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ANDREW BREMNER AND RICHARD K. GUY

6. Lemmas restricting the rank in specific cases

Although the generic rank of (7) is zero, the rank of the specialized curve, when m and n are given numerical values, is often positive. We obtain arithmetic information about the rank in the manner of §4 of [1]. To make the dependence on m, n more explicit, we introduce some notation and note some congruences which will be useful in the sequel. Write

$$K = m^2 - 2mn - n^2 = (m-n)^2 - 2n^2,$$

$$L = K^2 + 8m^2n^2 = m^4 - 4m^3n + 10m^2n^2 + 4mn^3 + n^4,$$
$$M = K^2 + 16m^2n^2 = m^4 - 4m^3n + 18m^2n^2 + 4mn^3 + n^4,$$

so that $\kappa = K^2/4m^2n^2$, $\kappa^2 + 2 = L/4m^2n^2$, $\kappa^2 + 4 = M/4m^2n^2$. We may take $m \perp n$, so that $K \perp mn, L \perp mn, M \perp mn$, and

Fact (a) If m, n are of opposite parity, then $K \equiv \pm 1 \bmod 8$, and each of K^2, L, and M is congruent to 1 mod 16.

Fact (b) If m, n are both odd, then $K \equiv 2 \bmod 4$, and $K^2 \equiv 4, L \equiv 12$, and $M \equiv 20 \bmod 32$.

Fact (c) Each odd prime factor of K is congruent to $\pm 1 \bmod 8$.

Fact (d) Each odd prime factor of M (the sum of two squares) is congruent to 1 mod 4.

We also change the scale, replacing $(4m^2n^2\sigma, 8m^3n^3\tau)$ by (σ, τ), so that (7), which is

$$64m^6n^6\tau^2 = 4m^2n^2\sigma(16m^4n^4\sigma^2 + L \cdot 4m^2n^2\sigma + 16m^4n^4),$$

becomes

$$\tau^2 = \sigma(\sigma^2 + L\sigma + 16m^4n^4); \tag{19}$$

and replace $(4m^2n^2S, 8m^3n^3T)$ by (S, T), so that (8), namely

$$64m^6n^6T^2 = 4m^2n^2S(4m^2n^2S - K^2)(4m^2n^2S - M),$$

becomes

$$T^2 = S(S - K^2)(S - M). \tag{20}$$

In (19) we substitute $\sigma = \delta a^2/b^2, \tau = \delta ac/b^3$, with $\delta, a, b, c \in \mathbf{Z}, \delta$ squarefree, $a \perp b$ and $a, b, c \geq 0$, and obtain

$$\delta c^2 = \delta^2 a^4 + L\delta a^2 b^2 + 16m^4n^4b^4, \tag{21}$$

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and, in (20), substitute $S = \Delta A^2/B^2$, $T = \Delta AC/B^3$ with corresponding conditions on Δ, A, B, C, yielding

$$\Delta C^2 = (\Delta A^2 - K^2B^2)(\Delta A^2 - MB^2). \tag{22}$$

From (21), $\delta|16m^4n^4$, and, because δ is squarefree, $\delta|2mn$. We have already noted the global solutions $\delta = 1, (a,b,c) = (1,0,1)$ and $\delta = -1, (a,b,c) = (2mn, 1, 2mnK)$, where $K = m^2 - 2mn - n^2$, so we only need to seek solutions with $\delta > 1$. See also Lemmas 12 and 13 in §10. Lemmas 1 to 3 concern δ: Lemma 3 refers to the case where m, n are both odd.

Lemma 1. *If an odd prime p divides $M = m^4 - 4m^3n + 18m^2n^2 + 4mn^3 + n^4$, and the Legendre symbol $(\delta|p) = -1$, then (21) has no nontrivial solutions.*

Proof. We may write (21) as

$$(2\delta a^2 + Lb^2)^2 - K^2Mb^4 = 4\delta c^2,$$

so that

$$(2\delta a^2 + Lb^2)^2 \equiv 4\delta c^2 \bmod p,$$

whereupon $(\delta|p) = -1$ implies that $2\delta a^2 + Lb^2 \equiv 0 \equiv 2c \bmod p$, and since $L = M - 8m^2n^2$, we have

$$2\delta a^2 - 8m^2n^2b^2 \equiv 0 \bmod p.$$

Again, $(\delta|p) = -1$ implies that $a \equiv 0 \equiv 2mnb \bmod p$. Now p odd, $M \perp mn$ and $p|M$ imply that $p|b$, and $a \equiv 0 \equiv b \bmod p$ contradicts $a \perp b$. □

Lemma 2. *If a prime $p \equiv 1 \bmod 8$ divides $K = m^2 - 2mn - n^2$, and $(\delta|p) = -1$, then (21) has no nontrivial solutions.*

Proof. As in Lemma 1, $2\delta a^2 + Lb^2 \equiv 0 \bmod p$, and since $p|K^2$ and $L = K^2 + 8m^2n^2$,

$$2\delta a^2 + 8m^2n^2b^2 \equiv 0 \bmod p,$$

$$-\delta a^2 \equiv (2mnb)^2 \bmod p.$$

As before, $p \nmid 2mn$, and $a \equiv 0 \equiv b \bmod p$ is not permitted, so $(-\delta|p) = -1, (-1|p) = -1, p \equiv 3 \bmod 4$, contradicting Fact (c) above. □

Lemma 3. *If m, n are both odd, so is δ.*

Proof. By Fact (b), $L \equiv 12 \bmod 32$, and (21) gives

$$\delta c^2 \equiv \delta^2a^4 + 12\delta a^2b^2 + 16b^4 \bmod 32.$$

If a is odd, then $\delta c^2 \equiv \delta^2 \bmod 4$, and $2|\delta$ would imply $2|c, 2^2|c^2$ and $4|\delta$, contradicting δ squarefree. If $2\,\|\,a$, then b is odd, $b^2 \equiv 1 \bmod 8, \delta c^2 \equiv 16(\delta^2 + 3\delta + 1) \bmod 32$, which implies $8|c^2, 16|c^2, \delta \equiv \delta^2 + 3\delta + 1 \bmod 2$, and δ odd. If $4|a$, then b is odd, $\delta c^2 \equiv 16 \bmod 32, 8|c^2, 16|c^2$, and δ odd. □

From (22), $\Delta|K^2M$, and, because Δ is squarefree, $\Delta|KM$. We have noted the global solutions $\Delta = 1, (A,B,C) = (1,0,1)$ [or $(K,1,0)$] and $\Delta = M, (A,B,C) = (1,1,0)$ [or $(0,-1,K)$], corresponding to the points $(S,T) = \infty, (K^2,0), (M,0)$ and $(0,0)$ on the curve (20). See also Lemmas 14 and 15 in §10. Lemmas 4 to 7 concern Δ: Lemma 7 refers to the case with m and n of opposite parity.

Lemma 4. *Δ is positive and odd.*

Proof. If Δ were negative, the left side of (22) would be negative, and the right side positive. If $m + n \equiv 1 \bmod 2$, Fact (a) states that M is odd, so that Δ, which divides MK, is odd. If $m \equiv n \equiv 1 \bmod 2$, Fact (b) states that $K^2 \equiv 4$ and $M \equiv 20 \bmod 32$, and (22) gives

$$\Delta C^2 \equiv (\Delta A^2 - 4B^2)(\Delta A^2 - 20B^2) \bmod 32.$$

If Δ were even, $2\,\|\,\Delta$, since it is squarefree; 2^2 divides the right side of (22); $2|C$; 8 divides the left side of (22); 4 divides at least one factor on the right; $2|A$; $2 \nmid B$, since $A \perp B$; 4 exactly divides each factor on the right of (22), whereas an odd power of 2 exactly divides the left; a contradiction. So Δ is odd. □

Lemma 5. *If P is an odd prime divisor of m or n, and $(\Delta|P) = -1$, then (22) has no nontrivial solution.*

Proof. We may write (22) as

$$(\Delta A^2 - LB^2)^2 - 64m^4n^4 = \Delta C^2,$$

so that

$$(\Delta A^2 - LB^2)^2 \equiv \Delta C^2 \bmod P$$

and $(\Delta|P) = -1$ now implies that $\Delta A^2 - LB^2 \equiv 0 \equiv C \bmod P$. Since $P|m$ or n, $\Delta A^2 \equiv m^4B^2$ or $n^4B^2 \bmod P$. But now $(\Delta|P) = -1$ implies that $A \equiv 0 \equiv B \bmod P$ in either case, contradicting $A \perp B$. □

Lemma 6. *Write $\Delta = D_1D_2$, where $D_1|K = m^2 - 2mn - n^2$, and $D_2|M = m^4 - 4m^3n + 18m^2n^2 + 4mn^3 + n^4$; then every prime factor of D_1 is congruent to 1 mod 8, and every prime factor of D_2 is congruent to 1 mod 4.*

Proof. As $\Delta | KM$, it is legitimate to write Δ in this way. By Lemma 4, D_1 and D_2 are odd. By Fact (d), the prime factors of D_2 are congruent to 1 mod 4. Let P be a prime which divides D_1. Then $P \| D_1$, because D_1 is squarefree. Moreover, $P|K$, so, by Fact (c), $P \equiv \pm 1 \bmod 8$. Suppose $P^\alpha \| K$. From (22),

$$C^2 = \left(D_2 A^2 - \frac{K^2}{D_1} B^2\right)\left(D_1 A^2 - \frac{M}{D_2} B^2\right), \tag{23}$$

$$C^2 \equiv -MA^2B^2 \equiv -16m^2n^2A^2B^2 \bmod P.$$

If $P \equiv -1 \bmod 8$, then $C \equiv 0 \equiv AB \bmod P$. Suppose first that $C \equiv 0 \equiv A \bmod P$, so that $B \not\equiv 0 \bmod P$. We claim that $C \equiv 0 \equiv A \bmod P^\alpha$. To see this, suppose, inductively, that $C \equiv 0 \equiv A \bmod P^k$ for some $k \leq \alpha - 1$ (true for $k = 1$), and let $C = P^k C_1, A = P^k A_1$. Then (23) is

$$P^{2k} C_1^2 = \left(D_2 P^{2k} A_1^2 - \frac{K^2}{D_1} B^2\right)\left(D_1 P^{2k} A_1^2 - \frac{M}{D_2} B^2\right),$$

where $P^{2\alpha-1}$ divides K^2/D_1 and $2\alpha - 1 \geq 2k+1$, so that

$$P^{2k} C_1^2 \equiv -P^{2k} M A_1^2 B^2 \bmod P^{2k+1},$$

$$C_1^2 \equiv -16m^2n^2A_1^2B^2 \bmod P,$$

$$C_1 \equiv 0 \equiv A_1 \bmod P,$$

$$C \equiv 0 \equiv A \bmod P^{k+1}$$

and, by induction, $C \equiv 0 \equiv A \bmod P^\alpha$. So

$$0 \equiv \frac{K^2}{D_1} B^2 \frac{M}{D_2} B^2 \bmod P^{2\alpha}$$

and $0 \equiv B \bmod P$, a contradiction. Suppose second that $C \equiv 0 \equiv B \bmod P$. Then $P \nmid A$, since $A \perp B$, and $P \nmid D_2$, since we are still assuming that $P \equiv -1 \bmod 8$. Then P^2 divides the left side of (23), while P does not divide the first factor on the right, and P exactly divides the second factor, again a contradiction. So every prime divisor of D_1 is congruent to 1 mod 8. □

Lemma 7. *If m, n are of opposite parity, then $\Delta \equiv 1 \bmod 8$.*

Proof. From Lemma 6, $\Delta \equiv D_1 D_2$, where $D_1 \equiv 1 \bmod 8$, and $D_2 \equiv 1 \bmod 4$. We need to show that $D_2 \not\equiv 5 \bmod 8$. Note that D_2 *may* contain factors congruent to 5 mod 8, but there will always be an even number of them. Suppose $\Delta \equiv 5 \bmod 8$. Then Fact (a) with (22) gives

$$5C^2 \equiv (5A^2 - B^2)^2 \bmod 8,$$

$$C \equiv 0 \equiv 5A^2 - B^2 \bmod 2.$$

Now $A \perp B$ implies $A \equiv B \equiv 1 \bmod 2$, so that Fact (a) gives

$$\Delta A^2 - K^2 B^2 \equiv 4 \equiv \Delta A^2 - MB^2 \bmod 8,$$

and (22) now gives $\Delta C^2 \equiv 16 \bmod 32$, so that $C \equiv 4 \bmod 8$. But (22) may also be written

$$\Delta C^2 = (\Delta A^2 - LB^2)^2 - 64m^4n^4B^4$$

where $\Delta A^2 - LB^2 \equiv 4 \bmod 8$, so that $\Delta C^2 \equiv 16 \bmod 128$, contradicting $\Delta C^2 \equiv 80 \bmod 128$. So $\Delta \equiv 1 \bmod 8$. □

We will use these lemmas in §10 to restrict the numbers of possibilities for δ and Δ, and hence impose bounds on $[\mathbf{G}_\nu^*]$ and $[\mathbf{G}_{\nu'}^*]$, thereby providing an upper bound on the rank of the curves (7) and (8).

We remark that Lemmas 1 to 7 are complete in the sense that if a curve corresponding to a value of δ or Δ cannot be shown by the Lemmas to be locally unsolvable, then the curve actually does have local solutions at all primes. This is made formal for the curve (21) as follows; details for the curve (22) are similar, and may safely be left to the reader.

Observe that if the curve (21) is nonsingular at the prime p, then the Weil estimates [11] show that there is a point on the curve defined over the field with p elements; and then by Hensel's lemma a point defined over $\mathbf{Q}_p$. It is necessary, therefore, only to consider further those primes for which (21) is singular, namely the primes p with

$$p | 2mn(m^2 - 2mn - n^2)(m^4 - 4m^3n + 18m^2n^2 + 4mn^3 + n^4).$$

The greatest common divisor of any pair of the factors

$$2, \quad m, \quad n, \quad m^2 - 2mn - n^2, \quad m^4 - 4m^3n + 18m^2n^2 + 4mn^3 + n^4$$

is at most 2. First, suppose that $p \neq 2$.

1. If $p | (m^4 - 4m^3n + 18m^2n^2 + 4mn^3 + n^4)$, then necessarily $\delta \perp p$. The case $(\delta|p) = -1$ is covered by Lemma 1; and if $(\delta|p) = +1$, then there is a p-adic solution $(a, b, c) = (1, 0, \sqrt{\delta})$.

2. If $p | (m^2 - 2mn - n^2)$, then $\delta \perp p$. If $(\delta|p) = +1$, then there is a p-adic solution $(a, b, c) = (1, 0, \sqrt{\delta})$. The case $(\delta|p) = -1$, $p \equiv 1 \bmod 8$ is covered by Lemma 2; and if $(\delta|p) = -1$, $p \not\equiv 1 \bmod 8$, then, since $(m-n)^2 - 2n^2 \equiv 0 \bmod p$, it follows that $(2|p) = +1$, so $p \equiv -1 \bmod 8$. Then there is the p-adic solution

$$(a, b, c) = \left(2mn\sqrt{-1/\delta}, 1, 2mn(m^2 - 2mn - n^2)\sqrt{-1/\delta}\right).$$

3. If $p | mn$ and $p | \delta$, then there is a p-adic solution given by $(a, b) = (1, 1)$. Suppose that $p | m$ and $p \nmid \delta$. Let $u, v \in \mathbf{Q}_p$ be a solution of the Pell equation $u^2 - \delta v^2 = 1$ for which $v \not\equiv 0 \bmod p$; then there is a p-adic solution of (21) given by $(a, b) = (n^2 v, 1)$. Similarly for $p | n$, $p \nmid \delta$.

Second, suppose that $p = 2$.

1. If m and n are of opposite parity, then the reader may check that there is a 2-adic solution given by $(a, b) = (4, 1)$.

2. If m and n are both odd, the following are readily verified to provide 2-adic solutions of (21):

$$\delta \equiv 1 \bmod 8 \quad : \quad (a, b) = (1, 0),$$
$$\delta \equiv 5 \bmod 8 \quad : \quad (a, b) = (4, 1),$$
$$\delta \equiv 3 \bmod 4 \quad : \quad (a, b) = (2, 1).$$

7. Table of solutions found by machine search

A search was undertaken by Richard Blecksmith and John Selfridge for numerical solutions to the equation (3). This extended a list of about eighty solutions, kindly supplied to us by John Leech. Parity considerations at (3) allow us to assume that at most one of the quantities $m + n, r + s$ and $u + v$ is even. We may also assume that $m, n; r, s; u, v$ are each positive, that $m \perp n, r \perp s, u \perp v$, and, by symmetry, that $m + n \leq r + s \leq u + v$. The search was made under these conditions over the range

$$3 \leq m + n \leq r + s \leq 250.$$

Of course, this does not ensure that we have found all slopes with $m + n \leq 250$ that do occur, since there will be many solutions with

$$3 \leq m + n \leq 250 < r + s \leq u + v.$$

The 468 solutions found in the search are listed in Table 1 with $m, n; r, s; u, v$ in the lexicographic order $1 \leq m < m + n \leq r + s$. The number of solutions

with	$m + n$ even	$r + s$ even	$u + v$ even	all odd	Total
is	104	78	37	249	468

	m	n	r	s	u	v
1.	1	2	3	1	3	2
2.	1	2	60	143	220	39
3.	3	1	91	60	52	105
4.	3	2	4	3	4	3
5.	5	1	1	6	10	3
6.	2	5	5	3	10	3
7.	2	5	21	5	15	14
8.	2	5	5	36	45	4
9.	3	4	4	5	10	3
10.	4	3	13	225	468	25
11.	7	1	105	136	56	255
12.	5	4	18	5	9	20
13.	3	7	28	15	35	12
14.	7	3	85	22	102	385
15.	1	10	3	10	15	1
16.	5	6	11	15	33	10
17.	7	4	6	7	12	7
18.	7	5	6	7	21	10
19.	2	11	3	40	165	8
20.	5	8	5	9	40	9
21.	6	7	119	9	14	153
22.	9	4	15	14	36	35
23.	10	3	3	11	30	11
24.	10	3	5	9	10	9
25.	10	3	7	12	21	20
26.	10	3	8	55	88	15
27.	10	3	63	26	35	117
28.	10	3	28	65	91	60
29.	10	3	72	23	45	184
30.	11	2	11	21	14	33
31.	11	2	119	33	34	231
32.	12	1	9	5	4	45
33.	5	9	105	74	259	90
34.	11	3	15	38	95	66
35.	13	1	27	22	26	297
36.	2	13	30	1	3	65
37.	11	4	35	33	60	77
38.	1	15	18	19	171	10
39.	1	15	65	2	5	78
40.	1	15	85	44	187	12
41.	9	7	22	15	99	70
42.	15	1	11	10	5	66
43.	15	1	9	20	4	45
44.	2	15	15	16	48	5
45.	4	13	13	18	52	9
46.	5	12	6	11	60	11
47.	5	12	8	9	40	9
48.	5	12	34	15	51	20
49.	6	11	55	38	95	33
50.	9	8	112	33	77	144
51.	10	7	77	39	165	182
52.	11	6	31	55	155	66
53.	12	5	11	10	11	12
54.	14	3	35	209	399	110
55.	13	5	45	38	130	171
56.	13	6	8	13	24	13
57.	13	6	57	13	[illegible]	78
58.	18	1	17	99	[illegible]	187
59.	1	19	50	3	[illegible]	25
60.	9	11	20	143	[illegible]	52
61.	8	13	27	4	[illegible]	104
62.	8	13	51	40	[illegible]	68
63.	8	13	77	75	[illegible]	429
64.	8	13	60	187	[illegible]	221
65.	10	11	87	44	[illegible]	116
66.	11	10	95	99	[illegible]	90
67.	13	8	28	45	[illegible]	117
68.	16	5	21	22	[illegible]	165
69.	7	15	16	15	[illegible]	21
70.	9	13	33	80	[illegible]	144
71.	3	20	5	39	[illegible]	4
72.	5	18	184	27	[illegible]	216
73.	6	17	27	4	[illegible]	68
74.	6	17	42	5	[illegible]	85
75.	6	17	51	11	[illegible]	33
76.	9	14	15	8	[illegible]	35
77.	20	3	7	24	[illegible]	40
78.	13	12	24	5	[illegible]	40
79.	22	3	60	31	[illegible]	372
80.	3	23	3	70	[illegible]	7
81.	7	19	15	14	[illegible]	30
82.	11	15	14	33	[illegible]	22
83.	11	15	65	36	[illegible]	52
84.	15	11	77	32	[illegible]	224
85.	25	1	87	50	6	145
86.	2	25	75	119	525	34
87.	7	20	45	8	40	63
88.	13	14	25	21	175	78
89.	17	10	27	85	135	34
90.	19	8	88	45	99	190
91.	25	2	189	5	14	675
92.	9	19	54	85	646	135
93.	25	3	132	35	28	275
94.	25	3	44	153	225	748
95.	4	25	91	90	819	100
96.	7	22	10	21	33	5
97.	8	21	21	20	35	8
98.	9	20	12	25	50	9
99.	9	20	33	20	55	18
100.	9	20	65	19	247	180
101.	10	19	69	19	115	114
102.	12	17	80	23	272	345
103.	21	8	10	21	40	21
104.	21	8	11	30	88	35
105.	21	8	69	80	115	112
106.	24	5	35	26	56	195
107.	11	19	66	85	646	165
108.	23	7	32	15	161	480
109.	3	28	35	16	48	5
110.	4	27	105	13	260	189
111.	6	25	26	5	39	25
112.	6	25	25	57	150	19
113.	9	22	65	18	55	39
114.	10	21	15	17	70	17
115.	13	18	36	5	13	60

Table 1. *Numerical solutions to equation* (3)

116.	13	18	11	192	1287	64
117.	15	16	30	17	80	51
118.	15	16	117	100	325	144
119.	16	15	33	8	33	80
120.	16	15	42	5	16	105
121.	18	13	65	69	230	117
122.	19	12	20	19	30	19
123.	19	12	42	5	19	140
124.	19	12	55	27	297	380
125.	20	11	9	35	252	55
126.	20	11	17	33	51	20
127.	20	11	88	57	152	165
128.	21	10	209	36	220	1197
129.	26	5	20	33	60	143
130.	26	5	65	27	26	135
131.	27	4	180	29	27	290
132.	28	3	76	35	21	190
133.	17	15	34	5	10	51
134.	21	11	35	6	21	110
135.	7	26	35	6	35	39
136.	13	20	66	65	165	52
137.	25	8	132	25	33	200
138.	25	8	55	156	429	200
139.	26	7	28	33	84	143
140.	19	15	65	38	95	78
141.	19	15	150	23	114	575
142.	2	33	25	11	50	3
143.	22	13	30	11	33	65
144.	22	13	105	104	429	280
145.	31	4	35	93	60	217
146.	1	36	166	45	415	12
147.	4	33	39	4	33	26
148.	4	33	27	22	108	11
149.	4	33	112	15	240	77
150.	7	30	4	65	156	7
151.	9	28	39	4	18	91
152.	15	22	15	119	374	35
153.	17	20	26	15	68	39
154.	20	17	12	31	255	62
155.	20	17	65	6	17	156
156.	25	12	15	152	200	19
157.	36	1	18	25	3	100
158.	5	33	95	12	76	55
159.	15	23	69	14	46	105
160.	29	10	15	26	39	29
161.	33	7	55	82	210	451
162.	1	40	165	47	1551	40
163.	2	39	87	8	312	29
164.	12	29	59	6	59	348
165.	12	29	5	68	510	29
166.	12	29	87	34	116	51
167.	17	24	24	23	68	23
168.	17	24	88	21	66	119
169.	17	24	164	51	123	136
170.	21	20	102	55	187	140
171.	23	18	36	25	100	69
172.	26	15	37	12	111	260
173.	30	11	105	116	203	220
174.	31	10	12	31	60	31
175.	31	10	31	69	186	115
176.	35	6	56	155	120	217
177.	38	3	120	7	24	665
178.	1	42	92	3	[illegible]	12
179.	4	39	135	14	[illegible]	140
180.	13	30	13	32	[illegible]	32
181.	13	30	60	19	[illegible]	95
182.	13	30	68	21	[illegible]	140
183.	13	30	129	65	[illegible]	78
184.	15	28	135	88	[illegible]	77
185.	17	26	20	39	[illegible]	15
186.	30	13	32	15	[illegible]	65
187.	39	4	32	39	[illegible]	96
188.	19	25	61	150	[illegible]	38
189.	23	21	19	46	[illegible]	14
190.	35	9	33	50	[illegible]	225
191.	7	38	28	171	[illegible]	19
192.	19	26	140	39	[illegible]	380
193.	11	35	84	23	[illegible]	231
194.	12	33	76	165	[illegible]	95
195.	27	19	71	30	[illegible]	710
196.	2	45	13	60	[illegible]	9
197.	3	44	16	75	[illegible]	25
198.	5	42	40	9	56	9
199.	10	37	74	135	185	27
200.	11	36	44	185	407	45
201.	14	33	143	56	231	104
202.	39	8	204	13	17	312
203.	41	6	123	95	38	205
204.	42	5	28	93	35	124
205.	42	5	117	98	65	441
206.	43	5	15	98	245	258
207.	4	45	80	11	99	16
208.	8	41	41	132	328	33
209.	12	37	20	69	185	23
210.	25	24	15	43	200	43
211.	27	22	27	119	374	63
212.	39	10	23	30	23	39
213.	11	40	48	5	11	48
214.	25	27	92	35	644	675
215.	41	11	44	45	205	396
216.	1	52	76	117	494	9
217.	5	48	57	112	456	35
218.	23	30	15	58	145	23
219.	50	3	75	4	3	100
220.	25	29	45	196	2900	441
221.	3	52	11	52	143	6
222.	42	13	155	91	186	455
223.	43	12	14	43	84	43
224.	46	9	39	40	184	585
225.	5	51	55	4	33	68
226.	45	11	1	78	990	13
227.	47	9	31	90	186	235
228.	44	13	78	77	84	143
229.	1	57	67	4	804	19
230.	1	57	77	4	836	21
231.	15	43	36	23	828	215
232.	55	3	44	35	5	84
233.	20	39	11	72	715	72
234.	28	31	111	124	259	93

Table 1 (continued)

235.	39	20	28	39	70	39
236.	39	20	40	183	312	61
237.	17	44	85	33	165	68
238.	20	41	153	76	1710	697
239.	25	36	18	65	156	25
240.	37	24	42	37	56	37
241.	5	57	44	19	380	33
242.	7	55	140	33	165	28
243.	17	45	63	22	170	77
244.	23	39	76	115	1196	285
245.	39	23	39	70	230	91
246.	45	17	11	60	187	36
247.	51	11	99	68	44	153
248.	51	11	150	17	50	561
249.	9	55	116	15	261	220
250.	39	25	21	58	1218	325
251.	45	19	35	78	266	117
252.	29	36	18	61	348	61
253.	32	33	116	55	435	352
254.	44	21	205	44	105	451
255.	51	14	15	98	255	49
256.	19	48	168	5	19	560
257.	27	40	65	18	104	135
258.	42	25	75	92	300	161
259.	52	15	15	58	145	52
260.	11	57	26	209	494	33
261.	4	65	85	87	2652	145
262.	29	41	45	116	1044	205
263.	5	66	161	30	231	23
264.	6	65	116	21	1092	145
265.	15	56	119	16	112	255
266.	26	45	65	99	234	55
267.	57	14	16	57	112	57
268.	57	14	35	39	91	190
269.	60	11	65	21	44	273
270.	10	63	95	51	1190	171
271.	13	60	99	104	585	88
272.	15	58	81	145	405	58
273.	23	51	85	22	374	345
274.	35	39	105	104	168	65
275.	52	23	16	105	1840	273
276.	56	19	48	35	133	240
277.	7	69	46	55	805	66
278.	37	39	23	78	598	111
279.	37	39	62	49	3038	1443
280.	12	65	99	8	99	520
281.	75	2	7	192	64	525
282.	49	30	28	65	156	49
283.	49	30	48	49	80	49
284.	55	24	24	65	156	55
285.	65	14	28	57	260	399
286.	68	11	42	187	132	119
287.	72	7	60	49	45	392
288.	47	33	145	22	290	1551
289.	13	69	131	30	1965	598
290.	17	65	102	13	51	130
291.	67	15	45	116	261	268
292.	69	13	54	161	299	378
293.	15	68	105	92	714	115
294.	32	51	26	85	195	32
295.	57	26	13	165	285	22
296.	62	21	90	77	99	155
297.	65	18	117	64	65	192
298.	68	15	25	68	51	50
299.	80	3	85	134	48	1139
300.	11	74	135	19	2090	999
301.	16	69	75	11	275	368
302.	11	75	95	6	38	275
303.	35	52	170	63	595	468
304.	68	19	48	85	272	285
305.	3	86	165	16	880	43
306.	24	65	81	10	104	405
307.	50	39	75	52	75	52
308.	73	16	18	73	144	73
309.	34	57	21	95	714	95
310.	40	51	213	17	120	1207
311.	60	31	17	80	816	155
312.	66	25	85	114	561	475
313.	75	16	159	50	48	265
314.	15	77	129	14	43	110
315.	17	75	100	11	60	187
316.	28	65	81	14	260	567
317.	28	65	39	88	840	143
318.	39	55	99	2	5	234
319.	67	28	36	67	126	67
320.	7	90	99	7	99	70
321.	9	88	129	8	43	198
322.	9	88	71	160	3905	288
323.	42	55	99	28	99	140
324.	93	4	5	114	155	76
325.	11	90	132	5	11	180
326.	61	40	72	61	90	61
327.	72	29	81	29	81	232
328.	72	29	64	63	192	203
329.	3	100	104	3	104	75
330.	25	78	21	130	273	25
331.	48	55	143	24	55	208
332.	48	55	154	15	48	385
333.	48	55	19	176	380	33
334.	74	31	65	93	806	555
335.	88	17	39	68	39	88
336.	35	72	35	124	279	40
337.	85	22	20	99	340	99
338.	57	52	117	124	403	171
339.	69	40	92	25	69	200
340.	75	34	125	21	125	714
341.	90	19	95	144	80	171
342.	91	18	20	91	180	91
343.	100	9	45	74	37	300
344.	100	9	88	45	25	264
345.	110	1	210	13	11	1365
346.	55	57	23	114	230	33
347.	103	9	143	90	206	2145
348.	5	108	40	123	1080	41
349.	8	105	150	11	175	88
350.	36	77	124	13	217	1287
351.	99	14	63	65	77	390
352.	24	91	107	14	312	749
353.	17	99	135	34	459	110
354.	11	106	110	93	1749	155

Table 1 (continued)

355.	40	77	185	18	385	2664
356.	85	33	54	85	270	187
357.	105	13	154	45	99	910
358.	75	44	99	70	275	252
359.	17	104	143	12	33	136
360.	79	42	60	79	140	79
361.	100	21	175	22	21	220
362.	13	110	124	33	1364	195
363.	38	85	37	153	703	90
364.	58	65	40	87	[illegible]	104
365.	55	69	115	118	[illegible]	177
366.	33	92	80	143	[illegible]	368
367.	12	115	110	87	[illegible]	115
368.	19	108	112	15	[illegible]	513
369.	45	82	104	53	[illegible]	2120
370.	92	35	9	160	[illegible]	115
371.	99	28	38	91	[illegible]	693
372.	51	77	88	45	[illegible]	495
373.	85	44	189	22	[illegible]	2380
374.	5	126	177	7	[illegible]	126
375.	26	105	91	44	[illegible]	44
376.	63	68	164	63	[illegible]	123
377.	76	55	133	45	[illegible]	693
378.	105	26	87	130	[illegible]	145
379.	110	21	44	135	[illegible]	220
380.	111	20	22	111	[illegible]	111
381.	48	85	131	51	[illegible]	80
382.	75	58	117	50	[illegible]	145
383.	119	15	203	20	[illegible]	812
384.	16	119	231	17	[illegible]	528
385.	119	16	95	48	[illegible]	969
386.	105	31	45	106	[illegible]	371
387.	77	60	175	12	[illegible]	275
388.	19	120	159	40	[illegible]	57
389.	54	85	155	32	[illegible]	992
390.	55	84	80	77	[illegible]	77
391.	103	36	44	103	[illegible]	103
392.	104	35	21	176	[illegible]	176
393.	108	31	113	42	[illegible]	3164
394.	1	140	30	119	[illegible]	7
395.	56	85	153	35	[illegible]	280
396.	56	85	196	3	[illegible]	2499
397.	56	85	168	37	[illegible]	510
398.	87	55	154	15	[illegible]	203
399.	38	105	118	35	[illegible]	95
400.	92	53	142	69	[illegible]	1219
401.	97	48	66	97	176	97
402.	117	28	126	107	364	963
403.	13	133	130	21	570	91
404.	105	41	65	84	195	164
405.	33	115	170	33	115	102
406.	45	104	120	121	484	117
407.	77	72	77	124	279	88
408.	93	56	88	93	154	93
409.	105	44	143	53	1092	2915
410.	140	9	9	164	205	42
411.	15	136	170	13	104	255
412.	39	112	35	129	1040	129
413.	46	105	115	61	915	322

414.	91	60	110	91	132	91
415.	23	130	115	132	260	33
416.	133	22	24	133	264	133
417.	148	7	84	143	111	2002
418.	42	115	161	40	161	120
419.	120	37	128	37	128	555
420.	85	78	39	176	1105	176
421.	88	75	141	88	141	100
422.	103	60	118	117	4635	3068
423.	31	135	154	27	310	231
424.	81	85	135	34	81	170
425.	133	33	15	224	3040	231
426.	155	11	18	155	90	341
427.	10	157	117	127	20410	1143
428.	104	63	88	117	156	77
429.	150	17	17	156	425	156
430.	37	132	132	59	222	59
431.	60	109	57	115	2180	437
432.	84	85	153	80	153	112
433.	121	48	55	122	1936	915
434.	135	34	92	153	391	540
435.	57	115	46	183	1403	190
436.	143	30	69	110	69	143
437.	13	165	156	35	220	21
438.	13	165	198	13	99	130
439.	109	70	120	109	168	109
440.	140	39	148	45	259	1170
441.	85	96	165	56	187	224
442.	125	56	40	183	875	183
443.	157	24	26	157	312	157
444.	57	130	209	39	209	390
445.	59	130	165	26	59	165
446.	136	53	78	125	7208	4875
447.	147	44	52	147	286	147
448.	182	9	8	195	455	72
449.	141	52	55	156	715	282
450.	106	91	91	132	371	132
451.	144	53	133	72	399	848
452.	63	136	153	58	232	119
453.	111	88	120	121	740	363
454.	139	60	78	139	260	139
455.	133	72	104	133	234	133
456.	153	52	119	108	238	351
457.	183	26	28	183	364	183
458.	130	81	81	148	481	180
459.	137	75	91	150	910	411
460.	162	55	41	195	4455	1066
461.	183	38	17	209	3553	366
462.	165	58	137	99	1507	2610
463.	133	94	165	76	420	517
464.	17	212	80	159	255	16
465.	163	66	84	163	308	163
466.	213	16	16	219	584	213
467.	52	183	149	90	9685	2196
468.	211	28	30	211	420	211

Table 1 (concluded)

8. A rank-three example considered in detail

In this section we shall examine a specific numerical instance in detail, in order to demonstrate how, for a given slope (m,n), it is usually possible to determine all the solutions of (3) corresponding to that slope. John Leech observed many solutions containing the slope $(m,n) = (10,3)$. This corresponds at (7) to the curve

$$\tau^2 = \sigma(\sigma^2 + \frac{8161}{3600}\sigma + 1), \tag{24}$$

and this is indeed a curve of rank 3:

Theorem 8. *The group of rational points on the elliptic curve* (24) *has rank* 3, *and is generated by the points*

$$\left(-\frac{3}{5}, -\frac{1}{100}\right), \quad \left(\frac{1}{30}, \frac{341}{1800}\right), \quad \left(\frac{6}{25}, \frac{31}{50}\right)$$

of infinite order, and $\left(-1, \frac{31}{60}\right)$ *of order* 4.

In fact, we work with the Néron minimal model of (24), or rather a translation thereof, in order to avoid a rational point with x-coordinate 0. This is necessary in order to apply a technique of Tate.

The curve (24) maps to the following (Figure 6):

$$E: \quad y^2 + xy - 500y = x^3 + 540x^2 - 480000x - 20000000 \tag{25}$$

via the maps

$$(x,y) = (900\sigma + 500, -450\sigma + 27000\tau),$$
$$(\sigma,\tau) = \left(\frac{x-500}{900}, \frac{y+\frac{1}{2}x-250}{27000}\right).$$

We actually prove the following theorem.

Theorem 9. *The group of rational points on* (25) *is generated by* $(-40,0)$, $(530,5100)$, $(716,16632)$ *of infinite order, and* $(-400,14400)$ *of order* 4.

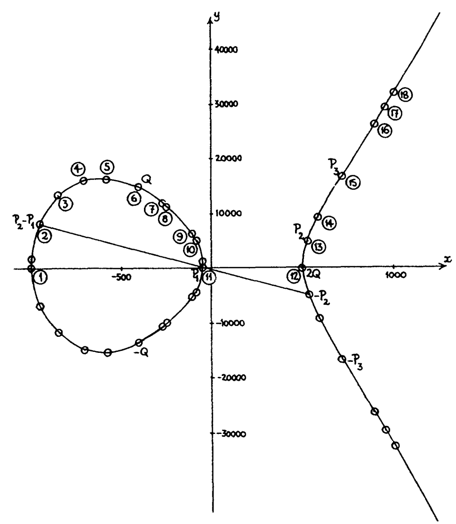

Figure 6. *The curve* $y^2 + xy - 500y = x^3 + 540x^2 - 480000x - 20000000$.
The circled numbers correspond to the line numbers of Table 3.

Proof. The determination of the torsion subgroup is quite straightforward and is left as an exercise for the reader. The group is cyclic of order 4 with generator $Q = (-400, 14400)$. Henceforth, when we indicate a rational point of (25) by P, it will be assumed that P is nontorsion.

Any determination of generators will involve an argument with heights; the proof we present is modelled on that of Buhler, Gross & Zagier [2].

Let $P = (x, y)$ be a point of (25), and write $x = x(P) = a/b$, $a \perp b$, $b > 0$. Then the naive height $h(P)$ is defined by

$$h(P) = \ln \max(|a|, b).$$

The canonical (or Tate) height $\hat{h}(P)$ can be written as a sum of local heights,

$$\hat{h}(P) = \sum_{\nu} \hat{h}_\nu(P),$$

one term for each distinct valuation on $\mathbf{Q}$, and it is this height which we estimate as follows.

For the archimedean valuation or ordinary absolute value, Tate has given an easily computed power series which allows computation of $\hat{h}_\infty(P)$. Specifically, for the curve (25),

$$\hat{h}_\infty(P) = \ln|x| + \sum_{n=0}^{\infty} 4^{-n-1} \ln(z_n)$$

with

$$z_n = 1 + \frac{960500}{x_n^2} + \frac{159500000}{x_n^3} + \frac{273725000000}{x_n^4} \quad (n \ge 0)$$

and

$$x_0 = x; \quad x_{n+1} = \frac{x_n^4 + 960500x_n^2 + 159500000x_n + 273725000000}{4x_n^3 + 2161x_n^2 - 1921000x_n - 79750000}.$$

The formula for x_{n+1} corresponds to the duplication formula on (25), so that

$$x_n = x(2^n P), \quad n \ge 0.$$

Note that $2^n P$ for $n \ge 1$ is a point lying in the right-hand branch of the graph of (25).

Rewriting (25) in the form

$$\left(y + \frac{1}{2}x - 250\right)^2 = (x - 500)\left(x^2 + \frac{4161}{4}x + 39875\right),$$

we see that for P in the loop of the graph (Figure 6),

$$-1000.390568 \le x(P) \le -39.859432,$$

and for P in the right-hand branch,

$$500 \le x(P).$$

It follows that if P is in the right hand branch, then $x_n \ge 500$ for $n \ge 0$. Since P is nontorsion, then $x_n > 500$ for $n \ge 0$. Hence,

$$1 < z_n \le 10.497600 \quad (n \ge 0),$$

so that

$$0 < \ln z_n \le 2.351147 \quad (n \ge 0)$$

and

$$0 < \sum_{n=0}^{\infty} 4^{-n-1} \ln z_n \le 2.351147 \sum_{n=0}^{\infty} 4^{-n-1} < 0.783716.$$

Thus, $x > 500$ implies

$$\ln|x| < \hat{h}_\infty(P) < \ln|x| + 0.783716. \quad (26)$$

On the other hand, if P is in the loop of the graph (which is compact), then the maximum and minimum of $(\hat{h}_\infty(P) - \ln|x|)$ can be readily computed, and we find (see Figure 7):

$$\ln|x| + 0.182338 < \hat{h}_\infty(P) < \ln|x| + 2.894039 \quad (26a)$$

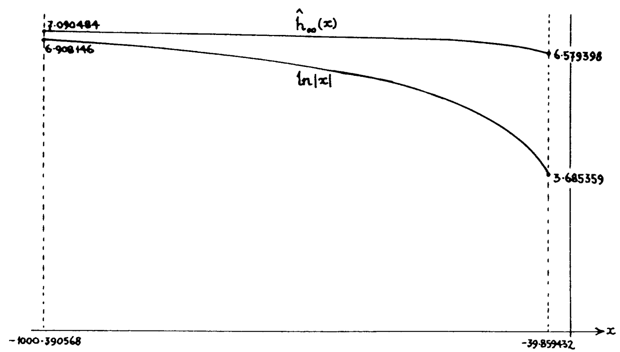

Figure 7. *Comparison of archimedean valuation with* $\ln|x|$

It remains now to estimate the local component of the canonical height at the prime p. The algorithms of Silverman [9] are useful here; he gives an explicit recipe for computing the p-component.

We summarize the results of the local computation. Recall the notation $x = x(P) = a/b$, $a \perp b$, $b > 0$. Denote the usual p-adic valuation by v_p.

1. If $p|b$, then $\hat{h}_p(P) = \mathrm{ord}_p(b) \cdot \ln p$.
2. If $p \nmid b$, and $p \neq 2, 3, 5$, or 31, then $\hat{h}_p(P) = 0$. Define c_i as $-7/8$, $-3/2$, $-15/8$, or -2, according as $i = 1, 2, 3$, or $i \geq 4$.
3. If $p = 2$, $2 \nmid b$, and $v_2(2y + x - 500) = i$, then $\hat{h}_2(P) = c_i \ln 2$.
4. If $p = 3$, $3 \nmid b$, and $x \equiv 1 \bmod 3$, then $\hat{h}_3(P) = 0$, while if $x \not\equiv 1 \bmod 3$ and $v_3(2y + x - 500) = i$, then $\hat{h}_3(P) = c_i \ln 3$.
5. If $p = 5$, $5 \nmid b$, and $x \not\equiv 0 \bmod 5$, then $\hat{h}_5(P) = 0$, while if $x \equiv 0 \bmod 5$ and $v_5(2y + x - 500) = i$, then $\hat{h}_5(P) = c_i \ln 5$.
6. If $p = 31$, and $31 \nmid b$, then $\hat{h}_{31}(P) = -\frac{1}{2}\ln 31$ or $\hat{h}_{31}(P) = 0$, according as $x \equiv 3$ or $x \not\equiv 3 \bmod 31$.

Notice that 1. implies that the sum of all local components for primes dividing b is simply $\ln b$.

The above local factors thus produce the following estimate for the sum over non-archimedean valuations:

$$\ln b - \ln(2^2 \cdot 3^2 \cdot 5^2 \cdot 31^{1/2}) \leq \sum \hat{h}_v(P) \leq \ln b. \quad (27)$$

Combining (26) with (27) we have that for P in the right-hand branch of Figure 6,

$$\ln|a| - 8.519388 < \hat{h}(P) < \ln|a| + 0.783716, \quad (28)$$

and combining (26a) with (27), we have that for P in the loop,

$$\ln|a| - 8.337050 < \hat{h}(P) < \ln|a| + 2.894039. \quad (28a)$$

Since $|a|/b = |x(P)| \geq 39.859$, we can use

$$h(P) = \ln\max(|a|, b) = \ln|a| \ln\min(1, |a|/b) = \ln|a|$$

to rewrite (28) and (28a) in the form

$$-8.519388 < \hat{h}(P) - h(P) < 0.783716 \quad (P \text{ in the right-hand branch}), \quad (29)$$

$$-8.337050 < \hat{h}(P) - h(P) < 2.894039 \quad (P \text{ in the loop}). \quad (29a)$$

It is now straightforward to find all the points P on the curve with given bounded height. For example, to find all points P satisfying $\hat{h}(P) < 2.144204$, we argue as follows.

If P is in the right-hand branch, then (29) implies

$$h(P) < 10.663592, \quad (30)$$

and if P is in the loop, then (29a) implies

$$h(P) < 10.481254. \quad (30a)$$

With $x = a/b$, suppose first $x > 0$. Then (30) holds, and

$$a/b \geq 500.$$

Thus,

$$0 < a < e^{10.663592} < 42770.$$

Moreover, $b \le a/500 < 86$ and b has to be a perfect square. It is a small machine search to find all a, b satisfying these conditions; the x-coordinates of the resulting points are displayed on the left of Table 2.

Suppose second $x < 0$. Then (30a) holds and

$$-1000.390568 < \frac{a}{b} < -39.859432.$$

Thus,

$$|a| < e^{10.481254} < 35642,$$

and as above, b is strongly bounded. The x-coordinates of the points resulting from the machine search are displayed on the right of Table 2. The third column of Table 2 expresses the point P in terms of the group law on the curve.

Table 2. *All points with height* $\hat{h}(P) < 2.144204$

$x(P)$	$\hat{h}(P)$		$x(P)$	$\hat{h}(P)$	
500	0	$2Q$	−40	2.071634	P_1
530	2.144203	P_2	−100	1.951139	$P_3 - Q$
716	1.951139	P_3	−250	2.144203	$P_2 - Q$
1460	2.071634	$P_1 + Q$	−400	0	Q
4250	1.951139	$P_3 + 2Q$	−580	2.144203	$P_2 + Q$
27500	2.144203	$P_2 + 2Q$	−850	1.951139	$P_3 + Q$
5375/4	2.071634	$P_1 - Q$	−1000	2.071634	$P_1 + 2Q$

Now a standard 2-descent on the curve (25) shows that P_1, P_2, P_3 are generators for $E(\mathbf{Q})/2E(\mathbf{Q})$, which is of rank 3 over $\mathbf{Z}/2\mathbf{Z}$. Together with the above computation of all points with small height, it follows immediately that P_1, P_2, P_3 are generators of infinite order for $E(\mathbf{Q})$, as required. □

The principal theorem is now an immediate corollary.

As an instance of height computations, we list the heights of 30 integer points P of $E(\mathbf{Q})$ in Table 3 together with the representation $P = n_0Q + n_1P_1 + n_2P_2 + n_3P_3$. Some of these are illustrated in Figure 6.

Table 3. *Thirty points of the curve* (25), *with their heights*

	x	y	n_0	n_1	n_2	n_3	$\hat{h}(P)$
1.	−1000	0	2	−1	0	0	2.071634
2.	−958	8262	0	−1	1	0	2.553675
3.	−850	13500	−1	0	0	−1	1.951139
4.	−710	15950	0	1	0	1	3.240451
5.	−580	16200	−1	0	−1	0	2.144203
6.	−400	14400	1	0	0	0	0
7.	−268	11712	0	1	−1	−1	4.393466
8.	−250	11250	−1	0	1	0	2.144203
9.	−100	6000	−1	0	0	1	1.951139
10.	−88	5376	0	−1	1	−1	4.616162
11.	−40	0	0	1	0	0	2.071634
12.	500	0	2	0	0	0	0
13.	530	5100	0	0	1	0	2.144203
14.	590	9450	−1	−1	0	−1	3.240451
15.	716	16632	0	0	0	1	1.951139
16.	892	26096	−1	−1	−1	0	5.878000
17.	950	29250	−1	1	−1	0	2.553675
18.	1000	32000	2	0	1	1	3.201672
19.	1460	59040	1	1	0	0	2.071634
20.	2120	103680	0	0	−1	−1	3.201672
21.	2300	117000	−1	−1	1	0	2.553675
22.	4250	288750	2	0	0	−1	1.951139
23.	7790	704700	−1	−1	0	1	4.805095
24.	9500	945000	−1	1	0	1	3.240451
25.	12020	1339200	0	0	1	−1	4.989012
26.	27500	4590000	2	0	−1	0	2.144203
27.	36800	7091700	1	−1	1	1	4.393466
28.	57632	13870656	1	1	1	1	6.153146
29.	922100	885254400	0	−2	0	0	8.286536
30.	1521500	1876329000	1	1	−2	−1	8.270243

We now have a well-ordering on $E(\mathbf{Q})$ determined by the canonical height. The appropriate quadratic form gives the following lemma.

Lemma 10. *The canonical height is given by the quadratic form* $\hat{h}(aP_1+bP_2+cP_3) = \hat{h}(P_1)a^2 + \hat{h}(P_2)b^2 + \hat{h}(P_3)c^2 + (\hat{h}(P_2+P_3) - \hat{h}(P_2) - \hat{h}(P_3))bc + (\hat{h}(P_3+P_1) - \hat{h}(P_3) - \hat{h}(P_1))ca + (\hat{h}(P_1+P_2) - \hat{h}(P_1) - \hat{h}(P_2))ab$, *in which the coefficients* $\hat{h}(P_1) \approx 2.071634, \hat{h}(P_2) \approx 2.144203, \hat{h}(P_3) \approx 1.951139$, $\hat{h}(P_2+P_3) \approx 3.201672, \hat{h}(P_3+P_1) \approx 3.240451, \hat{h}(P_1+P_2) \approx 5.878000$ *can be computed to any desired degree of accuracy.*

Table 4. *Points on* (25), *to within torsion, in order of height, up to height 10*

P	x	y	$\hat{h}(P)$
P_3	716	16632	1.951139
P_1	−40	0	2.071634
P_2	530	5100	2.144203
$-P_1+P_2$	−958	8262	2.553675
$-P_2-P_3$	2120	103680	3.201672
P_1+P_3	−710	15950	3.240451
$P_1-P_2-P_3$	−268	11712	4.393466
$-P_1+P_2-P_3$	−88	5376	4.616162
$-Q-P_1+P_3$	7790	704700	4.805095
P_2-P_3	12020	1339200	4.989012
$-Q-P_1-P_2$	892	26096	5.878000
$Q+P_1+P_2+P_3$	57632	13870656	6.153146
$-Q-2P_1+P_2$	$-46180/13^2$	$26017200/13^3$	7.106415
$Q-P_1+2P_2$	$-7015/8^2$	$-2994975/8^3$	7.324122
$-2P_3$	$25496/5^2$	$4137516/5^3$	7.804556
$-Q-P_2-2P_3$	$-103780/11^2$	$-15850800/11^3$	8.161419
$Q+P_1-2P_2-P_3$	1521500	1876329000	8.270243
$-2P_1$	922100	885254400	8.286536
$2Q+P_1+2P_3$	$-70375/16^2$	$48670875/16^3$	8.311546
$-2P_1+P_2-P_3$	$154205/4^2$	$61783155/4^3$	8.386580
$2P_2$	$817400/11^2$	$-768384000/11^3$	8.576812
$Q+2P_1+P_3$	$-42916/7^2$	$-3869424/7^3$	8.673031
$Q+2P_2+P_3$	$-24946/5^2$	$332748/5^3$	8.740611
$P_1+P_2-P_3$	$-350200/67^2$	$1443406500/67^3$	9.505131
$Q+2P_1-P_2-P_3$	$-309245/18^2$	$49827635/18^3$	9.728526

In Table 4 we list, up to torsion, points on the curve (25) in terms of increasing height, up to height 10.

The maps at (25) which recover the corresponding points on the curve (24) lead to the following formulas for recovery of the $r, s; u, v$ parameters of the geometric nu-configuration:

$$\frac{r}{s} = \frac{(500-x)(400+x)}{30(y-15x+7500)}, \quad \frac{u}{v} = \frac{y-15x+7500}{30(400+x)}.$$

Corresponding to the entries of Table 4 we can therefore compute the following parameters for geometric nu-configurations (Table 5).

Table 5. *Nu-configurations containing the slope* 91/60 ($m = 10, n = 3$)

P	r	s	u	v
P_3	3	−5	2	5
P_1	4	5	3	4
P_2	1	−5	1	6
$-P_1+P_2$	9	−10	9	−5
$-P_2-P_3$	12	−7	21	20
P_1+P_3	11	−30	11	−3
$P_1-P_2-P_3$	8	55	88	15
$-P_1+P_2-P_3$	28	65	91	60
$-Q-P_1+P_3$	117	−35	63	26
P_2-P_3	184	−45	72	23
$-Q-P_1-P_2$	238	−285	133	255
$Q+P_1+P_2+P_3$	552	−65	299	40
$-Q-2P_1+P_2$	154	1105	561	91
$Q-P_1+2P_2$	1003	560	357	944
$-2P_3$	1653	−1700	1292	2175
$-Q-P_2-2P_3$	5254	−2145	481	−781
$Q+P_1-2P_2-P_3$	3705	−89	2314	57
$-2P_1$	4000	−123	3936	125
$2Q+P_1+2P_3$	805	5856	1403	224
$-2P_1+P_2-P_3$	2451	−664	4731	1720
$2P_2$	4524	2035	5365	−1716
$Q+2P_1+P_3$	7102	−3045	1537	−2345
$Q+2P_2+P_3$	4187	−3525	3713	−2650
$P_1+P_2-P_3$	6789	14740	6820	4891
$Q+2P_1-P_2-P_3$	5833	−6588	18727	−10260

The eight points $\pm P$, $\pm P \pm Q$, $\pm P + 2Q$ each return the same nu-configuration. This is a general phenomenon, illustrated by the pattern on the right of Table 6, and is not restricted to the particular slope $(m, n) = (10, 3)$, nor the particular point $P = P_1$, which corresponds to solution **9.** of Table 1, or line 11 (or 1, or 19) of Table 3. We give the coordinates of the points (x, y) on the curve (25), and also of the corresponding points (σ, τ) on the curve (19), which, for $(m, n) = (10, 3)$, takes the form

$$\tau^2 = \sigma(\sigma^2 + 8161\sigma + 60^4). \tag{31}$$

Negation (change of sign of τ in (19) or (31)) corresponds to interchange of (r, s) with (u, v), and reflexion of Table 6 from top to bottom.

We shall see in the next section that the torsion group of E consists of, or contains as a subgroup, a group isomorphic to $\mathbf{Z}/4\mathbf{Z}$ and generated by $Q(-4m^2n^2, 4m^2n^2K)$. Addition of Q corresponds to interchange of (r, s) with (u, v) while replacing the latter by $(v, -u)$. The coordinates (x, y) on (25) and (σ, τ) on (31) are related by

$$\sigma = 4(x - 500), \quad \tau = 4(2y + x - 500) \quad \text{or} \quad x = (\sigma + 2000)/4, \quad y = (\tau - \sigma)/8.$$

Table 6. *Eight points give a single nu-configuration*

	x	y	σ	τ	r	s	u	v				
P_1	-40	0	-2160	-2160	4	5	3	4	r	s	u	v
P_1+Q	1460	59040	3840	476160	4	-3	4	5	v	$-u$	r	s
P_1+2Q	-1000	1500	-6000	6000	5	-4	4	-3	s	$-r$	v	$-u$
P_1-Q	5375/4	$-421875/8$	3375	-418500	3	4	5	-4	u	v	s	$-r$
$-P_1+Q$	5375/4	415125/8	3375	418500	5	-4	3	4	s	$-r$	u	v
$-P_1+2Q$	-1000	0	-6000	-6000	4	-3	5	-4	v	$-u$	s	$-r$
$-P_1-Q$	1460	-60000	3840	-476160	4	5	4	-3	r	s	v	$-u$
$-P_1$	-40	540	-2160	2160	3	4	4	5	u	v	r	s

Generally, on the curve

$$E: \quad \tau^2 = \sigma(\sigma^2 + L\sigma + 16m^4n^4), \tag{19}$$

the eight points lie one each on the eight arcs of the curve separated by the point at infinity, the three points with $\tau = 0$, and the two pairs of points $(-4m^2n^2, \pm 4m^2n^2K)$ and $(4m^2n^2, \pm 4m^2n^2\sqrt{M})$, of which the last is not, in general, rational (see Theorem 11 in §10). One could choose a canonical solution from this set of points: for example, that with $0 < \sigma < 4m^2n^2$ and $\tau > 0$.

9. The torsion group

We describe the torsion group, $\mathbf{T}$, of the curve (7), which we recall in the notation of §6 ($L = K^2 + 8m^2n^2$, $K = m^2 - 2mn - n^2$):

$$E: \tau^2 = \sigma(\sigma^2 + L\sigma + 16m^4n^4). \tag{19}$$

The points $\pm Q(-4m^2n^2, \pm 4m^2n^2K)$ are of order 4, so, by Mazur's theorem [7, 8], the only possibilities for $\mathbf{T}$ on E are

$$\mathbf{Z}/4\mathbf{Z}, \quad \mathbf{Z}/8\mathbf{Z}, \quad \mathbf{Z}/12\mathbf{Z}, \quad \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z} \quad \text{or} \quad \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}.$$

Now the doubling law on E gives the σ-coordinate of $2(\sigma, \tau)$ as the perfect square $(16m^4n^4 - s^2)^2/4t^2$, and since the σ-coordinate of Q, namely $-4m^2n^2$, is not a square, the points $\pm Q$ are not divisible by 2, and $\mathbf{T} \not\simeq \mathbf{Z}/8\mathbf{Z}$. A more tedious calculation shows that these points are not divisible by 3.

[From a more sophisticated standpoint, one can argue as follows. The singular fibres of our curve occur at $m = 0$, $n = 0$, and at the roots of $K = 0$ and $M = 0$. The Kodaira classification of each singular fibre is of Type I_b, where b is in each instance a (small) power of 2. Accordingly the torsion group, being killed by the least common multiple of these orders, is itself of order a power of 2. See §2D of Cox & Zucker [4] for full details.]

So we are left with the possibilities

$$\mathbf{Z}/4\mathbf{Z}, \quad \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}, \quad \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z},$$

each of which can occur.

The last two cases each contain three points of order 2, and the quadratic on the right of (19) splits into two linear factors, $(\sigma + r_1)(\sigma + r_2)$. Then $r_1 + r_2 = L$, $r_1 r_2 = 16m^4n^4$, $(r_1 - r_2)^2 = L^2 - 64m^4n^4 = (L - 8m^2n^2)(L + 8m^2n^2) = K^2M$, so that M is a perfect square, say $z^2 = M = K^2 + 16m^2n^2$, and $(z - 4mn)(z + 4mn) = K^2$. Fact (b) from §6 implies that m, n are of opposite parity, z is odd and $z \perp mn$, and $z - 4mn \perp z + 4mn$ implies that $z \pm 4mn$ are each squares, say w_+^2 and w_-^2. Also, $r_1 - r_2 = \pm Kz$ and $r_1, r_2 = (L \pm Kz)/2$, and, since $r_1 \perp r_2$, $r_1 r_2 = (2mn)^4$, r_1, r_2 are each fourth powers. The equation

$$M = z^2 = m^4 - 4m^3n + 18m^2n^2 + 4mn^3 + n^4$$

represents an elliptic curve, which maps into $v^2 = u(u^2 + 6u + 4)$, recognizable as the curve (4) of §3, where it arose from symmetrical nu-configurations: its rank is 1, with solutions $(m, n) = (1,0)$, $(3,2)$, $(50,39)$, $(221,5700)$, ...

The tangents from the origin to the curve (19) have slopes μ, given by repetition of roots in $\mu^2\sigma = \sigma^2 + L\sigma + 16m^4n^4$:

$$(L - \mu^2)^2 = 64m^4n^4,$$

$$\mu^2 = L \pm 8m^2n^2 = K^2 \text{ or } M\ (= z^2),$$

$$\mu = \pm K \text{ or } \pm z.$$

The four points of contact have order 4, and coordinates $(-4m^2n^2, \pm 4m^2n^2K)$ and $(4m^2n^2, \pm 4m^2n^2z)$.

There will be a point of order 8 just if a tangent from $(4m^2n^2, 4m^2n^2z)$ has a rational point of contact: i.e., if its slope μ is such that the quadratic

$$\sigma^2 + \sigma(L - \mu^2 + 4m^2n^2) + 4m^2n^2(z - \mu)^2 = 0$$

has equal roots, i.e., just if

$$L - \mu^2 + 4m^2n^2 = \pm 4mn(z - \mu),$$

i.e., just if

$$(\mu \pm 2mn)^2 = L + 4m^2n^2 + 4m^2n^2 \pm 4mnz = M \pm 4mnz = z(z \pm 4mn). \qquad (32)$$

Now $(z+4mn)(z-4mn) = M - 16m^2n^2 = K^2$, and $z+4mn,\ z-4mn$ are coprime. Hence $z \pm 4mn = w_\epsilon^2$, so that (32) implies that z is a square, say y^2. Then $\mu + 2\epsilon mn = \pm w_\epsilon y$, and the equal roots are $\sigma = 2\epsilon mn(z-\mu)$, leading to the eight points of order 8:

$$(2mn(2mn + y(\epsilon y \pm w_\epsilon)), \quad \pm 2mnw_\epsilon y(2mn + y(\epsilon y \pm w_\epsilon))),$$

where ϵ is to be read throughout either as + or as −.

Thus the group $\mathbf{T} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$ occurs only when

$$z^2 = M = m^4 - 4m^3n + 18m^2n^2 + 4mn^3 + n^4 = y^4. \qquad (33)$$

The equation (33) represents a curve of genus 3, and by the theorem of Faltings [5], has only finitely many rational solutions. The actual determination of such solutions may be very difficult: we conjecture that the only solutions are afforded by $\pm(m,n) = (0,1)$, $(1,0)$, $(3,2)$ and $(2,-3)$. To sum up:

Theorem 11. (a) $\mathbf{T} \simeq \mathbf{Z}/4\mathbf{Z}$ *when* $M = m^4 - 4m^3n + 18m^2n^2 + 4mn^3 + n^4 \neq z^2$;
(b) $\mathbf{T} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$ *when* $M = z^2$, *but* $z \neq y^2$;
(c) $\mathbf{T} \simeq \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$ *when* $M = y^4$.

Generators in each instance are (a) $(-4m^2n^2, 4m^2n^2K)$; (b) $(-4m^2n^2, 4m^2n^2K)$ and $(-(L+Kz)/2, 0)$; (c) $(2mn2(2mn + y^2 + wy),\ 2mnwy(2mn + y^2 + wy))$ and $(-(L+Kz)/2, 0)$, where $w = w_+$ in the above notation.

Numerical examples are provided by (a) $(m,n) = (2,1)$; (b) $(m,n) = (50,39)$ and $z = 8329$; (c) $(m,n) = (3,2)$, $y = 5$, and $w = 7$.

In terms of the nu-configurations, (a) returns four degenerate solutions, (b) returns (a) together with four copies of a symmetrical nu-configuration given by

$$\frac{u}{v} = \frac{K+z}{4mn} = -\frac{r}{s} \quad ,$$

while, if our conjecture is true, (c), with points of order 8, occurs only when $m/n = 3/2$, leading to the exceptional configuration with $r/s = 1/2$ and $u/v = 3/1$, and its permutations.

For the 2-isogenous curve (20) the torsion group is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ in case (a). Otherwise it is $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/4\mathbf{Z}$, except for the possibility $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/8\mathbf{Z}$ when $M = y^4$ and m and n are both perfect squares, an extremely unlikely situation, given the remark following equation (33).

10. Tables of ranks and information leading to the generation of nu-configurations when the rank is positive

Table 7 ($m+n$ odd; $3 \le m+n \le 49$) and Table 8 ($m+n$ even; $4 \le m+n \le 50$) give the ranks of the curves (7) and (8) for specific values of m and n.

Columns m and n are clear, while column K lists those factors of $K = m^2 - 2mn - n^2$ which are 1 mod 8. These are factors of potential values of Δ, but, on the other hand, they may, via Lemma 2, restrict the possibilities for δ.

Column M gives the prime factorization of $M = K^2 + 16m^2n^2$. In Table 8, the factor 2^2 is omitted. In Table 7 (but not in 8) the primes congruent to 1 mod 8, are printed in **bold**: Lemma 7 requires that, if $m+n$ is odd, then $\Delta \equiv 1 \bmod 8$, so that the other factors of M, congruent to 5 mod 8, must be taken in pairs. Any factor of M may serve, via Lemma 1, to restrict values of δ.

Column $\pm\delta$ contains ω entries, where ω is the number of distinct prime factors of mn. There are $2^\omega - 1$ possible values, other than ± 1, for $\pm\delta$. The entries, in order, are:

1. primes chosen from column M, which successively eliminate, via Lemma 1, $2^{\omega-1}, 2^{\omega-2}, \ldots$ of the $2^\omega - 1$ possibilities for δ;
2. primes from column K, which may continue this elimination process via Lemma 2;
3. solution numbers (in **bold**, with a period) from Table 1, for as many solutions as are known, whose δ-values are linearly independent;
4. the letter A, repeated if necessary, augmenting the list 3. of solutions by referring to auxiliary Tables 7A or 8A, where further independent values of δ are given, with the relevant solutions (a,b,c) of (21), beyond the range of Table 1;
5. sufficient query marks (?) to make the total number of entries up to ω, indicating (along with similar items in column Δ) the extent of our lack of knowledge of the precise rank.

To calculate solutions corresponding to values of δ not specifically listed in Tables 7A and 8A, use Lemmas 12 and 13:

Lemma 12. *If* (δ, a, b, c) *satisfies equation* (21), *then there is a solution* $(-\delta, a', b', c')$ *with* $a' : b' = 2mn(c \pm abK) : \delta a^2 + 4m^2n^2b^2$.

Lemma 13. *If $(\delta_1, a_1, b_1, c_1)$ and $(\delta_2, a_2, b_2, c_2)$ each satisfy equation (21), and the* $\gcd(\delta_1,\delta_2) = \delta$, *with* $\delta_1 = \delta\delta'_1$, $\delta_2 = \delta\delta'_2$, *then there is a solution* $(\delta'_1\delta'_2, a', b', c')$, *with* $a' : b' = a_1b_1c_2 - a_2b_2c_1 : \delta'_1a_1^2b_2^2 - \delta'_2a_2^2b_1^2$.

Column Δ contains Ω entries, which concern potential values of Δ. These parallel those in column $\pm\delta$, but the situation is more complicated. If there are ω_k entries in column K, and ω_1 and ω_5 distinct prime factors respectively congruent to 1 and 5 mod 8, in column M, then, in Table 7, $\Omega = \omega_k + \omega_1 + \max(\omega_5 - 1, 0) - 1$, while in Table 8, $\Omega = \omega_k + \omega_1 + \omega_5 - 1$. The term $\max(\omega_5 - 1, 0)$ arises from Lemma 7, which requires that, when $m+n$ is odd, the number of factors of Δ which are 5 mod 8 must be even. The final -1 in each formula reflects the fact that possible values of Δ occur in complementary pairs, of which the prototype is 1 and M (more precisely, the squarefree part of M). In fact, if we extend the notation of Lemma 6, a little manipulation leads to the following formula.

Lemma 14. *If* $K = D_1E_1F_1^2$ *and* $M = D_2E_2F_2^2$, *and there is a solution* (A, B, C) *of* (22) *with* $\Delta = D_1D_2$, *then a complementary solution* $(BE_1F_1^2F_2, A, CE_1F_1^2F_2)$ *exists with* $\Delta = D_1E_2$.

Here we assume that D_1E_1 and D_1E_2 are separately squarefree, but make no other assumptions of coprimality. If it is required that $A \perp B$, then, while removing a common factor from A and B, its *square* can and must be removed from C. The two pairs of points $(S, \pm T)$ corresponding to this complementary pair of solutions form a quadrangle on the curve (20) whose diagonal point triangle comprises the points $(K^2, 0)$, $(M, 0)$, and the point at infinity, each of finite order. So they are the same, modulo torsion, and yield the same nu-configuration.

For brevity, Tables 7A and 8A list only one solution from each complementary pair, and list only a minimal set of linearly independent values of Δ. Solutions for other possible values of Δ may be constructed by Lemmas 14 and 15.

Lemma 15. *If* (A_1, B_1, C_1) *and* (A_2, B_2, C_2) *are solutions of* (22) *with* $\Delta = \Delta_1$ *and* Δ_2, *then* (A, B, C) *will be a solution for* $\Delta = \Delta_1\Delta_2/(\gcd(\Delta_1, \Delta_2))^2$, *where* $A : B = A_1B_1C_2 - A_2B_2C_1 : \Delta_1A_1^2B_2 - \Delta_2A_2^2B_1$.

The entries in column Δ are, in order, prime factors of mn which successively eliminate $2^{\Omega-1}, 2^{\Omega-2}, \ldots$ of the $2^{\Omega} - 1$ pairs of possibilities for Δ, other than 1 and M, much as in 1., 2. above, but using Lemma 5 in place of Lemmas 1 and 2, followed by letters A and query marks, as in 4., 5. above.

Column r indicates the rank, either exactly, or in the form $r_1 < r_2$ (meaning $r_1 \le r \le r_2$), where r_1 is the total number of solution numbers from column $\pm\delta$ and letters A from columns $\pm\delta$ and Δ, while r_2 is the same total, augmented by the number of query marks in those two columns.

Remark. As observed at the end of §6, if a value of δ (or Δ) is not eliminated by means of Lemmas 1 to 7, then the corresponding curve is actually everywhere locally solvable. In particular, the order of the corresponding Selmer group (see, for example, the Cassels survey article [3]) will be equal to 2^{r_2}. The well-known Selmer Conjecture implies that the actual rank satisfies $r \equiv r_2 \bmod 2$. Accordingly, for those rank entries in Tables 7 and 8 such as $0 < 1$ and $0 < 3$, it seems very likely that they should read 1 and 1 or 3 respectively, indicating the existence of at least one rational solution corresponding to a "?" that lies beyond the bounds of our calculations to date. In fact, this explains many of the larger entries in Tables 7A and 8A, which resulted from searching for precisely such solutions.

Illustrative examples

(m,n)=(19,22). Prime factors of mn: 2, 11, 19 ($\omega = 3$); $K = -959 = -7 \cdot \mathbf{137}$ ($\omega_k = 1$); $M = 3715265 = 5 \cdot \mathbf{17} \cdot 109 \cdot \mathbf{401}$ ($\omega_1 = 2$, $\omega_5 = 2$). Potential values for $\pm\delta$ (other than 1): 2, 11, 22, 19, 39, 209, 418. Of these, $2^{\omega-1} = 4$, namely 2, 22, 38, 418, are eliminated by Lemma 1 with $p = 5$ (see the table of Legendre symbols at left); $2^{\omega-2} = 2$ of them (11, 209) by $p = 17$; and finally $2^{\omega-3} = 1$ of them, namely 19, is eliminated by $p = 109$, so the $\pm\delta$ column reads: 5 17 109 and there is no positive contribution to the rank. Next, $\Omega = 1+2+(2-1)-1 = 3$, and there are $2^{\Omega} - 1 = 7$ pairs of potential values for Δ, other than 1 and M. The $2^{\Omega-1} = 4$ pairs. 17 and $5 \times 109 \times 401$, 5×109 and 17×401, and 137 times each of these, are all ruled out by Lemma 5 with $P = 11$; and the $2^{\Omega-2} = 2$ pairs 401 and $5 \times 17 \times 109$, 401×137 and $5 \times 17 \times 109 \times 137$ by $P = 19$. There remains the possibility $\Delta = 137$ (and $137 \times M$), so our Δ column entry originally read: 11 19 ? and the rank was listed as $0 < 1$. The remark preceding this example induced us to search for, and find, the solution $(A, B, C) = (79247, 407, 35579222616)$ corresponding to $\Delta = 137$.

	2	11	19
5	−	+	+
17	+	−	+
109	−	−	−
401	+	+	−
137	+	+	+

(m,n)=(21,20). Prime factors of mn: 2, 3, 5, 7 ($\omega = 4$); $K = -799 = -\mathbf{17}.47$ ($\omega_k = 1$); $M = 3460801 = 1733.1997$ ($\omega_1 = 0$, $\omega_5 = 2$). Potential values for $\pm\delta$: 2, 3, **6**, 5, 10, 15, 30, 7, 14, 21, 42, **35**, 70, 105, 210. From the table of Legendre symbols we see that $2^{\omega-1} = 8$ of these are eliminated by Lemma 1 with $p = 1733$ (or with $p = 1977$), and $2^{\omega-2} = 4$ more by Lemma 2 with $p = 17$. This leaves $\delta = 15$, 42 and 70, which are represented by solutions **97.**, **170.** and **25.** Only two of these are independent, so the entries 1733 17 **25. 97.** suffice for our $\pm\delta$ column. $\Omega = 1 + 0 + (2 - 1) - 1 = 1$ and the only possible pair of values for Δ, apart from 1 and M, is 17 and $17M$, and these are ruled out by Lemma 5 with $P = 3$ (or 5 or 7), so the Δ column reads simply: 3, and the rank is 2.

	2	3	5	7
1733	−	−	−	+
1977	−	−	−	+
17	+	−	−	−

(m,n)=(31,4). Prime factors of mn: 2, 31 ($\omega = 2$); $K = 697 = \mathbf{17} \cdot \mathbf{41}$ ($\omega_k = 2$); $M = 731825 = 5^2\mathbf{73} \cdot \mathbf{401}$ ($\omega_1 = 2$, $\omega_5 = 1$; $\Omega = 2 + 2 + \max(1-1, 0) - 1 = 3$).

	2	31
5	−	+
73	+	−
401	+	−
17	+	−
41	+	−

Potential values for δ: 2, 31, 62. The first and third fall to Lemma 1 with $p = 5$, and the second to Lemma 1 with $p = 73$. The $\pm\delta$ entry is 5 73. There are, of course, just two independent rows in the table of Legendre symbols. There are $2^\Omega - 1 = 7$ pairs of potential Δ-values, $2^{\Omega-1} = 4$ of which are eliminated by Lemma 5 with $P = 31$ (the *only* nontrivial odd divisor of mn). The Δ entry reads 31 A A because the solutions (85,1,44880) and (15,1,8680), corresponding to $\Delta = 41$ and $\Delta = 17 \times 73$ can be found in Table 7A. This suffices to determine the rank as 2. Lemma 14 enables the complementary solutions, $\Delta = 41 \times 73 \times 401$ (1,1,528) and $\Delta = 17 \times 401$ (41,3,71176) to be found, and Lemma 15 provides the missing $\Delta = 41 \times 17 \times 73$ (83,1,1551240) and $\Delta = 41 \times 17 \times 401$ (5,83,7756200).

The numbers of entries of given rank in Tables 7 and 8 are

rank	0	0<2	0<4	1	0<1	0<3	1<3	2	1<2	2<4	1<4	3	2<3	Total
Table 7	81	96	4	123	34	9	68	65	10	6	0	12	2	510
Table 8	48	53	0	72	40	2	14	28	4	0	1	0	0	262
Total	129	149	4	195	74	11	82	93	14	6	1	12	2	772

So, if we assume the truth of the Selmer conjecture, but otherwise take a pessimistic view, the numbers and percentages of various ranks are

rank	0	1	2	3	Total
Table 7	181 (35.5%)	234 (45.9%)	81 (15.9%)	14 (2.7%)	510
Table 8	101 (38.5%)	128 (48.9%)	33 (12.6%)	0 (0%)	262
Total	282 (36.53%)	362 (46.89%)	114 (14.77%)	14 (1.81%)	772

It may be of interest to note that the numbers of curves of even rank (396) and of odd rank (376) are roughly equal; further that those of rank 2 or more seem to be at least 0.45 times as numerous as those of rank 0. Compare the concluding remarks of Lecture 1 of Washington [10, esp. pp. 254–255].

Table 7. *Outline of rank calculations, $m + n$ odd*

m	n	K	M	±δ	Δ	r	m	n	K	M	±δ	Δ	r
1	2		113	1.		1	1	16	41	5.17293	5	A	1
2	1		5.13	5		0	2	15	281	89.1049	89 ? ?	3 A	1<3
1	4		5 157	5		0	3	14		5 20333	5 ? ?		0<2
2	3	17	5 173	5 17	5	0	4	13	257	109313	45. ?	?	1<3
3	2		5^4	5 1 *		0*	5	12		89 1289	89 46 47.	3	2
4	1		5.61	5		0	6	11		5 23357	5 46 49		2
1	6		5 557	5 5.		1	7	10		13.8837	13 ? ?		0<2
2	5	41	17.193	17 6.	5 A	2	8	9		5.21773	5 47		1
3	4		5 653	5 9.		1	9	8		13 7621	13 50.		1
4	3	17	2593	17 4	3	1	10	7	89	37.2333	37 89 51	7	1
5	2		1601	? ?		0<2	11	6		5.73.197	5 73 52.	11	1
6	1		5.13.17	5 17	3	0	12	5		57601	53 ? ?		1<3
1	8		5 1453	5		0	13	4		5 9133	5 A		1
2	7	73	5 1693	5 73	7	0	14	3		38833	54. ? ?		1<3
4	5		13 677	13 9.		1	15	2		61.661	61 ? ?		0<2
5	4		17 433	17 12	5	1	16	1		$5^2$2153	5		0
7	2	17	$5^2$137	5 137	7	0	1	18		5.26813	5 A		1
8	1		53 61	53		0	2	17	353	5 28621	5 ?	?	0<2
1	10	17	15761	17 15	5	1	3	16		17.61.149	17 61	3	0
2	9	113	13.1381	13 113	3	0	4	15		13 12757	13 ? ?		0<2
3	8		$5^2$13.61	5 13	3	0	5	14		37.4733	37 ? ?		0<2
4	7	89	5 4093	5 89	7	0	6	13	17	5.61.593	5 61 17	3 13	0
5	6		19441	16. ? ?		1<3	7	12		5.13.2801	5 13 25	3	1
6	5		53 317	53 ? ?		0<2	8	11	233	73.2441	73 A	11 A	2
7	4		17 769	17 17	7	1	9	10		17.37.269	17 37 A	3	1
8	3		5 17.109	5 17	3	0	10	9		155521	24. ? ?		1<3
9	2	41	5 1373	5 41	3	0	11	8	17	5.53.521	5 17	11 A	1
10	1		7841	? ?		0<2	12	7	73	$5^2$4729	5 73 17.	7	1
1	12		109 277	109 A		1	13	6		97.1009	97 56.57	13	2
2	11		5.6733	5 19		1	14	5		61 1301	61 ? ?		0<2
3	10		37201	15 ? ?		1<3	15	4	89	65521	89 ? ?	3	0<2
4	9	137	5 7901	5 137	3	0	16	3		5.11933	5 A		1
5	8	17	39761	17 20	5	1	17	2		5.13.1009	5 13	17	0
6	7	97	37633	97 17 18.	7	2	18	1	41	87553	41 58.	3	1
7	6		5.6653	5 ? ?		0<2	1	20		13 $17^2$53	13 17	5	0
8	5	41	27281	? ?	A	1<3	2	19	433	197 1069	197 433	19	0
9	4		5 4157	5 22		1	4	17	409	5.73 661	5 73	17 A	1
10	3		15361	5.6 9.		3	5	16	17	13.73.269	13 73	5 A	1
11	2	73	17.769	17 30	11 A	2	8	13	313	$5^2$37 293	5 61	A A	3
12	1	17	5.37 89	5 37	3 A	1	10	11	241	109.2309	109 241 65	11	1
1	14		5 97.109	5 97	7	0	11	10		233201	42.53 66.		3
2	13		5 37 313	5 36	A	2	13	8		5.109.337	5 67	A	2
4	11	193	$5^2$2729	5 193	11	0	16	5		107441	68. A		2
7	8		5 89.149	5 89	7	0	17	4	137	92753	? ?	A	1<3
8	7	97	5.17.701	5 17	7 A	1	19	2	281	5 137.149	5 281	19 A	1
11	4	17	5 $13^2$37	5 13	11 A	1	20	1		135281	? ?		0<2
13	2	113	5 53 89	5 53	13 A	1	1	22	17	285473	17 A	11	1
14	1		$5^2$17 73	5 17	7	0	2	21	521	5 73 821	5 73 521	3 7	0

Table 7 (continued)

m	n	K	M	±δ	Δ	r	m	n	K	M	±δ	Δ	r
3	20	73	13 24517	13 73 71	5	1	13	14	17	5.136573	5 17 88	7	1
4	19		$5^2$13577	5 A		1	14	13	337	643553	? ? ?	?	0<4
5	18		359041	72 ? ?		1<3	16	11		5.108541	5 A		1
6	17	457	89 4217	89 73 74	3 A	3	17	10		485201	89 ? ?		1<3
7	16		5 37.2089	5 37	7	0	19	8		457.809	90 ?	?	1<3
8	15	401	37 97 109	37 97 401	3 5	0	20	7		318641	A ? ?		1<3
9	14		5 $17^2$269	5 17 76	3	1	22	5		250721	A ? ?		1<3
10	13		17 22273	17 ? ?	5	0<2	23	4		5.48733	5 A		1
11	12	41	$13^2$2137	13 41 53		1	25	2	521	13.23957	13 91	A	2
12	11	2'1	5 89.757	5 89 241	3 11	0	26	1	89	5.73.1093	5 73	13 A	1
13	10		397 773	397 ? ?		0<2	1	28		5.17.8429	5 17	7	0
14	9	137	5.89 613	5 89 137	3 7	0	2	27	17	5 13 11393	5 13	3 A	1
15	8		236641	76 ? ?		1<3	3	26		17 45569	17 ? ?	3	0<2
16	7	17	13 15461	13 17	7	0	4	25	809	37.22013	37 95	A	2
17	6		5 33773	5 ? ?		0<2	5	24	113	856081	113 ? ?	3	0<2
18	5	17	233 617	233 ? ?	3 A	1<3	6	23	769	5 179213	5 769 ?	23	0<1
19	4	193	5 25933	5 193	19	0	7	22		5 186301	5 96 A		2
20	3		131041	77 ? ?		1<3	8	21		959953	97 ? ?		1<3
21	2	353	152833	353 ? ?	3	0<2	9	20	97	73 13417	73 12 43	5 A	3
22	1		5.40093	5 A		1	10	19	641	$13^2$5849	13 101 ?	19	1<2
1	24		5 13 6113	5 13	3 A	1	11	18		5 53 61^2	5 61 A	3	1
2	23	617	5 17 4877	5 17	23 A	1	12	17		5 373 521	5 373 102	3	1
3	22		5 87629	5 ? ?		0<2	13	16		945233	A A		2
4	21	593	5 53 1753	5 53 593	3 7	0	14	15	449	53 17117	53 449 ? ?	3	0<2
6	19		5 37 2777	5 37 A	3	1	15	14	17	13 66037	13 17 7 22	3	2
7	18	17	5 106349	5 17 A	3	1	16	13		5 160093	5 A		1
8	17		5 13 8353	5 13	17	0	17	12		$5^2$29401	5 ? ?		0<2
9	16		5.109229	5 A		1	18	11	193	17.39089	17 193 A	3 11	1
11	14		5.105229	5 ? ?		0<2	19	10	17	61 89 109	61 89 17	5 19	0
12	13	337	5 17 61.97	5 61 17	3 13 A	1	20	9	41	17 30593	17 41 A	3 5	1
13	12	41	5.94349	5 41 78	3	1	21	8	41	5 269.337	5 104 ?	3 A	2<3
14	11	233	5 13 6673	5 13 ?	11	0<1	22	7		5.61 1297	5 61 A	11	1
16	9	113	5.68909	5 113	3	0	23	6		13.27061	13 ? ?		0<2
17	8		5 59629	5 A		1	24	5		373.877	373 78.106		2
18	7		5.50909	5 ? ?		0<2	25	4	409	293 1117	293 ?	?	0<2
19	6	97	5 17.2557	5 17 97	3 19	0	26	3	73	5.71693	5 73 A	13	1
21	4	257	5 13 2753	5 13 ?	3 7	0<1	27	2	617	5 85469	5 617	3	0
22	3		5 89.421	5 89 79	3	1	28	1		13 41621	13 A		1
23	2	433	5 44269	5 433	23	0	1	30	137	149 6269	149 137 A	3	1
24	1	17	5 57389	5 17	3	0	2	29	953	962033	? ?	A	1<3
1	26		5 269 401	5 269	13	0	3	28	41	5 37 5417	5 37 109	3 A	2
2	25		559841	86 A		2	4	27	929	5 17.53 233	5 17	3 A A	2
4	23	17 41	389 1597	389 17	23 ?	0<1	5	26		1100321	A ? ?		1<3
5	22	97	13 37 1361	13 37 97	5 11	0	6	25		233 4937	233 111 112	3	2
7	20		197 3613	197 87 A		2	7	24		1013 1181	1013 77 A		2
8	19	601	5 146173	5 601	19	0	8	23	17	5 13 19009	5 13	23 A	1
10	17		742241	A ? ?		1<3	9	22		5 253133	5 113 ?		1<2
11	16		5 17 37 233	5 17	11 A	1	10	21	761	61.21061	61 761 96 103	3	2

Table 7 (continued)

m	n	K	M	±δ	Δ	r	m	n	K	M	±δ	Δ	r
11	20		97 13313	97 ? ?	5	0<2	12	23	937	5 137 3061	5 137 ?	3 23	0<1
12	19	673	13 17.5813	13 17 ?	3 19	0<1	13	22		5.17.89.277	5 17 89	11 13	0
13	18	89	$5^2$61 829	5 61 115	3 A	2	16	19		$5^2$79481	5 A		1
14	17	569	5 37.61.109	5 37 ?	7 A ?	1<3	17	18		5.13.37.797	5 13 37	3 17	0
15	16	73	89.97.137	89 97 44	3 5 A	2	18	17	577	5.366221	5 ? ?	A	1<3
16	15	449	401.2801	401 119 120.	3 A	3	19	16		5.13.26641	5 13	19	0
17	14		1052993	? ? ?		0<3	22	13		1416977	143 144 ?		2<3
18	13	313	5 194813	5 121. ?	?	1<3	23	12		5 17.53.281	5 17 ?	3 ?	0<2
19	12		5 389.457	5 122 123.	A	3	24	11	73	5.277.809	5 277 73	3 11	0
20	11		421 1901	421.125.126		2	26	9		$5^2$89.401	5 89 A	3	1
21	10		13.17 3221	13 17 18.128	3	2	27	8	233	5.17.9421	5 17	3 A	1
22	9		293 2141	293 ? ?		0<2	29	6	457	5.61.2273	5 61 ?	3 ?	0<2
23	8	97	5.110221	5 97	23	0	31	4	17.41	$5^2$73.401	5 73	31 A A	2
24	7		5.97613	5 ? ?		0<2	32	3		5 13.12689	5 13	3	0
25	6	17^2	13 109.313	13 109 17	3 5	0	33	2	953	5.195581	5 953 ?	3	0<1
26	5	17	423281	17 111 129	5	2	34	1		5.389.617	5 389	17	0
27	4		433633	61.73		2	1	36		1951633	146. ?		1<2
28	3		5 96269	5 132 A		2	2	35	1361	13.148517	13 1361 A	7	1
29	2		5 $17^2$397	5 17	29	0	3	34	193	5.$13^2$2357	5 13 193	3 17	0
30	1		718321	36. ? ?		1<3	4	33		401.5153	401 147.148.	3	2
1	32		1197953	?		0<1	5	32		17.126433	17 A	5	1
2	31		5 37.61.109	5 37	31 ?	0<1	6	31	1297	5.149.3001	5 1297 ?	31 ?	0<2
4	29		5 17.61 257	5 17	29 ?	0<1	7	30	41	2321041	41 150. ? ?	3	1<3
5	28		1393121	A ? ?		1<3	8	29	17.73	5.61.7873	5 61	29 ? ?	0<2
7	26		5.17.17789	5 17 135.	7	1	9	28	17	397.6229	397 17 151.	3	1
8	25		521.3001	521 A		1	10	27		97.26113	97 ? ?	5	0<2
10	23		1636721	A ? ?		1<3	11	26		5.515773	5 ? ?		0<2
13	20		1645601	136. ? ?		1<3	12	25		2608561	98. ? ?		1<3
14	19	697	5 323581	5 ? ?	?	0<3	13	24		5.17 30829	5 17 A	3	1
16	17	577	761.1993	? ?	A A	2<4	14	23	977	73.35801	73 977 A	7 23	1
17	16	73	5.288973	5 73	17	0	15	22		13.198997	13 152 ? ?		1<3
19	14		5 13 19489	5 ? ?	?	0<3	16	21	857	5.13.39089	5 13 857	3 7	0
20	13	17^2	1165121	17 ? ?	5	0<2	17	20	113	109.22709	109 113 153	5	1
23	10		847361	A ? ?		1<3	18	19		5.478253	5 38. A		2
25	8		665921	137.138.		2	19	18		2290033	? ? ?		0<3
26	7		599153	139 ? ?		1<3	20	17	569	2173361	154 ? ?	A	2<4
28	5		257 2113	257 ? ?	5	0<2	21	16		5.408701	5 ? ?		0<2
29	4	593	5.149 761	5 ?	A ?	1<3	22	15	401	17.111953	17 401 41. A	3 5	2
31	2	17	755393	17 ?	31	0<1	23	14		5 351133	5 ? ?		0<2
32	1	137	5.13 14401	5	A A	2	24	13		1604593	56 ? ?		1<3
1	34		$5^2$37.1637	5 ?	?	0<2	25	12	17	53.27437	53 17 156.	3	1
2	33	1217	5 269 1153	5 1217 142	3 A	2	26	11	17	$5^2$52361	5 17 A	11	1
3	32	17	5 320861	5 17	3	0	27	10	89	53 22157	53 89 A	3	1
4	31	1193	5 13 61.421	5 13	31 ? ?	0<2	28	9		5.13 109.149	5 13 109	3 7	0
6	29	1153	$5^2$13.5581	5 13 ?	3 ?	0<2	29	8	313	13.89 829	13 89	29 A	1
8	27	1097	5.389981	5 1097	3	0	30	7		17.52433	17 A ? ?	3	1<3
9	26		5 421.953	5 421 A	3	1	31	6		5.171869	5 ? ?		0<2
11	24		$5^2$83257	5 ? ?		0<2	32	5	97	870641	97 A	5	1

Table 7 (continued)

m	n	K	M	$\pm\delta$	Δ	r	m	n	K	M	$\pm\delta$	Δ	r
33	4	809	5 186653	5 809 A	?	1<2	21	20	17	1733 1997	1733 17 25 97	3	2
34	3	41	1055713	41 ? ?	3	0<2	22	19		3303953	A ? ?		1<3
35	2		1246961	A ? ?		1<3	23	18	89	5 13.17 2833	5 13 171	3 ? ?	1<3
36	1		5 303293	5 157		1	24	17		5 13 45281	5 13 ?	3	0<1
1	38		5 89 5237	5 89	19	0	25	16		2745761	? ?		0<2
2	37	17 89	5 53 8969	5 53	37 ? ?	0<2	26	15		2541841	153 172 ? ?		2<4
4	35	1489	857 2953	857 1489 A	5 7	1	27	14		61 149.257	61 149 A	3	1
5	34		337 7793	337 A A	5	2	28	13	113	5 426541	5 ? ?	A	1<3
7	32		5 565549	5 A		1	29	12		5 61 6353	5 61 ?	3	0<1
8	31		2924513	? ?		0<2	30	11	17	149 11789	149 17 23.143	3	2
10	29	1321	3090641	? ? ?		0<3	31	10	241	89 17929	89 174 175	31 A	3
11	28		5 630733	5 ? ?		0<2	32	9		53 27581	53 A		1
14	25	1129	17 149 1277	17 149 ?	5 ?	0<2	33	8		5 89 3061	5 89 119.	3	1
16	23	1009	5 17 89 421	5 17	23 A ?	1<2	34	7		5 260893	5 ? ?		0<2
17	22	41	5 53 11801	5 53 41	11 17	0	35	6	769	37 35053	37 769 134 135	7	2
19	20		61 48341	61 A ?		1<2	36	5		13 17 6101	13 17 115	3	1
20	19		37 76493	37 122 A		2	37	4		13 112901	13 ?		0<1
22	17		5 97 1049	5 97 A	17	1	38	3	17	5 233 1429	5 17 177	3 A	2
23	16		2381153	? ?		0<2	39	2	1361	5 149 2617	5 1361 ?	3 ?	0<2
25	14		2033441	? ? ?		0<3	40	1		17.37 3709	17 37	5	0
28	11		1217 1249	1217 ? ?	7	0<2	1	42		3439633	178 ? ?		1<3
29	10		373 3677	373 160 A		2	2	41		5 699373	5 A		1
31	8	401	5 17 13469	5 17	31 ?	0<1	3	40		137.26153	137 19 A	3	2
32	7	17	5 197 1097	5 197	7 A	1	4	39		5 738173	5 179 A		2
34	5	113	1088081	113 133 A	5	2	5	38	257	17 157 1429	17 157 257	5 19	0
35	4	929	1176641	929 ? ?	7	0<2	6	37	1777	61 64693	61 ? ?	A	1<3
37	2	1217	5 313741	5 ?	?	0<2	7	36	17	5 13 62801	5 13 17	3 7	0
38	1		61 31013	61 ?		0<1	8	35	1721	4216241	1721 ? ?	7	0<2
1	40	73	2844641	73 162	5	1	9	34	241	$5\ 13^2 33\ 97$	5 13 53	3 17 A	1
2	39		17 170369	17 163 A	3	2	10	33	17 97	61.73141	61 17 97 A	3 5	1
3	38		5 37 16073	5 37 ?	3	0<1	11	32		149 30637	149 A		1
4	37	17 97	5 193 3181	5 193	37 ? ?	0<2	12	31		5 930173	5 154 174.		2
5	36		809 3929	809 8 ?	3	1<2	13	30		257 18353	257 180 181 183	3	3
6	35	1609	17 193793	17 1609 ? ?	3 7	0<2	14	29		$5^2 190409$	5 ? ?		0<2
7	34		53 64381	53 ? ?		0<2	15	28		17 281153	17 184 ? ?	3	1<3
8	33	1553	5 705389	5 1553 ?	3	0<1	16	27		4773553	? ?		0<2
9	32		5 726893	5 A		1	17	26	41	5 948253	5 41 185	13	1
10	31	1481	13 109 2633	13 109 ?	5 ?	0<2	18	25	1201	4682401	157 ? ?	A	2<4
11	30		13 137 2141	13 137 104 A	3	2	19	24		5 919421	5 ? ?		0<2
12	29		3878113	164 165 166		3	20	23	1049	13 61 5657	13 61 1049	5 23	0
13	28	17	5 784717	5 17 A	7	1	21	22		421 10333	421 68 ? ?		1<3
14	27	1289	5 789533	5 1289 A	3	1	22	21	881	5 13 17. 3793	5 13 17 881	3 7 11	0
15	26		109 36229	109 160 ? ?		1<3	23	20	113	37 108413	37 113 ?	5	0<1
16	25		3926561	? ?		0<2	24	19	17 41	5 762557	5 ? ?	3 19	0<2
17	24		3880033	167 168 169		3	25	18		3598801	A ? ?		1<3
18	23	1033	$5^2 152377$	5 1033 A	23	1	26	17		89 37897	89 ? ?	13	0<2
19	22	137	5 17 109 401	5 17 109	11 19 A	1	27	16	17	5 627773	5 17	3	0
20	21	881	313 11497	313 881 ? ?	3 5	0<2	28	15	281	2901361	281 13 ? ?	3	1<3

Table 7 (continued)

m	n	K	M	$\pm\delta$	Δ	r	m	n	K	M	$\pm\delta$	Δ	r
29	14		5 533053	5 ? ?		0<2	11	36	281	5.17.75037	5 17 A	3 A	2
30	13		149 16349	149 186 ? ?		1<3	12	35	17.113	109 149 401	109 149 17 113	3 5 7	0
31	12	73	1249 1777	73 ? ?	31 A	1<3	13	34		5.1325293	5 ? ?		0<2
32	11		5 17 23789	5 17	11	0	14	33		13 193.2677	13 193 30.82	11	2
33	10		13 142357	13 16 ? ?		1<3	15	32		6780481	? ? ?		0<3
34	9		5 37.9257	5 37 A	3	1	16	31	1697	$5.17^2 53.89$	5 17	31 ? ?	0<2
35	8	601	13 124277	13 97 ?	7	1<2	17	30	233	89.76649	89 233 ? ?	3 5	0<2
36	7		73 21481	73 ? ?	7	0<2	18	29		5 13 104561	5 13 ?	3	0<1
37	6		5 109.2897	5 109 A	3	1	19	28		853 7901	853 A A		2
38	5		1657121	? ? ?		0<3	20	27	1409	89 74729	89 ? ?	3 A	1<3
39	4	1193	$5^3 17\ 853$	5 17 147	5 A	2	21	26		5.1306157	5 A ? ?		1<3
40	3	193	157.13093	157 193 A	5	1	22	25	17 73	6380081	17 ? ?	5 A	1<3
41	2	17 89	421.5693	421 17	41 A	1	23	24		5 1240013	5 ? ?		0<2
42	1	73	5 521 1093	5 521 73	3 7	0	24	23		269.22277	269 167. A		2
1	44	17	5 824701	5 ?	11	0<1	25	22	137	61. 94421	61 137 A	5	1
2	43	2017	$5^3 33493$	5 ?	43	0<1	26	21	857	5.37 29753	5 37 ? ?	3 A	1<3
4	41	1993	5 13 89 761	5 13	41 A ?	1<2	27	20		13 402277	13 ? ?		0<2
7	38	41	$5^2 13.17\ 877$	5 13 17	17 19 A	1	28	19	641	5 17.58109	5 17 641	7 19	0
8	37		5 17 89 661	5 17	37 ?	0<1	29	18	17	97 47809	97 17 A	3 29	1
11	34		5 61 17761	5 61 ?	11	0<1	30	17	409	53 81677	53 117 ? ?	A	2<4
13	32	241	5 797 1409	5 797	13 A	1	31	16	41	$5^3 13\ 2473$	5 13	31 A	1
14	31		5 1136077	5 ? ?		0<2	32	15		37 100333	37 108 186		2
16	29	17 89	5 1146781	5 ?	29 ?	0<2	33	14		5 17.40189	5 17 ? ?	3	0<2
17	28		$5^2 13\ 17597$	5 13 A	17	1	34	13		3136433	A ? ?		1<3
19	26		5 73 15349	5 73 192	13	1	35	12	241	61 47221	61 241 13 A	7	2
22	23		$5^2 208553$	5 ? ?		0<2	36	11		5.137 3877	5 137 A	3	1
23	22		5 1006333	5 ? ?		0<2	37	10		2470241	? ? ?		0<3
26	19	673	5 149.5849	5 673 ?	19 ?	0<2	38	9	97	5.613 761	5 613 97	3 19	0
28	17	457	5 766813	5 ? ?	A	1<3	39	8	17	73 30841	73 17 202	3 13	1
29	16		5 712477	5 ?		0<1	40	7		13 89.1933	13 89 A	7	1
31	14		5 604861	5 ? ?		0<2	41	6	1153	5 193 2381	5 193 203	41 ?	1<2
32	13		$5^2 110777$	5 ?		0<1	42	5		197.12413	197 74 120 123		3
34	11	41	5 389 1193	5 389 41	11 17	0	43	4	1489	5.538093	5 1489	43	0
37	8		$5^2 109.701$	5 A	A	2	44	3		13.234181	13 ? ?		0<2
38	7		5 375373	5 ? ?		0<2	45	2		17 206993	17 ? ?	3	0<2
41	4		5.17.97 269	5 17	41 ?	0<1	46	1	17	5.825277	5 17	23	0
43	2		5 13 37 1213	5 13	43 ?	0<1	1	48		5 313 3701	5 ?	?	0<2
44	1		5.688477	5 ?		0<1	2	47	2393	$5^2 234713$	5 2393	47	0
1	46		5 13 61 1237	5 13	23 ?	0<1	3	46		13 149 3089	13 149 A	3	1
2	45		61 73 1117	61 73 196	5	1	4	45		61 100501	61 32 43		2
3	44	313	5.1015853	5 197 ?	?	1<3	5	44		733 8597	733 ? ?		0<2
4	43		5212673	? ?		0<2	6	43	17 137	5.1297853	5 17 137	3 43	0
5	42	17	13 293 1409	13 293 17 198	3 5	1	8	41	2273	17 97 4177	17 208.	41 ? ?	1<3
6	41	2137	$5^2 221401$	5 ? ?	A	1<3	9	40		37.191533	37 ? ?		0<2
7	40		5710721	A ? ?		1<3	10	39		7278001	? ? ? ?		0<4
8	39	2081	5 1177613	5 2081 ?	3	0<1	11	38	17	5.13.89 1289	5 13 89	11 19 A	1
9	38	89	1193.5081	1193 ? ?	3 A	1<3	12	37	2113	5 1523789	5 209 ?	?	1<3
10	37	41	6226481	199 ? ?	A	2<4	13	36		157 49429	157 ? ?		0<2

Table 7 (concluded)

m	n	K	M	±δ	Δ	r
15	34		**7968001**	? ? ? ?		0<4
16	33	1889	5 13 149 829	5 13 149	3 11 A	1
17	32		5 **233** 6917	5 233	17	0
18	31	1753	**8054833**	1753 ? ?	31	0<2
19	30	73	**8017441**	73 A ? ?	5	1<3
20	29	1601	53 197 **761**	53 ? ?	? ?	0<4
22	27	1433	5 1539773	5 1433 ?	3	0<1
23	26	17	**7525313**	17 ? ?	23	0<2
24	25	1249	13 563077	13 ? ?	A	1<3
25	24		**17** 149 2797	17 149 **210**	3	1
26	23	1049	5 **257**.5309	5 1049 ?	23 ?	0<2
27	22	41	$5^3$61 **857**	5 61 **35**	3 A	2
29	20		13 453797	13 ? ?		0<2
30	19	601	61 91141	61 601 **122** A	19	2
31	18		5 **17** 37 **1657**	5 17 ?	3 ?	0<2
32	17	353	5 971917	5 ?	?	0<2
33	16		**4510273**	A ? ?		1<3
34	15	89	**4169521**	89 **48** ? ?	3	1<3
36	13		5 **73 89** 109	5 73 89	3 13	0
37	12	337	5 13 **50273**	5 13 **172**	3 A	2
38	11		1069 2837	1069 A A		2
39	10	641	**2844481**	641 **212** ? ?	3	1<3
40	9	17	**2712001**	17 **20 47**	3	2
41	8		5.528973	5 ?		0<1
43	6	1297	**521 5273**	521 1297 A	3 43	1
44	5		**2938241**	A ? ?		1<3
45	4	17 97	**3237601**	17 97 **8**	5 3	1
46	3		5 37 53 373	5 37 A	3 A	2
47	2	2017	5 841933	5 2017	47	0
48	1		**17 288689**	17 ?	3	0<1

Auxiliary Table 7A *Additional solutions of* (21) *and* (22)

m	n	δ	Δ	(a,b,c) or (A,B,C)	m	n	δ	Δ	(a,b,c) or (A,B,C)
2	5		41	1,1,360	2	27		1105	25,1,384
1	12	3		312,11,610008	4	25		809	27,1,4240
8	5		41	3,1,928	7	22	14		10612,961,25011213724
11	2		1241	6,1,1001	9	20		7081	3,1,7168
12	1		1513	3,1,32	11	18	66		76,13,3361588
2	13		313	11,1,768	13	16	2		806,1,1117246
8	7		1649	9,4,2905			26		88,1,85736
11	4		1105	7,1,1056	16	13	26		75352,851,57537838792
13	2		113	33,2,4277	18	11	2		186,1,163794
1	16		41	7231,173,59001600	20	9	2		168,1,131736
2	15		25009	1,1,384	21	8		337	1,1,1344
13	4	26		13156,1475,5579082964	22	7	14		9460,107,638199892
1	18	6		158684,2833,174334974548	26	3	6		598,5,1896986
8	11	2		992,1,1432352	28	1	14		904,1,3128392
			17009	13,17,191400	1	30	30		3692,29,127277932
9	10	30		476,1,1250516	2	29		953	42953,1339,2133624192
11	8		521	21,1,6160	3	28		7585	1,1,10752
16	3	6		318,5,417666	4	27		3961	73,5,24624
4	17		409	1459,3,42959000				15793	7,1,1248
5	16		1241	11,1,480	5	26	65		28876,1301,40108730756
8	13		185	23,1,312	7	24	6		812,5,4464796
			313	14,1,4953	8	23		1105	49,4,361767
13	8		337	47,1,34944	13	18		27145	1,1,4056
16	5	5		44,1,16084	14	17		173545	8,3,1309
17	4		137	11,1,1104	15	16		7081	4,1,4725
19	2		137	24,1,91	16	15		180049	1,1,336
1	22	2		1012,7,4029124	19	12		457	11,1,1824
4	19	19		537,17,5325670	28	3	21		8,17,1782152
6	17		457	35,1,11904	5	28	2		35300,167,6961375300
18	5		3961	1,1,600	8	25	5		1900,11,26140100
22	1	11		5,1,2294	10	23	115		2083,3,47036206
1	24		5785	151,19,127968	16	17		761	83,1,155040
2	23		10489	1271,171,51696640				577	55,1,23664
6	19	114		5388,55,363701148	23	10	5		16316476,21451,638938413934996
7	18	21		2133,10,24914979	29	4		593	19,1,9048
9	16	6		1504,5,7274656	32	1		65	7621,29,371462640
12	13		337	469,47,23319816				8905	7,1,5208
17	8	34		35008,50259,32057105403584	2	33		1153	16,1,35937
2	25	10		30,37,4405310	9	26	26		17482,2893,365041444662
7	20	10		294,1,351134	16	19	19		160,9,7056032
10	17	5		1701428,4499,8540511125732	18	17		577	163,1,592344
11	16		185	1553,13,26600640	26	9	39		2196,1,30152340
16	11	11		16256,1,876487552	27	8		3961	191,13,559680
20	7	5		385,2,474845	31	4		41	85,1,44880
22	5	2		302,1,178454				1241	15,1,8680
23	4	46		41252,3639,92021083612	2	35	10		7011674,22809,269371681131806
25	2		521	20,1,3531	5	32	2		101321920,118801,22216757072348480
26	1		6497	44,7,83121	13	24	26		67548,1021,125051739852

Table 7A (continued)

m	n	δ	Δ	(a,b,c) or (A,B,C)	m	n	δ	Δ	(a,b,c) or (A,B,C)
14	23	2		12658786,14249,336072709987126	40	3	3		2400,83,359973600
18	19	19		21584,225,8244880672	41	2		1513	77,3,311240
20	17		569	5,1,34272	4	41		1993	619,9,9500192
22	15	2		930,1,1575930	7	38		45305	9,1,1064
26	11	26		1534,23,45946550	13	32		339569	1,1,6240
27	10	10		740,3,2569180	17	28	14		1233928,27665,197361858166472
29	8		313	67,1,43152	28	17		457	6781,17,955340064
30	7	2		322,1,304794	37	8	74		508,5,3710452
32	5	2		6592,1,61688896				545	29,1,11544
33	4	66		2200,1,39369176	6	41		2137	635,11,5269320
35	2	14		1810,1,12418390	7	40	7		616040,5217,7995204186520
4	35	2		1880,1,5777320	9	38		106177	13,7,701784
5	34	85		4320,37,301810160	10	37		41	209,1,492840
		170		364,1,1817084	11	36	66		110,17,22712074
7	32	14		724640,1237,2411753813728				4777	7,1,68376
16	23		1513	583,29,22116432	19	28	14		1456,31,306145168
19	20	19		508,3,2617900			266		518,5,7223846
20	19	38		436,1,1303988	20	27		125401	91,5,2622456
22	17	11		23596,1,1846816012	21	26	21		84,1,313068
29	10	29		1888,13,30014000	22	25		1241	271,1,2473800
32	7		16745	2449,269,326645760	24	23	6		688,1,1809584
34	5	85		555,2,3020435	25	22	22		2900,3,42593300
2	39	2		2028,47,165377316	26	21		25498321	1,2,1755
9	32	6		1155760,5017,11055171441104	29	18	2		2436,1,9252276
11	30	22		345540,2653,1792541261820	30	17		409	8593,25,1422386328
13	28	26		118832,127,76416779152	31	16		41	5,1,89280
14	27	14		11222,143,4147427106	34	13	34		1326,29,123025838
18	23	6		93794,389,74188855522	35	12	3		1344,1,3550848
19	22		137	79247,407,35579222616	36	11	11		6993,43,524681658
22	19	11		14896,37,1097198624	40	7	10		9030,37,514357690
27	14	42		14716,53,1660334012	3	46	69		6996,281,4819198092
28	13		113	159,1,134992	11	38		21913	47,1,285912
31	10		21449	827,95,13308672	16	33		122785	17,1,84480
32	9	6		13389680,12049,462621736887664	19	30	114		1250,13,46175950
38	3		3961	22,1,5415	24	25		1249	193,1,1188000
2	41	41		144832,21899,6221857650512	27	22		12505	1,1,21384
3	40	2		112800,2177,496687855200	30	19	5		16188,29,1111169172
4	39	39		780,31,47780148	33	16	66		2480,17,90127024
6	37		1777	1093,23,2998536	37	12		337	257,1,1117104
9	34		23377	37,1,185736	38	11	22		1958,9,31252738
10	33	66		570,1,2852430			418		2428,3,120887036
11	32	22		53112,313,35634890744	43	6	2		796876,863,1368781629732
18	25		1201	29,1,36360	44	5	10		10406,39,738976766
25	18	3		2256,1,9380832	46	3	46		118,15,4168414
31	12		1249	4891,47,772805448				265	365,3,215096
34	9	34		25380,283,9501371172					
37	6	74		1186,5,13725678					
39	4		20281	5,1,7680					

Table 8 *Outline of rank calculations, m and n odd*

m	n	K	M/4	±δ	Δ	r	m	n	K	M/4	±δ	Δ	r
1	3		5 17	5	3	0	9	13		80677	70. A		2
3	1		37	1.		1	13	9	73	5.61 197	5 73	3 13 A	1
1	5	17	389	17	5	0	15	7	17	44389	17 ? ?	3	0<2
5	1		149	5.		1	17	5		13 2393	13 A	5	1
1	7		13.89	13	7	0	19	3	17	13.2089	13 17	3 19	0
3	5		1429	? ?		0<2	21	1		5.8273	5 A	3	1
5	3		13 73	13 6.	5	1	1	23	41	5.61.277	5	23 A A	2
7	1	17	5.97	5	7 A	1	5	19		105269	A A		2
1	9		5^{2}109	5	3	0	7	17		5.61 373	5 ?	7 ?	0<2
3	7	41	5.13.53	5 13	3 7 A	1	11	13		5.21937	5 A	13	1
7	3		5.353	5 14	3	1	13	11	17	95957	17 A	11	1
9	1		5.257	5	3	0	17	7		5 11329	5 A	7	1
1	11		5^{2}13 17	13	11 A	1	19	5	73	17.2437	17 A	5 A	2
5	7		7109	? ?		0<2	23	1	241	17.3541	17	23 ?	0<1
7	5		61 89	61 18	7	1	1	25	337	37.3137	37	5 A	1
11	1		5.577	?	?	0<2	3	23		5.25457	5 80	3	1
1	13	97	5.2017	5	13 A	1	5	21	313	17.61.137	17 61 313	3 5 7	0
3	11	89	12277	89 23.	3	1	7	19	17	154277	17 81	7	1
5	9	73	13 1033	13 24.	5 A	2	9	17	257	5.109.293	5 109	3 17 A	1
9	5	17	8389	17 32.	3	1	11	15		389.401	389 16. 82.	3	2
11	3		5 977	5 34.	3	1	15	11	113	17^{2}421	17 113 84.	3 11	1
13	1		5717	35.		1	17	9		137.701	137 A	3	1
1	15		17029	38 39		2	19	7		5 53.269	5 53	7 19	0
3	13	17	5.4049	5 17	3 13	0	21	5		54709	7 ? ?		1<3
5	11		22709	? ?		0<2	23	3		5^{2}2221	5 A	3	1
7	9		17.1301	17 A	3	1	25	1	41	84869	85	A	2
9	7		5.3617	5 41	3	1	1	27	17	155797	17	3	0
11	5		12149	? ?		0<2	3	25		89.1901	89 ?	3	0<1
13	3	41	5.1553	5 ?	3 ?	0<2	5	23		109 1721	109 A	23	1
15	1	97	13^{2}61	13 15	5 A	2	9	19		5.42737	5 92.	3	1
1	17		27077	A		1	11	17		13.61.269	13 61	11 17	0
5	13	137	53 673	53 137	5 13	0	13	15		201829	A ? ?		1<3
7	11	113	5.7297	5 ?	7 ?	0<2	15	13		179989	A ? ?		1<3
11	7	41	109.233	41 109	7 11	0	17	11		5.30097	5 A	17	1
13	5		17.997	17 55	5	1	19	9		5^{2}53 89	5 53	3 19	0
17	1		5 3457	5	17	0	23	5	137	13 37.149	13 37	5 23 A	1
1	19		5 8209	59.	A	2	25	3	233	17 4517	17 93.	3 A	2
3	17		5 9377	5 A	3	1	27	1	337	5.23297	5	3 A	1
7	13		5^{2}2237	5 A	7	1	1	29	449	5.40993	449	29 ?	0<1
9	11	17	5.13 821	5 13	3 11 A	1	7	23	401	5 13^{2}313	5 13	7 23 A	1
11	9		5 61.149	5 61	3 11	0	11	19		5 17 3329	17 107	11 A	2
13	7		5 17 401	5 17	7 13	0	13	17	281	5^{2}10973	5 281	13 17	0
17	3	89	5^{2}733	5 89	3 17	0	17	13		5 44257	5 ?	13	0<1
19	1		5.13 421	13	19 A	1	19	11	89	5 36529	89 ?	19 ?	0<2
1	21	241	5.11969	5 241	3 7	0	23	7		5^{2}4397	5 108	7	1
3	19	233	5 13457	5 A	3 A	2	29	1	17	5 31249	17	29 A	1
5	17		75989	A A		2	1	31	73	5 197 269	197	31 ? ?	0<2
7	15	193	81349	193 69 A	5	2	3	29		5.53.1069	5 A	3 A	2

Table 8 (continued)

m	n	K	M/4	±δ	Δ	r	m	n	K	M/4	±δ	Δ	r
5	27		149 2081	149 A	3	1	13	25		197 3697	197 ?	13	0<1
7	25		13 25913	13 ?	5	0<1	15	23		723109	**159** ? ?		1<3
9	23		89 4013	89 ?	3	0<1	17	21	433	5 139457	5 433 ?	3 7	0<1
11	21	17	$5^2$14653	5 17 **30**	3 ?	1<2	21	17	281	13 45289	13 281 A	3 7	1
13	19		5 72337	5 ?	13	0<1	23	15	193	17 30197	17 193 A	3 5	1
15	17	41	342469	41 **114** A	3	2	25	13	97	521 829	97 ?	5 ?	0<2
17	15		13 23833	13 **133** A	5	2	27	11		5 13 61 89	5 13	3 11 A	1
19	13		266837	? ?		0<2	29	9	17	5 57329	5 17	3 29	0
21	11		5 37 1181	5 **134** ?	3 ?	1<3	31	7		245477	? ?		0<2
23	9	17	5 34337	5 ?	3 ?	0<2	33	5		243589	**96** ? ?		1<3
25	7	113	17 73 109	17 113	5 7 A	1	35	3		17 17477	17 ? ?	3	0<2
27	5		97 1237	97 A	5	1	37	1		5 89 953	5	37 ?	0<1
29	3		138517	? ?		0<2	1	39	17	5 157 821	5 17	3 13 A	1
31	1	449	5 17 2417	17	31 ? ?	0<2	3	37	113	5 134993	5 ?	3 ?	0<2
1	33	577	5 13 5189	5 13	3 11 ?	0<1	7	33		5 61 2549	5 61 ?	3 7	0<1
3	31	569	73 4909	73 ?	3 31	0<1	9	31		$5^2$17 1949	5 17	3 31	0
5	29		13 89 337	13 89	5 29	0	11	29	97	5 173617	97 ?	29 ?	0<2
7	27		$5^2$37 457	5 ?	3 ?	0<2	13	27		5 73 2441	5 73	3 13	0
9	25		61 7369	? ?	A	1<3	17	23	73	5 37 53 89	5 53	17 23 ? ?	0<2
11	23	457	5 109 853	5 109	11 23 ?	0<1	19	21		$5^2$33181	5 ? ?	3	0<2
13	21	409	337 1381	409 ? ?	7 A	1<3	21	19		5 153137	5 ? ?	3	0<2
15	19	353	61 7369	353 ? ?	A ?	1<4	23	17		5 136993	5 ?	23	0<1
19	15		397 937	397 **140 141**	5	2	27	13		5 17 5857	5 17	3 13	0
21	13	137	5 63377	5 137 ?	3 7	0<1	29	11	41	$5^2$16349	41 A	11 A	2
23	11		258437	? ?		0<2	31	9		5 13 5189	5 13	3 31	0
25	9		269 761	269 ?	3	0<1	33	7	17	5 59393	5 17 **161**	3 11	1
27	7		5 13 2549	5 13	3 7	0	37	3	569	5 74609	5 ?	3 ?	0<2
29	5		153269	? ?		0<2	39	1		$5^2$109 193	5 109	3 13	0
31	3		5 13 2789	5 13	3 31	0	1	41	881	5 156577	?	A ?	1<3
33	1	73	53 5009	53 73	3 11	0	5	37	857	871349	857 ?	5	0<1
1	35		423509	? ?		0<2	11	31	761	5 457^2	457 ?	11 ?	0<2
5	31	89	484229	89 A	31	1	13	29		5 37 5821	5 37	13 29	0
7	29		523637	? ?		0<2	17	25	593	61 17609	61 593	5 17	0
11	25	17	13 44633	13 A	5 A	2	19	23	521	17 60901	17 521	19 23	0
13	23		5 137 857	5 ?	13 ?	0<2	23	19	353	5 13 13669	5 13	19 23 A	1
17	19		546197	? ?		0<2	25	17	257	788549	257 A	5	1
19	17	41	5 37 2701	5 37	17 19 A	1	29	13	41	$17^2$1973	17 ?	13 29	0<1
23	13	17	5 74353	5 17	13 23	0	31	11		5 94273	A ?	?	1<3
25	11		303029	**142** A		2	37	5		374069	? ?		0<2
29	7	193	5 13 3109	5 193	7 29 ?	0<1	41	1	17	$5^3$13 397	13	41 ? ?	0<2
31	5	313	194069	313 ?	5	0<1	1	43		5 233 809	5	43 ?	0<1
35	1	577	293 1153	293 ?	5 ?	0<2	3	41	137	980197	137 ?	3	0<1
1	37		13 17 2377	13	37 A	1	5	39	41	1041349	41 **71** A	3	2
3	35		552469	A ? ?		1<3	7	37		5 73 3049	5 73	7 37	0
5	33	17 41	594709	17 41 **158**	3 5	1	9	35		193 6133	193 **125** A	5	2
7	31	673	5 128257	5 673	7 31	0	13	31	17	1288037	17 A	31	1
9	29	641	5 97 1409	5 97	3 29 ?	0<1	15	29		17 37 2081	17 37 A	3 5	1
11	27	601	714037	601 ?	11	0<1	17	27	97	5 260753	5 97	3 17	0

Table 8 (concluded)

m	n	K	M/4	±δ	Δ	r	m	n	K	M/4	±δ	Δ	r
19	25		137 9277	137 **188**	5	1	43	5	17 41	13 51593	13 **206**	5 ? ?	1<3
21	23	17	5 $13^2$1433	5 13 ?	3 7 ?	0<2	47	1		5 225217	5	47	0
23	21		61 18457	61 **189.** ?	7	1<2	1	49	1249	5.313921	5	7	0
25	19		13 97 809	13 ?	5 ?	0<2	3	47	17 73	5 13 24917	5 13	3 47 A ?	1<2
27	17		5 179969	5 ?	3	0<1	7	43	1201	5.$17^2$1249	5 17	7 43 A	1
29	15		773029	? ? ?		0<3	9	41		5 382241	5 ?	3	0<1
31	13		5 129937	5 ?	13	0<1	11	39	1129	5 397 1013	5 397 1129	3 11 13	0
35	9	257	269 1721	269 257 **190**	3 5	1	13	37		5 73 5737	5 73	13 37	0
37	7	401	$5^3$3433	5 401	7 37	0	17	33		5 436481	5 **126** ?	3	1<2
39	5		109 4201	109 ? ?	13	0<2	19	31		5 37.61 193	37 61	19 31 ?	0<1
41	3		5 113777	5 ?	3	0<1	21	29	809	5 17 25153	5 17 809	3 7 29	0
43	1	881	783557	?	?	0<2	23	27		5 412481	5 A	3	1
1	45		17 53 1249	17 ?	3 ?	0<2	27	23	521	5 362801	5 A	3 A	2
3	43	1049	5 17 13729	5 17	3 43 A	1	29	21	409	5 13 109 233	5 13 409	3 7 29 A	1
5	41	1033	61 20249	1033 ?	5 ?	0<2	31	19	17	5.294241	17 ?	31 ?	0<2
7	39	1009	61 21577	61 1009 A	7 13	1	33	17		5 293 877	5 293 ?	3 11	0<1
9	37	977	5 13 137 157	5 13	3 37 ? ?	0<2	37	13	17	5 187921	5 17	13 37	0
11	35	937	1470869	937 **193** A	5	2	39	11		5 161921	5 ? ?	3	0<2
13	33		5 305297	5 **194** A	3	2	41	9		5 13 17 661	5 13	3 41 A	1
15	31	17	1558789	17 ? ?	?	0<3	43	7		5 144241	5 ?	7	0<1
17	29	769	1213 1289	1213 769	17 29	0	47	3	137	5 337 593	5 ?	3 A ?	1<3
19	27	17 41	5 13 23669	5 13	3 19 ? ?	0<2	49	1		5 37 7213	5	7 ?	0<1
21	25	617	761 1949	761 617 ?	3 5	0<1							
25	21	433	277 4657	277 433 **88**	5 7	1							
27	19		313 3709	**195** ?	?	1<3							
29	17		5 203857	5 A	17	1							
31	15	97	797 1097	797 97 ?	3 5	0<1							
33	13		$5^3$5897	5 ? ?	3	0<2							
35	11		13 17 53^2	13 53 A	5 7	1							
37	9		17 61 521	17 61	3 37	0							
39	7		5 102497	5 ? ?	3	0<2							
41	5	89	556229	89 A	41	1							
43	3	113	5 138449	5 ?	3 ?	0<2							
45	1		$13^2$5581	13 A	5	1							
1	47		73 18269	73	47	0							
5	43		1455029	? ?		0<2							
7	41		5 373 829	5 ?	7 ?	0<2							
11	37		1725557	? ?		0<2							
13	35		37 48497	37 ? ?	5	0<2							
17	31		5 13.28549	5 13	17 31	0							
19	29	113	5 13 28309	13 113	19 29 ?	0<1							
23	25	89	1710629	89 A	23	1							
25	23	17	53 109 277	53 109	5 23 ?	0<1							
29	19		$5^3$17 617	17 A	29 A	2							
31	17		53 21649	53 ?	31	0<1							
35	13	73	833429	73 ? ?	5	0<2							
37	11		5 141937	5 ?	37	0<1							
41	7		157 3881	157 ?	7	0<1							

Auxiliary Table 8A *Additional solutions of* (21) *and* (22)

m	n	δ	Δ	(a,b,c) or (A,B,C)	m	n	δ	Δ	(a,b,c) or (A,B,C)
7	1		85	8,1,420	5	27	5		4550340,2321,47599568264460
3	7		205	4,1,420	15	17	5		1479,1,5079039
1	11		5	82,1,5568	17	15	17		480,1,1035420
1	13		485	1718,61,58412640	25	7		1241	6,1,1600
5	9		949	2,1,960	27	5	3		1782,23,32974452
7	9	21		135,1,89559	9	25		61	28,3,1529500
15	1		1261	4,1,540	13	21		337	34,1,34944
1	17	17		2911792,284939,273322377019972	15	19		61	306,5,2083616
1	19		5	11362900,1951,288436463656788	5	31	5		899,1,2162839
3	17	51		278290,14567,1772421670556	11	25	55		134,3,655588
7	13	91		702,11,5627180				221	64,1,36300
9	11		1105	14,1,672	19	17		1517	54,1,80896
19	1		5	144,1,68	25	11	5		126445,769,115837963945
3	19	19		1026,1,4616316	1	37		221	8306,77,191879040
			1165	230,1,1798464	3	35	3		350,23,17859100
5	17	5		4,1,13076	21	17	17		168,47,273281316
		17		415,2,821345	23	15	69		156,11,7168020
7	15	21		595,41,20136655	27	11		793	409,8,2660427
9	13	13		87,1,50883	1	39		157	11866,47,1311902784
13	9		61	584,65,33604236	29	11	319		110,3,422092
17	5	17		185905,5878,398152283585				5	18,1,40832
21	1	21		2747,1655,2111818049	1	41		5	798,1,28864
1	23		41	213,2,81995	23	19		353	233,1,908523
			305	293,4,1193553	25	17	17		584800,4127,4566371688100
5	19	19		610,9,3687260	31	11	341		59,1,89953
		95		1178,7,14376844	5	39	5		1452,1,5510892
11	13	11		2241070,7357,18816133011884	9	35	21		1161,5,13253031
13	11	13		583,9,3307843	13	31	13		21452,207,11761411036
17	7	119		5713666,381713,1112713229940284	15	29	145		417,2,2652639
19	5	19		270,1,328420	3	43		17833	421,26,233877
			1241	34,11,131040	7	39	3		19194,59,2598703716
1	25		12469	22,3,17024	11	35	77		7,1,69251
9	17		140065	698,1843,3526931520	13	33	11		27742,1193,323850516132
17	9	17		471,2,1009743	29	17	29		105604,3899,2810231646436
23	3	69		381,11,2182425	35	11	77		836,23,42365444
25	1		41	6,1,52000	41	5	5		26213555,5401,1548787404136105
5	23	5		1589047,5731,9270354645007	45	1	3		101758,6347,1266015388012
13	15	39		138,7,1381260	23	25	5		10731100,49619,1837597753966100
15	13	39		402,7,2394780	29	19		5	162,1,511328
17	11	11		469634,2677,1065684531692	29	19	19		1057318,4509,10946515201636
23	5		5069	324,43,399740	3	47		1105	73,1,12045
25	3		3961	6,1,1760	7	43		1201	481,3,6328525
27	1		337	14866,311,1632509856	23	27	69		11097,221,10694449521
7	23		26065	95,12,673463	27	23	69		10692,149,5337118836
11	19		5	96,1,289852				2605	90,1,326144
29	1		5	340,1,17748	29	21		109	818,3,2879360
3	29	29		21663,7,2532055221	41	9		1105	52505,808,51376132359
			265	1553,22,7148115	47	3		337	98,1,31584