frac{u^2 - v^2}{2uv}, \quad Z = \frac{m^2 - n^2}{2mn}$$

with $r \perp s$ (r prime to s), $u \perp v$, and $m \perp n$, so that we seek integer solutions m, n, r, s, u, v to the equation

$$mnuv(r^2 - s^2) + mnrs(u^2 - v^2) + rsuv(m^2 - 2mn - n^2) = 0. \tag{3}$$

Geometrically, (3) represents a quartic surface in affine three-dimensional space with coordinates m/n, r/s, u/v. It seems an unlikely hope that there is any straightforward description of all the rational points on this surface. In order to obtain results, we shall assume, much as in [1], that the ratio m/n is predetermined, so that (3) may be considered as an elliptic curve over $\mathbb{Q}(m/n)$.

3. SYMMETRICAL NU'S

The special case where $r/s = m/n$, so that Figure 1(ν) is symmetric under rotation through π, is considered in §7 of [6], so we merely summarize the results. The solutions correspond to rational points on the curve

$$\tau^2 = \sigma(\sigma^2 + 6\sigma + 4) \tag{4}$$

which has rational rank 1, with a generator $P = (-1, 1)$ of infinite order. The solutions $(r\colon s; u\colon v)$ corresponding to P, $2P$, $3P$, and $4P$ are $(1\colon 0; 0\colon 1)$, $(-3\colon 4; 3\colon 2)$, $(75\colon 52; 50\colon 39)$, and $(425\colon -5928; 221\colon 5700)$. The first is degenerate, the second and third correspond to the proper tilings of Figures 2 and 3, and the fourth to the improper tiling of Figure 4.

All symmetrical nu-configurations are given by a single infinity of solutions satisfying the recurrence relations

$$r_{n+1}\colon s_{n+1} = r_n(2(r_n - s_n)u_n - s_n v_n)\colon -s_n(2r_n u_n - s_n v_n),$$
$$u_{n+1}\colon v_{n+1} = v_n(2(r_n - s_n)u_n - s_n v_n)\colon u_n(2r_n u_n - s_n v_n).$$

It was observed in Table 7.1 of [6] that these solutions are almost periodic, with a period close to 282. This is explained as follows. It is well known that

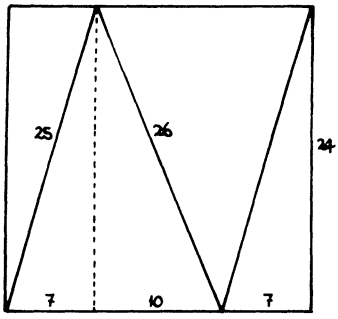

FIGURE 2. $n = 2$

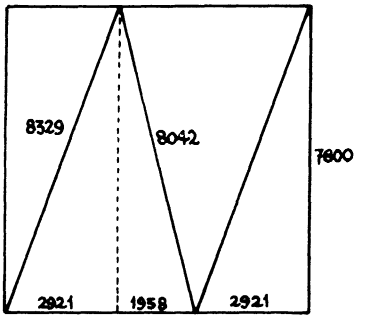

FIGURE 3. $n = 3$

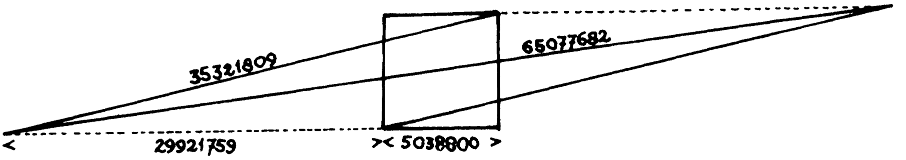

FIGURE 4. $n = 4$

an elliptic curve over $\mathbb{C}$ is isomorphic to the quotient of $\mathbb{C}$ by some lattice Λ, the isomorphism being given by the Weierstrass $\wp$-function, $(\wp(z), \wp'(z)) \mapsto z \bmod \Lambda$; and the points of the curve over $\mathbb{C}$ are in one-to-one correspondence with the points of the fundamental parallelogram of the lattice (Figure 5), where ω_1 and ω_2 are known as the *periods* of the curve.

For the curve (4), then, O corresponds to the point at infinity $\mathbf{o}$, and ω_2 is actually real. Further, as a point Q traverses the right-hand branch of the curve (containing $(0, 0)$), then the associated parameter z of the $\wp$-function traverses the edge of the fundamental parallelogram from O to ω_2 (containing $\frac{1}{2}\omega_2$). It follows that the associated parameter z of Q is of type $\alpha\omega_2$, $0 \leq \alpha < 1$. Suppose now that a/b is a good rational approximation to the real number α. Then Q will display a 'period' of b, in the sense that bQ is close to $\mathbf{o}$. But α is simply determined as the ratio of the integrals

$$\int_{x_Q}^{\infty} \frac{dx}{y} \Big/ 2\int_0^{\infty} \frac{dx}{y}$$

(the denominator here being the real period ω_2).

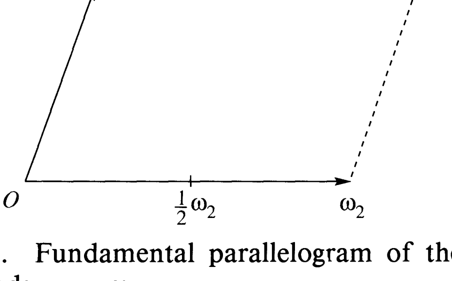

FIGURE 5. Fundamental parallelogram of the lattice with periods ω_1, ω_2

Performing the integration (kindly computed by John Hebron) for the point $Q = 2P = (9/4, 57/8)$, we find

$$\alpha = 0.24113288400309607871925 12\ldots.$$

In order to find good rational approximations to α, compute the continued fraction. The partial quotients are

$$4, 6, 1, 3, 1, 26, 7, 2, 1, 2, 1, 62, 9, 4, 5, 1, 1, 50, 2, 2, \ldots.$$

The large partial quotients $6, 26, 62, 50, \ldots$ ensure that the denominators of the convergents obtained by truncating them, namely

$$\frac{1}{4}, \frac{34}{141}, \frac{74165}{307569}, \frac{1980618265}{8213804074},$$

give very good 'periods' of $2P$. It follows that $4, 282, 615138, 8213804074$ give very good estimates for a 'period' of P.

It is not clear to us whether the large partial quotients in the above continued fraction are simply a curiosity. For a continued fraction without large partial quotients, there would be no obvious 'period' of small size. A similar example is worked by Don Zagier in Method 3 of [12].

4. Fixing (m, n) leads to an elliptic curve

In order to find solutions containing a given *slope*, (m, n), write equation (3) as a quadratic in r and s:

$$r^2 + 2rs\left(\frac{u^2 - v^2}{2uv} + \frac{m^2 - 2mn - n^2}{2mn}\right) - s^2 = 0,$$

and r/s will be rational just if the discriminant is square:

$$\left(\frac{u^2 - v^2}{2uv} + \frac{m^2 - 2mn - n^2}{2mn}\right)^2 + 1 = \alpha^2. \tag{5}$$

Write $u/v = U$ and

$$\frac{m^2 - 2mn - n^2}{2mn} = \kappa, \tag{6}$$

so that there is a birational map between (5) and the elliptic curve

$$E\colon \tau^2 = \sigma(\sigma^2 + (\kappa^2 + 2)\sigma + 1) \tag{7}$$

given by

$$\sigma = U\alpha + \tfrac{1}{2}(U^2 + 2\kappa U - 1), \qquad \tau = U(\sigma + 1) + \kappa\sigma,$$

with inverse

$$U = \frac{\tau - \kappa\sigma}{\sigma + 1}, \qquad \alpha = \frac{\tau + \kappa\sigma}{2\sigma} - \frac{\kappa}{\sigma + 1}.$$

There is a 2-isogeny, ν', between the curve (7) and the curve

$$E'\colon T^2 = S(S - \kappa^2)(S - (\kappa^2 + 4)) \tag{8}$$

given by

$$(S, T) = \nu'(\sigma, \tau) = \left(\frac{\tau^2}{\sigma^2}, \left(1 - \frac{1}{\sigma^2}\right)\tau\right). \tag{9}$$

The corresponding isogeny $\nu: E' \to E$, where $\nu\nu'$ is multiplication by 2 on E, is given by

$$(\sigma, \tau) = \nu(S, T) = \left(\frac{T^2}{4S^2}, \frac{1}{8}\left[1 - \frac{\kappa^2(\kappa^2+4)}{S^2}\right] T\right). \tag{10}$$

For terminology and details of the calculation of the rank of E (and E') over the field $k = \mathbb{Q}(m/n)$, we refer the reader to Bremner and Guy [1]. Here we simply compute the groups $\mathbf{G}_\nu^*$ and $\mathbf{G}_{\nu'}^*$ of ν-covers of E, and ν'-covers of E', which actually possess points defined over k. Then the rank g of E can be determined from the relation

$$2^{g+1} = [\mathbf{G}_\nu^*][\mathbf{G}_{\nu'}^*]/2.$$

Consider first the curve (7), which, by putting $\lambda = m/n$ in (6), we may write in the form

$$\tau^2 = \sigma(\sigma^2 + (\lambda^4 - 4\lambda^3 + 10\lambda^2 + 4\lambda + 1)\sigma + 16\lambda^4). \tag{11}$$

The ν-covers of (11) are obtained in the classical manner by putting $\sigma = \delta a^2/b^2$, with $\delta, a, b \in \mathbb{Z}[\lambda]$, δ squarefree, $a \perp b$. This gives

$$\delta^2 a^4 + (\lambda^4 - 4\lambda^3 + 10\lambda^2 + 4\lambda + 1)\delta a^2 b^2 + 16\lambda^4 b^4 = \delta c^2. \tag{12}$$

The elements of the group of ν-covers of E are in one-one correspondence with the curves (12). For such a cover to be locally solvable, it is clearly necessary that δ divide $16\lambda^4$, and δ squarefree forces the only possibilities for δ to be

$$\delta = \pm 1, \pm 2, \pm\lambda, \pm 2\lambda. \tag{13}$$

Rewrite (12) as

$$\begin{aligned}[2\delta a^2 &+ (\lambda^4 - 4\lambda^3 + 10\lambda^2 + 4\lambda + 1)b^2]^2 \\ &- (\lambda^4 - 4\lambda^3 + 18\lambda^2 + 4\lambda + 1)(\lambda^2 - 2\lambda - 1)^2 b^4 = 4\delta c^2.\end{aligned} \tag{14}$$

Suppose that δ is a quadratic nonresidue modulo the polynomial

$$p(\lambda) = \lambda^4 - 4\lambda^3 + 18\lambda^2 + 4\lambda + 1.$$

Then (14) forces

$$2\delta a^2 - 8\lambda^2 b^2 \equiv c \equiv 0 \mod p(\lambda).$$

In particular, $\delta a^2 \equiv 4\lambda^2 b^2$, so that $a \equiv b \equiv 0 \mod p(\lambda)$, a contradiction.

Accordingly, for local solvability, δ must be a quadratic residue modulo $p(\lambda)$.

Let θ denote a root of $p(\lambda)$. Then δ is a square $\mod p(\lambda)$ if and only if $\delta(\theta)$ is a square in $\mathbb{Q}(\theta)$.

Take $\theta = 1+2i+\sqrt{-2+4i}$. We claim that the possibilities $\delta = \pm 2, \pm\lambda, \pm 2\lambda$ from the list (13) contradict the above local criterion. Since -1 belongs to $\mathbb{Q}(\theta)^2$, it suffices to show that $2, \theta$, and 2θ do not. Consider, in $\mathbb{Q}(\theta)$, the first-degree prime ideal factor $\mathbf{p}_5$ of the principal ideal generated by 5, for which $i \equiv 2$ and $\theta \equiv 1$. If either 2 or 2θ were a square in $\mathbb{Q}(\theta)$, it would now follow that 2 is a square $\mod \mathbf{p}_5$, which is impossible.

Similarly, consider the first-degree prime ideal $\mathbf{p}_{13}(\theta)$ of the ideal generated by 13, for which $i \equiv 8$ and $\theta \equiv 2$. If θ were a square in $\mathbb{Q}(\theta)$, it would imply that 2 is a square $\mod \mathbf{p}_{13}$, which is also impossible.

We conclude that the only possibilities for δ in the list (13) are $\delta = \pm 1$; and there exist respective global points given by $(a, b, c) = (1, 0, 1)$ and $(2\lambda, 1, 2\lambda(\lambda^2 - 2\lambda - 1))$. Hence $[\mathbf{G}_\nu^*] = 2^1$.

In a similar way, consider the curve (8) in the form

$$T^2 = S(S - (\lambda^2 - 2\lambda - 1)^2)(S - (\lambda^4 - 4\lambda^3 + 18\lambda^2 + 4\lambda + 1)) \tag{15}$$

and put $S = \Delta A^2/B^2$, with $\Delta, A, B \in \mathbb{Q}[\lambda]$, Δ squarefree, $A \perp B$. This gives

$$(\Delta A^2 - (\lambda^2 - 2\lambda - 1)^2 B^2)(\Delta A^2 - (\lambda^4 - 4\lambda^3 + 18\lambda^2 + 4\lambda + 1)B^2) = \Delta C^2, \tag{16}$$

and, for local solvability of (16), Δ must divide

$$(\lambda^2 - 2\lambda - 1)^2(\lambda^4 - 4\lambda^3 + 18\lambda^2 + 4\lambda + 1);$$

it is also clear that both the leading coeffcient and the constant term in Δ must be positive.

Hence, there are just the possibilities

$$\Delta = 1 \quad \text{and} \quad \Delta = \lambda^4 - 4\lambda^3 + 18\lambda^2 + 4\lambda + 1,$$

with respective global points $(A, B, C) = (1, 0, 1)$ and $(1, -1, 0)$. So $[\mathbf{G}_{\nu'}^*] = 2^1$.

Finally, the rank g satisfies

$$2^{g+2} = [\mathbf{G}_{\nu'}^*][\mathbf{G}_\nu^*]$$

and accordingly $g = 0$. So numerical instances of the nu-configuration may be less common than the delta-configurations of Bremner and Guy [1]. But we shall see in §10 (in the Supplement at the end of this issue) that, as often as not, the ranks for specific slopes are positive. And in the next section we find an infinite family of parametrized nu-solutions.

5. Families of curves of rank one

Take the ratios m/n, r/s, u/v to satisfy $n = s = v$. Then (3) becomes

$$n^2(mr + ru + um) + 2nmru - mru(m + r + u) = 0$$

and accordingly, a solution is furnished by setting $mr + ru + um = 0$. There results

$$u = -\frac{mr}{m + r}, \qquad n = \frac{m^2 + mr + r^2}{2(m + r)},$$

leading to the parametrization

$$\frac{m}{n} = \frac{2\mu(\mu + 1)}{\mu^2 + \mu + 1}; \quad \frac{r}{s} = \frac{2(\mu + 1)}{\mu^2 + \mu + 1}; \quad \frac{u}{v} = \frac{-2\mu}{\mu^2 + \mu + 1}. \tag{17}$$

From this solution we may generate infinitely many further solutions in the standard manner, as follows. Set

$$\frac{m}{n} = \frac{2\mu(\mu + 1)}{\mu^2 + \mu + 1}$$

and work over the field $\mathbb{Q}(\mu)$. From (6), κ now takes the form

$$\kappa = -\frac{\mu^4 + 2\mu^3 + 7\mu^2 + 6\mu + 1}{4\mu(\mu+1)(\mu^2+\mu+1)}.$$

From the values (17) and the maps to the curve (7) we see that the curve possesses the $\mathbb{Q}(\mu)$-point

$$(18) \quad P \equiv (\sigma, \tau) = \left(\frac{4\mu(\mu+1)}{(\mu^2+\mu+1)^2}, -\frac{(2\mu+1)(\mu^4+2\mu^3+7\mu^2+6\mu+1)}{(\mu^2+\mu+1)^3}\right).$$

We have verified that the rank of E defined over $\mathbb{Q}(\mu)$ is equal to one, and it seems plausible that the point P at (18) is indeed a generator for the group of points of E defined over $\mathbb{Q}(\mu)$, although this has not been specifically checked. It is, however, easy to see that P is of infinite order, and accordingly, the points NP, $N \in \mathbb{N}$, furnish an infinite family of examples of parametrized nu-solutions. The case $N = 2$, for example, leads to the solution

$$\frac{u}{v} = \frac{(\mu+1)(\mu^2+\mu-1)^2(\mu^4+2\mu^3+7\mu^2+2\mu+1)}{2\mu(2\mu+1)(\mu^2+\mu+1)(\mu^4+2\mu^3+7\mu^2+10\mu+5)},$$
$$\frac{r}{s} = \frac{2(\mu+1)(2\mu+1)(\mu^2+\mu+1)(\mu^4+2\mu^3+7\mu^2+2\mu+1)}{\mu(\mu^2+\mu-1)^2(\mu^4+2\mu^3+7\mu^2+10\mu+5)}.$$

Another infinite family of parametrized nu-solutions is given by

$$\frac{m}{n} = \frac{\mu^2+1}{\mu(\mu^2-2\mu+2)}, \quad \frac{r}{s} = \frac{\mu(\mu^2-2\mu-2)}{2\mu^2-2\mu-1}, \quad \frac{u}{v} = \frac{\mu(\mu^2+1)}{2\mu^2-2\mu-1}$$

and may be handled in a similar manner.

Sections 6–10 of this paper appear in the Supplement at the end of this issue.

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