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# EXCEPTIONAL GRAPHS WITH SMALLEST EIGENVALUE $-2$ AND RELATED PROBLEMS

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ABSTRACT. This paper summarizes the known results on graphs with smallest eigenvalue around $-2$, and completes the theory by proving a number of new results, giving comprehensive tables of the finitely many exceptions, and posing some new problems. Then the theory is applied to characterize a class of distance-regular graphs of large diameter by their intersection array.

## INTRODUCTION

This paper presents a theory of graphs with smallest eigenvalue around $-2$ (in §§1 and 2, with tables in the appendix of the microfiche section) and their application to a characterization problem for distance-regular graphs (§3).

Apart from the classical line graph theorem of Cameron, Goethals, Seidel, and Shult [10]—which is introduced here in a new way by means of root lattices—and consequences of it observed by Bussemaker, Cvetković, Doob, Kumar, Rao, Seidel, Simić, Singhi, and Vijayan [8, 16, 18, 20, 21, 30, 35, 44], we obtain a number of new results, namely

(i) a classification of graphs $\Gamma$ with smallest eigenvalue $-2$ such that $\Gamma$ or its complement are edge-regular (Theorem 1.2),

(ii) a complete list of minimal graphs with smallest eigenvalue $-2$ (Theorem 1.7 and Table 3),

(iii) a complete list of minimal forbidden subgraphs for the class of graphs with smallest eigenvalue $\geq -2$ (Table 4), and

(iv) the computation of the eigenvalue gap at $-2$ (Theorem 2.4).

The importance of the eigenvalue gap is demonstrated by the characterization of a class of distance-regular graphs (folded cubes, folded half-cubes, and folded Johnson graphs of large diameters) by their intersection arrays, in the spirit of earlier work of Terwilliger [41] and Neumaier [34].

The proofs for (i)–(iv) are based on extensive computer calculations which enumerate the finitely many exceptions arising from the exceptional root lattices (or root systems) $E_6$, $E_7$, and $E_8$. We challenge the reader at several places to provide conceptional proofs of some remarkable observations deduced here from lists of graphs generated by computer. We also point out a number of open questions.

*Notation.* If $\Gamma$ is a graph and $S$ a set of vertices of $\Gamma$, we denote by $\Gamma\backslash S$ the

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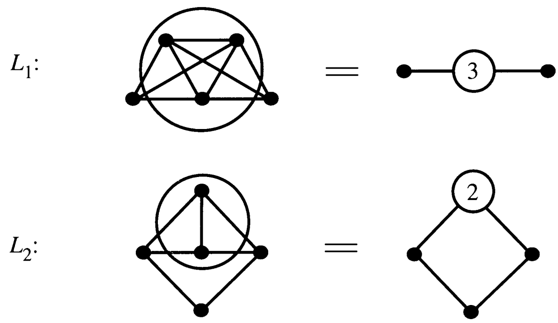



FIGURE 1. The two minimal forbidden line graphs with five vertices.

graph obtained by deleting from $\Gamma$ the vertices in $S$ and all edges containing a vertex of $S$. A *subgraph* of $\Gamma$ always refers to an induced subgraph, i.e., a graph of the form $\Gamma\backslash S$. For a vertex $\gamma \in \Gamma$, $\Gamma(\gamma)$ denotes the subgraph induced on the set of neighbors of $\gamma$, and $\gamma^{\perp}$ denotes the subgraph induced on the set consisting of $\gamma$ and its neighbors. The relation $\equiv$ defined by $\gamma \equiv \delta$ if and only if $\gamma^{\perp} = \delta^{\perp}$ is an equivalence relation on the set of vertices, and if we identify equivalent vertices, we obtain a *reduced graph* $\overline{\Gamma}$. $\Gamma$ can be recovered from $\overline{\Gamma}$ as a *clique extension*, i.e., by replacing each vertex $\bar{\gamma}$ of $\overline{\Gamma}$ by a suitable clique $C_{\bar{\gamma}}$, and joining the vertices of $C_{\bar{\gamma}}$ with the vertices of $C_{\bar{\delta}}$ when $\bar{\gamma}$ and $\bar{\delta}$ are adjacent. We shall draw a clique extension of $\overline{\Gamma}$ by drawing vertices of $\overline{\Gamma}$ replaced by an $i$-clique as circles with label $i$ if $i > 1$, and as black dots if $i = 1$ (cf. Figure 1). The *eigenvalues* of a graph $\Gamma$ are the eigenvalues of its $(0, 1)$-adjacency matrix; the *spectrum* of $\Gamma$ is the collection of its eigenvalues (together with their multiplicities). For a general discussion of graph spectra, see the book by Cvetković, Doob, and Sachs [17]. We denote the largest eigenvalue of $\Gamma$ by $\lambda_{\max}(\Gamma)$ and the smallest eigenvalue of $\Gamma$ by $\lambda_{\min}(\Gamma)$. By interlacing (cf. [17]), we have for a subgraph $\Gamma'$ of $\Gamma$ the relations

$$\lambda_{\min}(\Gamma) \leq \lambda_{\min}(\Gamma'), \qquad \lambda_{\max}(\Gamma') \leq \lambda_{\max}(\Gamma).$$

The minimal valency of a graph $\Gamma$ is denoted by $k_{\min}(\Gamma)$. A graph $\Gamma$ is called *regular* if every vertex has the same valency $k$, *edge-regular* (*coedge-regular*) if $\Gamma$ is regular and any two adjacent (nonadjacent) vertices have the same number $\lambda$ ($\mu$) of common neighbors, *amply regular* if $\Gamma$ is edge-regular and any two vertices at distance 2 have the same number of common neighbors, and *strongly regular* if it is edge-regular and coedge-regular.

Since isomorphic graphs have the same spectrum, we do not distinguish between different isomorphic graphs.

## 1. GRAPHS WITH SMALLEST EIGENVALUE $\geq -2$

The well-known fact that all line graphs have smallest eigenvalue $\geq -2$ prompted a great deal of interest in the characterization of certain classes of graphs $\Gamma$ with $\lambda_{\min}(\Gamma) \geq \lambda^*$ for $\lambda^*$ around $-2$. The work done on this problem culminated in a beautiful theory of Cameron, Goethals, Seidel, and Shult [10] who related the question to root systems. Together with computer calculations by Bussemaker, Cvetković, and Seidel [8], this theory implies a complete classification of all regular graphs with smallest eigenvalue $\geq -2$, and, as noted by Doob and Cvetković [21], a classification of all graphs (whether regular or not) with smallest eigenvalue $> -2$. In this section we summarize these results, and give some numerical information on the exceptional graphs.

A *root lattice* is an additive subgroup $\mathbb{L}$ of $\mathbb{R}^n$ generated by a set $X$ of vectors such that $(x, x) = 2$ and $(x, y) \in \mathbb{Z}$ for all $x, y \in X$; here, $(x, y) = \sum x_i y_i$ is the standard inner product in $\mathbb{R}^n$. The vectors in $\mathbb{L}$ of norm $(x, x) = 2$ are called the *roots* of $\mathbb{L}$. A root lattice is called *irreducible* if it is not a direct sum of proper sublattices. Every irreducible root lattice is isomorphic to one of the lattices

$$A_n = \{x \in \mathbb{Z}^{n+1} \mid \sum x_i = 0\} \qquad (n \geq 1),$$
$$D_n = \{x \in \mathbb{Z}^n \mid \sum x_i \text{ even}\} \qquad (n \geq 4),$$
$$E_8 = D_8 \cup (c + D_8), \quad \text{where } c = \tfrac{1}{2}(1, 1, 1, 1, 1, 1, 1, 1),$$
$$E_7 = \{x \in E_8 \mid \sum x_i = 0\},$$
$$E_6 = \{x \in E_7 \mid x_7 + x_8 = 0\}$$

(cf. Witt [45], Cameron et al. [10], Neumaier [33]). In terms of basis vectors $e_1, \ldots, e_n$ of $\mathbb{Z}^n$, the roots of $A_n$ are the $n(n+1)$ vectors

$$e_i - e_j \quad (1 \leq i < j \leq n+1),$$

those of $D_n$ are the $2n(n-1)$ vectors

$$\pm e_i \pm e_j \quad (1 \leq i < j \leq n),$$

and those of $E_8$ are the $240 = 112 + 128$ vectors

$$\pm e_i \pm e_j \quad (1 \leq i < j \leq 8)$$

and, with an even number of $+$ signs,

$$\tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8).$$

From this, one finds that $E_7$ contains $126 = 56 + 70$ roots and $E_6$ contains $72 = 32 + 40$ roots.

If $\Gamma$ is a connected graph with $\lambda_{\min}(\Gamma) > -2$ and adjacency matrix $A$, then $G = A + 2I$ is a symmetric positive semidefinite matrix. Thus, $G$ is the Gram matrix of a set $X$ of vectors of $\mathbb{R}^n$, i.e., there is a bijection $-: \Gamma \to X$ such that

$$(\bar{\gamma}, \bar{\delta}) = \begin{cases} 2 & \text{if } \gamma = \delta, \\ 1 & \text{if } \gamma, \delta \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

Such a mapping is called a spherical $(2, 1, 0)$-representation, and in this paper simply a *representation* of $\Gamma$. The additive subgroup $\mathbb{L}^+(\Gamma)$ generated by $X$ is a root lattice (whose isomorphism type depends on $\Gamma$ but not on $X$), and since $\Gamma$ is connected, $\mathbb{L}^+(\Gamma)$ is irreducible. This implies the basic observation of Cameron et al. [10] that every connected graph $\Gamma$ with $\lambda_{\min}(\Gamma) \geq -2$ has a representation by roots of $A_n$ $(n \geq 1)$, $D_n$ $(n \geq 4)$, or $E_n$ $(n = 6, 7, 8)$. Conversely, if $\Gamma$ is represented by roots of $A_n$, $D_n$, or $E_n$, then the Gram matrix $A+2I$ of the image of $\Gamma$ is positive semidefinite, so that $\lambda_{\min}(\Gamma) \geq -2$.

The *line graph* of a graph $\Delta$ is the graph $L(\Delta)$ whose vertices are the edges of $\Delta$, two edges being adjacent if they intersect. *Generalized line graphs* $L(\Delta, a_1, \dots, a_m)$, introduced by Hoffman [25], are obtained from the line graph $L(\Delta)$ of a graph $\Delta$ with vertex set $\{1, \dots, m\}$ by adding the vertices $(i, \pm l)$ $(i = 1, \dots, m;\ l = 1, \dots, a_i)$ and joining $(i, l)$ with all vertices $i, j$ of $L(\Delta)$ and all $(i, l')$, $l' \neq \pm l$. We get a representation of a generalized line graph by representing the vertices $ij$ of $L(\Delta)$ by $e_i + e_j$ and the adjoined vertices $(i, \pm l)$ by $e_i \pm e_{(i,l)}$, $l = 1, \dots, a_i$, where the $e_i$, $e_{(i,l)}$ form a set of orthonormal vectors. Moreover, if $\Delta$ is bipartite with bipartite parts $A$, $B$, then $L(\Delta)$ has also a representation by roots of $A_{m-1}$ obtained by representing an edge $ij$ with $i \in A$, $j \in B$ by $e_i - e_j$.

By analyzing the possible representations by roots of $A_n$ and $D_n$, Cameron, Goethals, Seidel, and Shult [10] arrived at the following result.

1.1. **Theorem.** *Let $\Gamma$ be a connected graph with smallest eigenvalue $\geq -2$. Then one of the following holds:*

(i) *$\Gamma$ is a generalized line graph.*

(ii) *$\Gamma$ has a representation by roots of $E_8$. The number $v$ of vertices, and the average valency $k$, are restricted by $v \leq \min(36, 2k + 8)$. Moreover, every vertex has valency at most* 28.

An as yet unsolved problem is the characterization of the graphs under (ii). Since every subgraph of such a graph is again represented by roots of $E_8$, it suffices to determine the (finitely many) maximal graphs under (ii) which are not generalized line graphs. It seems that most graphs under (ii) can be obtained by switching.

*Switching* a graph $\Gamma$ with respect to a set $S$ of vertices (or with respect to its complement $S^c$) is the operation of removing all edges of $\Gamma$ between $S$ and $S^c$ and adding the new edges $\gamma_1, \gamma_2$ $(\gamma_1 \in S,\ \gamma_2 \in S^c,\ \gamma_1 \not\sim \gamma_2)$ (cf. Seidel [36]). If $\Gamma^1$ is obtained from $\Gamma$ by switching with respect to $S_1$, and $\Gamma^2$ is obtained from $\Gamma^1$ by switching with respect to $S_2$, then $\Gamma^2$ can be directly obtained from $\Gamma$ by switching with respect to the symmetric difference $(S_1 \cap S_2) \cup (S_1^c \cap S_2^c)$. Therefore, switching defines an equivalence relation on the set of graphs with a given vertex set. If $\Gamma$ is the line graph of a graph $\Delta$ with vertex set $\{1, 2, \dots, 8\}$, then the graph $\Gamma'$ obtained from $\Gamma$ by switching with respect to $S$ can be represented in $E_8$ by the roots $e_i + e_j$ (if $ij$ is an edge $\notin S$) and $c - e_i - e_j$ (if $ij$ is an edge in $S$); here, $c = \frac{1}{2}(e_1 + \cdots + e_8)$. Therefore, $\Gamma'$ has smallest eigenvalue $\geq -2$. The maximal graphs obtainable from this construction are the graphs which are switching-equivalent to the triangular graph $T(8)$, the line graph of the complete graph on eight vertices.

A graph with 36 vertices, maximal valency 28, and smallest eigenvalue $-2$ which is not a generalized line graph can, e.g., be obtained by adding to $K_8 + L(K_8)$ edges joining $i \in K_8$ with $jk \in L(K_8)$ whenever $i \notin \{j, k\}$; a (2, 1, 0)-representation in $E_8$ is given by the vectors $\frac{1}{2}(f_1 + \cdots + f_8) - f_i$ $(i \leq 8)$ and $f_i + f_j$ $(i < j \leq 8)$, where $f_1, \dots, f_8$ are obtained from $e_1, \dots, e_8$ by reversing the sign of one $e_i$. Thus, examples satisfying equality in (ii) of the theorem exist.

If we restrict ourselves to regular graphs, sharper results are possible. Bussemaker, Cvetković, and Seidel [8], supported by a computer, used this theorem to show that, up to isomorphism, there are precisely 187 regular graphs with smallest eigenvalue $-2$ (and none with $\lambda_{\min}(\Gamma) > -2$) which are not generalized line graphs, namely

4 graphs generating $E_6$ (nos. 5, 185–187 in [8]),
24 graphs generating $E_7$ (nos. 19, 69, 164–184 in [8]),
159 graphs generating $E_8$ (the remaining ones).

Reference [8] also contains explicit adjacency matrices and representations by roots of $E_8$. To simplify the application of their results, we describe here the most important ones, and give (in Table 1) a list of relevant numerical invariants of these graphs. As shown in [8], all 187 graphs are subgraphs of the *Gosset graph* $E_7(1)$ with 56 vertices $e_i + e_j$, $c - e_i - e_j$ $(1 \le i < j \le 8)$, two vertices being adjacent if their inner product is 1. (Note that this defines not a representation in our sense, since $(e_i + e_j, c - e_i - e_j) = -1$; indeed, the smallest eigenvalue of $E_7(1)$ is $-9$. However, all subgraphs not containing such an antipodal pair of vertices are switching-equivalent to a line graph and hence have smallest eigenvalue $\ge -2$.) The Gosset graph is the skeleton of the Gosset polytope $3_{21}$ (cf. Coxeter [15]), and is related to the 28 bitangents of a quartic surface (cf. Dickson [32]). A modern description of the structure of $E_7(1)$ is given by Taylor [39] in terms of a regular two-graph with 28 points. We are interested in the graphs $E_n(1)$ $(n = 1, \dots, 6)$, the subgraphs of $E_7(1)$ induced on the set of common neighbors of $e_i + e_8$ $(i = n+1, \dots, 7)$, and some other graphs.

(i) The Schläfli graph $E_6(1)$ (no. 184 in Table 1) has the 27 vertices $e_i + e_7, e_i + e_8$ $(i \le 6)$, $c - e_i - e_j$ $(i < j \le 6)$ and valency 16. The graph $E_6(1)$ is the complement of the point graph of the generalized quadrangle of order $(2, 4)$ with 27 points, and is related to the 27 lines on a cubic surface (see Baker [1]; one easily recognizes a double six in the description given).

(ii) The Clebsch graph $E_5(1)$ (no. 187 in Table 1) has the 16 vertices $e_6 + e_7, e_i + e_8$ $(i \le 5)$, $c - e_i - e_j$ $(1 < j \le 5)$ and valency 10. The complement is a triangle-free graph obtained by identifying antipodal points of the 5-dimensional cube. The graph $E_5(1)$ contains two regular proper subgraphs which are not line graphs (nos. 185, 186 in Table 1); namely a graph with the 12 vertices $e_i + e_8$ $(i = 2, 3, 4)$, $c - e_i - e_j$ $(i < j \le 5, (i, j) \ne (1, 5))$ and valency 7, and a graph with the eight vertices $e_6 + e_7$, $e_i + e_8$ $(i = 2, 3, 4)$, $c - e_i - e_{i+1}$ $(i \le 4)$ and valency 4 (cf. Figure 2).

(iii) The graph $E_4(1)$ is isomorphic to the triangular graph $T(5)$ with ten vertices and valency 6.

(iv) The Petersen graph (no. 5 in Table 1) has ten vertices and valency 3. It is obtained from the triangular graph $T(5)$ with vertices $e_i + e_j$ $(i < j \le 5)$ by switching with respect to $\{e_i + e_j \mid i, j \le 5, j \equiv i + 1 \pmod 5\}$. This graph is strongly regular with parameters $(v, k, \lambda, \mu) = (10, 3, 0, 1)$.

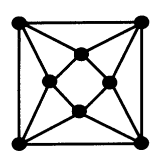


FIGURE 2. A subgraph of the Clebsch graph.

(v) The Shrikhande graph $L_2'(4)$ (no. 69 in Table 1) has 16 vertices and valency 6. It is obtained from the $(4 \times 4)$-grid $L_2(4)$ with vertices $e_i + e_j$ $(i \leq 4 < j)$ by switching with respect to $\{e_i + e_{i+4} \mid i \leq 4\}$. This graph is strongly regular with the same parameters $(\nu, k, \lambda, \mu) = (16, 6, 2, 2)$ as $L_2(4)$ (cf. Shrikhande [37]). It is a quotient of the triangular lattice in $\mathbb{R}^2$.

(vi) The three Chang graphs $T'(8)$, $T''(8)$, $T'''(8)$ (nos. 161–163 in Table 1) have 28 vertices and valency 12. They are obtained from the triangular graph $T(8)$ with vertices $e_i + e_j$ $(i < j \leq 8)$ by switching with respect to one of the sets

$$\{e_i + e_{i+4} \mid 1 \leq i \leq 4\} \quad \text{for } T'(8),$$
$$\{e_i + e_j \mid 1 \leq i \leq 8,\ j \equiv i + 1 \pmod 8\} \quad \text{for } T''(8),$$
$$\{e_1 + e_2, e_2 + e_3, e_3 + e_1, e_4 + e_5, e_5 + e_6, e_6 + e_7, e_7 + e_8, e_8 + e_4\} \quad \text{for } T'''(8).$$

These graphs are all strongly regular with the same parameters $(\nu, k, \lambda, \mu) = (28, 12, 6, 4)$ as $T(8)$ (cf. Chang [11, 12], and Seidel [36] for equivalence under switching).

The information collected by Bussemaker et al. [8] about the 187 regular exceptional graphs can be summarized together with the analysis of regular generalized line graphs in Cameron et al. [10] in the following theorem.

1.2. **Theorem.** *Let $\Gamma$ be a connected regular graph with $v$ points, valency $k$, and smallest eigenvalue $\geq -2$. Then one of the following holds:*

(i) *$\Gamma$ is the line graph of a regular or a bipartite semiregular connected graph $\Delta$.*

(ii) $v = 2(k + 2) \leq 28$, *and $\Gamma$ is a subgraph of $E_7(1)$, switching-equivalent to the line graph of a graph $\Delta$ on eight vertices, where all valencies of $\Delta$ have the same parity* (*graphs nos.* 1–163 *in Table* 1).

(iii) $v = \frac{3}{2}(k + 2) \leq 27$, *and $\Gamma$ is a subgraph of the Schläfli graph* (*graphs nos.* 164–184 *in Table* 1).

(iv) $v = \frac{4}{3}(k + 2) \leq 16$, *and $\Gamma$ is a subgraph of the Clebsch graph* (*graphs nos.* 185–187 *in Table* 1).

(v) $v = k + 2$, *and* $\Gamma \cong K_{m\times 2}$ *for some* $m \geq 3$.

*Moreover,* $\mathbb{L}^+(\Gamma) \cong A_n$ *if and only if* (i) *holds with a bipartite graph $\Delta$ with $n + 1$ vertices, and* $\mathbb{L}^+(\Gamma) \cong D_n$ *if and only if either* (i) *holds with a graph $\Delta$ with $n$ vertices with is not bipartite or* (v) *holds with* $m = n - 1$.

New and computer-free proofs of Theorems 1.1 and 1.2 are contained in Brouwer, Cohen and Neumaier [6].

A glance through Table 1, together with a straightforward analysis of line graphs, leads to the following application of the preceding result, which generalizes the characterization of strongly regular graphs with smallest eigenvalue $-2$ by Seidel [36].

1.3. **Theorem.** *Let $\Gamma$ be a connected regular graph with smallest eigenvalue $-2$.*

(i) *If $\Gamma$ is strongly regular, then $\Gamma$ is a triangular graph $T(n)$, a square grid $n \times n$ (also called a lattice graph $L_2(n)$), a complete multipartite graph $K_{n\times 2}$, or one of the graphs of Petersen, Clebsch, Schläfli, Shrikhande, or Chang.*

(ii) *If* $\Gamma$ *is edge-regular, then* $\Gamma$ *is strongly regular or the line graph of a regular triangle-free graph.*

(iii) *If* $\Gamma$ *is amply regular, then* $\Gamma$ *is strongly regular or the line graph of a regular graph of girth* $\geq 5$.

(iv) *If* $\Gamma$ *is coedge-regular, then* $\Gamma$ *is strongly regular, an* $(m \times n)$*-grid, or one of the two regular subgraphs of the Clebsch graph with eight and* 12 *vertices, respectively.*

The multiplicity of the eigenvalue $-2$ of a graph with $\lambda_{\min}(\Gamma) \geq -2$ can be found quite easily from the following results of Doob [19] (case (i)) and Cvetković, Doob, and Simić [18] (case (ii)).

1.4. **Theorem.** *Let* $\Gamma$ *be a connected graph with* $\lambda_{\min}(\Gamma) \geq -2$.

(i) *If* $\Gamma$ *is the line graph of a graph* $\Delta$ *with* $n$ *vertices and* $e$ *edges, then* $-2$ *is an eigenvalue of* $\Gamma$ *with multiplicity* $e - n + 1$ *if* $\Delta$ *is bipartite, and* $e - n$ *otherwise.*

(ii) *If* $\Gamma = L(\Delta; a_1, \ldots, a_n)$ $(\sum a_i > 0)$ *is a generalized line graph of a graph* $\Delta$ *with* $n$ *vertices and* $e$ *edges, then* $-2$ *is an eigenvalue of multiplicity* $e - n + \sum a_i$.

(iii) *If* $\mathbb{L}^+(\Gamma) \cong E_n$ $(n = 6, 7, 8)$, *then* $-2$ *is an eigenvalue of multiplicity* $|\Gamma| - n$.

The case when the multiplicity of $-2$ is zero corresponds to the graphs $\Gamma$ with $\lambda_{\min}(\Gamma) > -2$. Since $\lambda_{\min}(\Gamma) > -2$ implies that $A + 2I$ is positive definite, so that $\Gamma$ is represented by a linearly independent set of roots, we get the following results of Doob and Cvetković [21].

1.5. **Theorem.** *Let* $\Gamma$ *be a connected graph with* $\lambda_{\min}(\Gamma) > -2$. *Then* $\Gamma$ *is one of the following cases*:

(i) *The line graph of a connected graph without cycles of even length and with at most one cycle of odd length.*

(ii) *The generalized line graph* $L(\Delta; 1, 0, \ldots, 0)$ *obtained from the line graph of a tree* $\Delta$ *by adding two nonadjacent vertices* $\infty^+$, $\infty^-$ *which are adjacent with all edges of* $\Delta$ *containing a fixed vertex* $\infty$ *of* $\Delta$.

(iii) *A graph represented by a set of* $n \in \{6, 7, 8\}$ *linearly independent roots generating* $E_n$.

1.6. **Corollary.** *A connected regular graph with smallest eigenvalue* $> -2$ *is a complete graph or a polygon with an odd number of vertices.*

Calculations of the first author (quoted in [21]) imply that, up to isomorphism, there are precisely 573 graphs of the form (iii), namely

20 graphs with six vertices generating $E_6$,

110 graphs with seven vertices generating $E_7$,

443 graphs with eight vertices generating $E_8$.

Their adjacency matrices and smallest eigenvalues are listed in Table 2. The 20 graphs with six vertices generating $E_6$ are drawn in Figure 3 (see next page). It is a useful fact that every graph with $n$ vertices generating $E_n$ $(n = 7, 8)$ contains a subgraph with $n-1$ vertices generating $E_{n-1}$. It would be interesting to have a simple explanation of this fact.

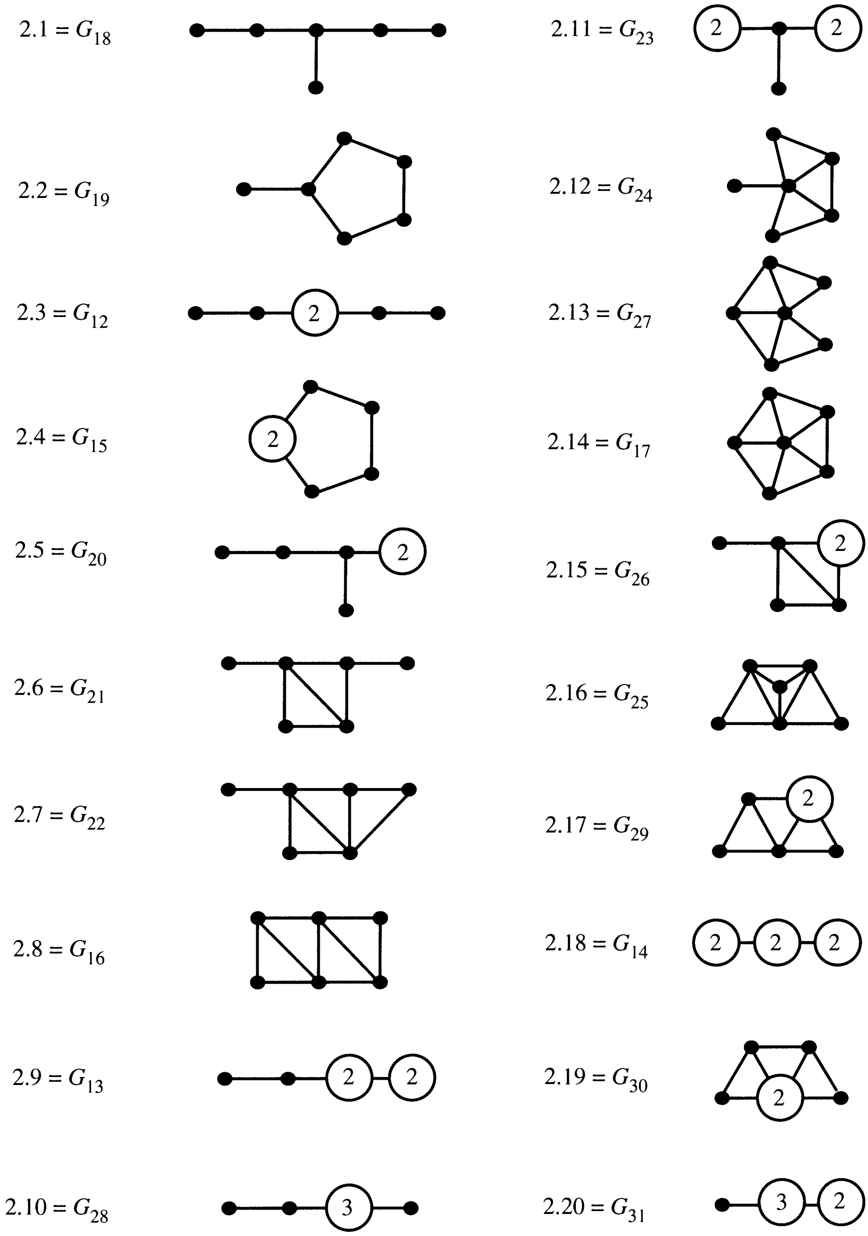



FIGURE 3. The graphs with six vertices generating $E_6$. (2.$i$ is graph number $i$ in Table 2; $G_i$ is the notation of [18].) $G_{12}$–$G_{17}$ are the minimal forbidden line graphs with six vertices.

As another consequence of Theorem 1.4 we determine the minimal graphs with smallest eigenvalue $-2$. The proof is straightforward and left to the reader.

1.7. **Theorem.** *Let* $\Gamma$ *be a connected graph with* $\lambda_{\min}(\Gamma) = -2$ *such that* $\lambda_{\min}(\Gamma') > -2$ *for all proper subgraphs* $\Gamma'$ *of* $\Gamma$. *If* $\Gamma$ *is a generalized line*

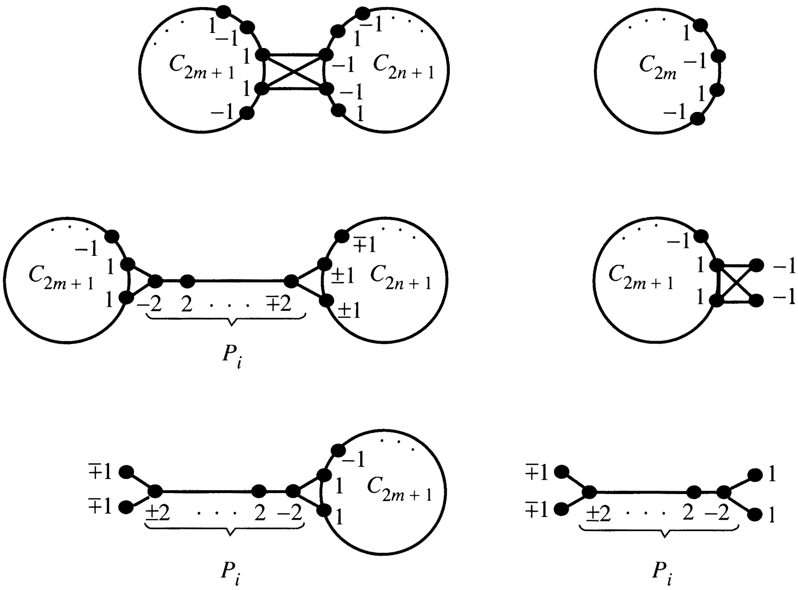



FIGURE 4. The minimal generalized line graphs with smallest eigenvalue $-2$ and associated eigenvectors ($C_i$ is a cycle with $i \geq 3$ vertices, $P_i$ a path with $i \geq 1$ vertices).

*graph, then* $\Gamma$ *is one of the graphs drawn in Figure* 4. *Otherwise,* $\Gamma$ *contains* $n+1$ *vertices* ($n \in \{6, 7, 8\}$) *and has a representation by roots of* $E_n$.

There are precisely 777 minimal graphs $\Gamma$ with smallest eigenvalue $-2$ which are not generalized line graphs, namely

12 graphs with seven vertices generating $E_6$,
79 graphs with eight vertices generating $E_7$,
686 graphs with nine vertices generating $E_8$.

Their adjacency matrices are listed in Table 3 together with an eigenvector belonging to the eigenvalue $-2$, normalized such that its absolutely smallest entries have the value $\pm 1$. It is a useful fact that the normalized eigenvectors (belonging to $\lambda = -2$) of all minimal graphs with smallest eigenvalue $-2$ are integral, and it implies that one can delete a vertex whose normalized eigenvector coefficient is $\pm 1$ without changing the lattice generated. It would be interesting to have a simple explanation of this fact.

A reader who wants to check the information given for the exceptional graphs in Theorem 1.5 and Theorem 1.7 can use the fact that a quadrangle has smallest eigenvalue $-2$; thus it is sufficient to check all graphs $\Gamma$ with $v \leq 9$ vertices and without quadrangles for their smallest eigenvalue, and if $\lambda_{\min}(\Gamma) = -2$ to determine the multiplicity of $-2$ ($\Gamma$ is minimal if and only if $-2$ is a simple eigenvalue and the corresponding eigenvector contains no zero entry).

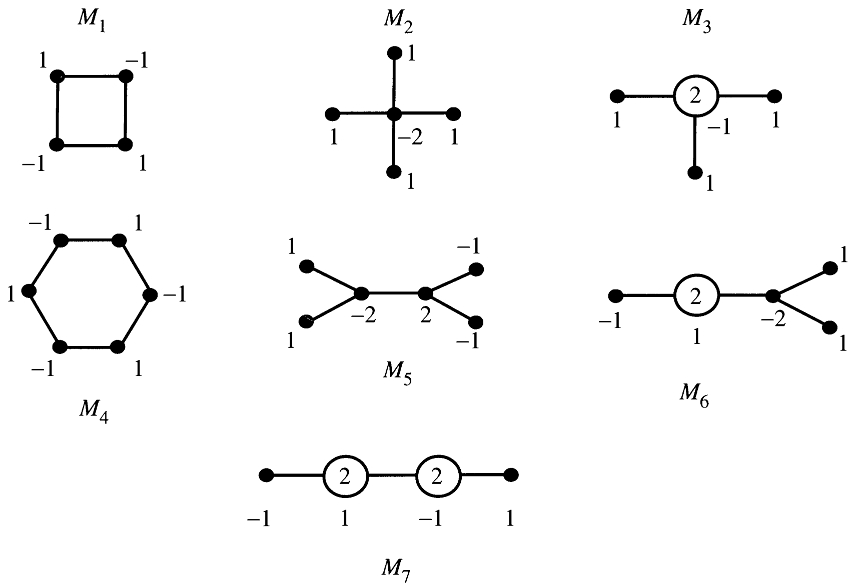



FIGURE 5. The minimal graphs with smallest eigenvalue $-2$ and $v \leq 6$ vertices, and associated eigenvectors.

## 2. MINIMAL FORBIDDEN SUBGRAPHS

Let $\mathscr{G}$ be a class of graphs such that if $\Gamma$ is in $\mathscr{G}$ then every subgraph of $\Gamma$ is also in $\mathscr{G}$. A *minimal forbidden subgraph* for $\mathscr{G}$ is a graph $\Gamma \notin \mathscr{G}$ all of whose proper subgraphs are in $\mathscr{G}$. A *complete list of minimal forbidden subgraphs* is a list $\mathscr{G}^\#$ of pairwise nonisomorphic minimal forbidden subgraphs for $\mathscr{G}$ such that every graph $\notin \mathscr{G}$ contains a subgraph isomorphic to some graph of $\mathscr{G}^\#$. Thus, $\Gamma \in \mathscr{G}$ if and only if $\Gamma$ contains no subgraph isomorphic to some graph of $\mathscr{G}^\#$; in particular, if $\mathscr{G}^\#$ is a known finite list, then we have an obvious finite algorithm for deciding whether a given graph is in $\mathscr{G}$ or not.

A complete list of minimal forbidden subgraphs for the class $\mathscr{L}$ of line graphs has been found by Beineke [3] (who also gives credit to unpublished work by N. Robertson). The list $\mathscr{L}^\#$ consists of nine graphs, the 3-claw $K_{1,3}$ (with four vertices), the two graphs drawn in Figure 1 (with five vertices), and the graphs $G_{12}$–$G_{17}$ in Figure 3 (with six vertices).

A complete list of minimal forbidden subgraphs for the class $\mathscr{L}_0$ of generalized line graphs has been found independently by Rao, Singhi, and Vijayan [35] and Cvetković, Doob, and Simić [18]. The list $\mathscr{L}_0^\#$ consists of 31 graphs, namely the 20 graphs drawn in Figure 3 and the 11 graphs drawn in Figure 6. Their adjacency matrices and smallest eigenvalues are given as the first 20 entries of Table 2 and the first 11 entries of Table 4.

We discuss some properties of the list $\mathscr{L}_0^\#$.

1. The minimal forbidden subgraphs $\Gamma$ for $\mathscr{L}_0$ with $\lambda_{\min}(\Gamma) > -2$ are precisely the graphs represented by a set of linearly independent generators for the lattice $E_6$. Indeed, $\mathbb{L}^+(\Gamma) \ncong A_n$ or $D_n$, since $\Gamma$ is not a generalized line graph. Moreover, $\mathbb{L}^+(\Gamma) \ncong E_7$ or $E_8$, since (as observed above) any set of linearly independent generators for $E_7$ and $E_8$ contains a subset generating $E_6$.

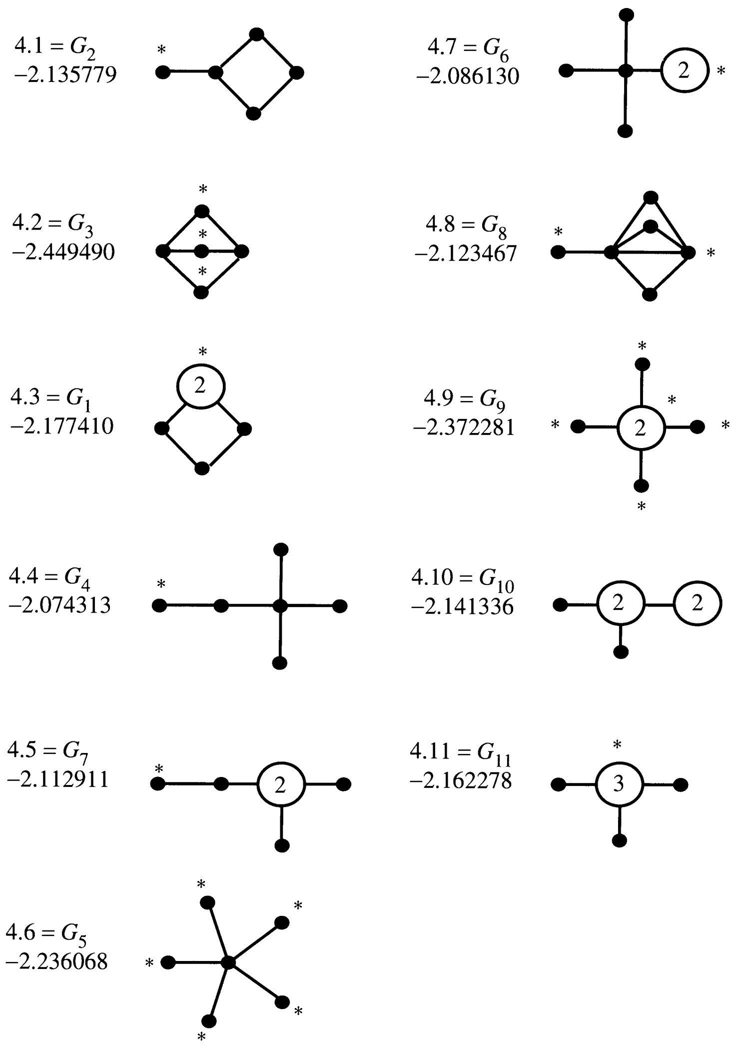



FIGURE 6. The minimal graphs with smallest eigenvalue $< -2$ and up to six vertices, and their smallest eigenvalue. A vertex is starred when its deletion leaves a graph with smallest eigenvalue $-2$. (4.$i$ is graph number $i$ in Table 4; $G_i$ is the notation of [18].)

2. There is no minimal forbidden subgraph for $\mathscr{L}_0$ with $\lambda_{\min}(\Gamma) = -2$. Indeed, as observed above, the minimal graphs $\Gamma$ with smallest eigenvalue $-2$ have a proper subgraph generating the same lattice as $\Gamma$. (Note that the argument given in Cvetković et al. [18, Corollary 4.2] to prove $\lambda_{\min}(\Gamma) \neq -2$ for $\Gamma \in \mathscr{L}_0^{\#}$ is incorrect, since it does not cover the case where some $\Gamma\backslash\{\gamma\}$ is disconnected or generates $A_n$; however, their argument can be replaced by the simple fact that such a $\Gamma$ would have at most nine vertices and thus is ruled out by McKay's computer search mentioned in Proposition 4.5 of [18].)

3. The set of minimal forbidden subgraphs for $\mathscr{L}_0$ with $\lambda_{\min}(\Gamma) < -2$ coincides with the set of minimal graphs with smallest eigenvalue $< -2$ and at

most six vertices. Indeed, if $\Gamma$ is a graph with $\lambda_{\min}(\Gamma) < -2$ and at most six vertices, then $\Gamma\backslash\{\gamma\}$ generates a lattice of dimension $\leq 5$, hence $A_n$ or $D_n$, so that each $\Gamma\backslash\{\gamma\}$ is a generalized line graph and $\Gamma \in \mathscr{L}_0^\#$. However, the remarkable fact that there are no graphs in $\mathscr{L}_0$ with $\lambda_{\min}(\Gamma) < -2$ and more than six vertices has not yet found a simple explanation.

4. Every minimal forbidden subgraph for $\mathscr{L}_0$ with $\lambda_{\min}(\Gamma) < -2$ contains one of the minimal graphs with smallest eigenvalue $-2$ and $\leq 5$ vertices (cf. Figure 5). Again, a simple explanation is missing.

Rao et al. [35] observed that the complete list of minimal forbidden subgraphs for the class $\mathscr{G}_{-2}$ of graphs with smallest eigenvalue $\geq -2$ is finite, since $\mathscr{L}_0^\#$ is finite and there are only finitely many graphs in $\mathscr{G}_{-2}\backslash\mathscr{L}_2$ (by Theorem 1.1). In particular, $\mathscr{G}_{-2}^\#\backslash\mathscr{L}_0^\#$ consists of graphs with at most 37 vertices. Kumar, Rao, and Singhi [30] improved this estimate by showing that the maximal number of vertices of a graph in $\mathscr{G}_{-2}^\#$ is ten. (Note, however, that the graph with ten vertices they give is *not* in $\mathscr{G}_{-2}^\#$.) They also determine the graphs in $\mathscr{G}_{-2}^\#$ with at most seven vertices (see Figures 6 and 7), but incorrectly state that $\mathscr{G}_{-2}^\#$ contains more than 100 graphs with eight vertices. Their complicated arguments were simplified in Vijayakumar [44]. We shall give a new proof of the results in [35] and [30], together with a complete list $\mathscr{G}_{-2}^\#$, based on the following variation of Lemma 4.3 of Cvetković, Doob, and Simić [18].

2.1. **Proposition.** *Let $\Gamma$ be a minimal forbidden subgraph for the class $\mathscr{G}_{-m}$ of graphs with smallest eigenvalue $\geq -m$ $(m \geq 1)$. Then for any two distinct vertices $\gamma, \delta \in \Gamma$, the graph $\Gamma\backslash\{\gamma, \delta\}$ has smallest eigenvalue $> -m$.*

*Proof.* Let $v$ be the number of vertices of $\Gamma$, and denote by $p(x)$ and $p_\alpha(x)$ $(\alpha \in \Gamma)$ the characteristic polynomials of $\Gamma$ and $\Gamma\backslash\{\alpha\}$, respectively. By Clarke [14], the derivative $p'(x)$ can be expressed as $p'(x) = \sum_{\alpha\in\Gamma} p_\alpha(x)$. Since $\Gamma \in \mathscr{G}_{-m}^\#$, all proper subgraphs of $\Gamma$ have smallest eigenvalue $\geq -m$; in particular, $(-1)^{v-1}p_\alpha(x)$ is positive for all $x < -m$. Hence, $(-1)^{v-1}p(x)$ is strictly increasing for $x < -m$, and since $\lambda_{\min}(\Gamma) < -m$, it follows that $\lambda_{\min}(\Gamma)$ is a simple eigenvalue of $\Gamma$ and all other eigenvalues are $> -m$.

Now suppose that $\Gamma' = \Gamma\backslash\{\gamma, \delta\}$ has smallest eigenvalue $\leq -m$ (and hence equal to $-m$). Let $z = (z_\alpha \mid \alpha \in \Gamma')$ be a corresponding eigenvector, and denote by $x_{c,d}$ the vector $x = (x_\alpha \mid \alpha \in \Gamma)$ with $x_\gamma = c$, $\gamma_\delta = d$, $x_\alpha = z_\alpha$ for $\alpha \in \Gamma'$. Writing $A$ for the adjacency matrix of $\Gamma$, we have $x_{0,0}^T(A + mI)x_{0,0} = 0$, and therefore

$$x_{c,0}^T(A + mI)x_{c,0} = c^2m + 2c \sum_{\substack{\alpha\in\Gamma(\gamma)\\ \alpha\neq\delta}} x_\alpha.$$

Since $\Gamma\backslash\{\gamma\}$ has smallest eigenvalue $\geq -m$, this expression must be nonnegative for all $c \in \mathbb{R}$, and this is possible only if $\sum_{\alpha\in\Gamma(\gamma),\ \alpha\neq\delta} x_\alpha = 0$. By the same reasoning we find that $\sum_{\alpha\in\Gamma(\delta),\ \alpha\neq\gamma} x_\alpha = 0$. Now, by construction of $z$, we find the relation $(A + mI)x_{0,0} = 0$, which is impossible, since $-m$ is not an eigenvalue of $\Gamma$. Therefore, the smallest eigenvalue of $\Gamma\backslash\{\gamma, \delta\}$ is $> -m$. □

In the special case $-m = -2$, this result can be combined with the results of §1 and yields the following restrictions on graphs in $\mathscr{G}_{-2}^\#$.

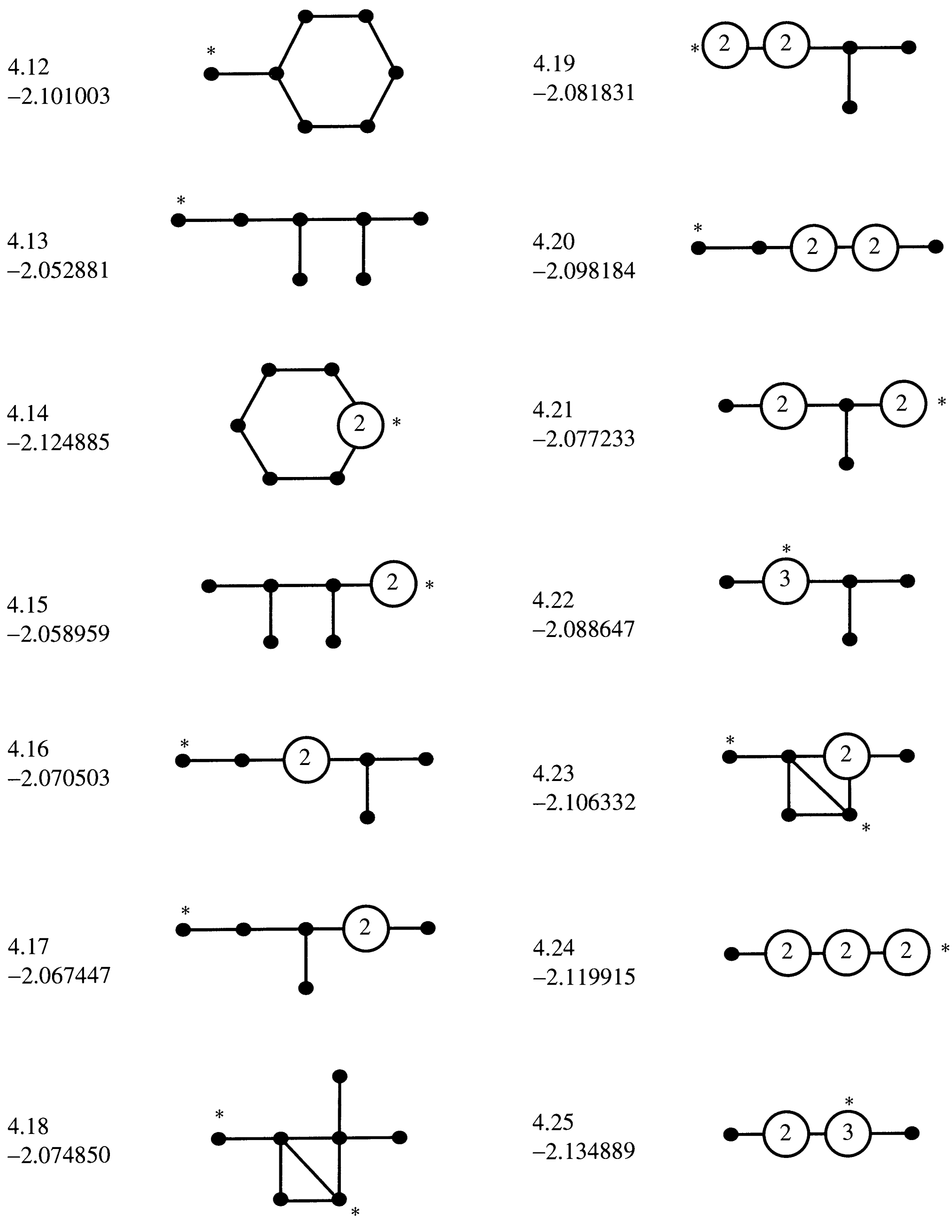



FIGURE 7. The minimal graphs with smallest eigenvalue $< -2$ and seven vertices, and their smallest eigenvalue. A vertex is starred when its deletion leaves a graph with smallest eigenvalue $-2$. (4.$i$ is graph number $i$ in Table 4.)

2.2. **Theorem.** *Let $\Gamma$ be a graph with $v$ vertices, and suppose that $\Gamma$ is a minimal forbidden subgraph for the class $\mathcal{G}_{-2}$ of graphs with smallest eigenvalue $\geq -2$. Then every subgraph of $\Gamma$ with smallest eigenvalue $-2$ has $v-1$ vertices, and one of the following holds:*

(i) $v \leq 10$, *and there exists a vertex* $\gamma \in \Gamma$ *such that* $\Gamma\backslash\{\gamma\}$ *is a minimal graph with smallest eigenvalue* $-2$.

(ii) $v \in \{7, 8, 9\}$, $\lambda_{\min}(\Gamma\backslash\{\gamma\}) > -2$ *for every vertex* $\gamma \in \Gamma$, *and for some* $\gamma \in \Gamma$, *the graph* $\Gamma\backslash\{\gamma\}$ *has a representation by* $n = v - 1$ *linearly independent roots generating* $E_n$.

*Proof.* We showed already that subgraphs with smallest eigenvalue $-2$ have $v - 1$ vertices. To prove the remainder, we distinguish two cases.

*Case* 1. All proper subgraphs of $\Gamma$ with $v - 1$ vertices are generalized line graphs. Then $\Gamma$ is a minimal forbidden subgraph for $\mathscr{L}_0$ with smallest eigenvalue $< -2$, and hence one of the graphs of Figure 6. This implies that $\Gamma$ satisfies (i).

*Case* 2. The graph $\Gamma$ contains a subgraph $\Gamma\backslash\{\delta\}$ which is not a generalized line graph. Then $\mathbb{L}^+(\Gamma\backslash\{\delta\}) \cong E_{v-1}$; in particular, $\Gamma\backslash\{\gamma, \delta\}$, having smallest eigenvalue $> -2$, is represented by $v - 2$ linearly independent roots of $E_n \subseteq E_8$, so that $v \leq 10$. If $\Gamma$ contains a vertex $\gamma \in \Gamma$ such that $\lambda_{\min}(\Gamma\backslash\{\gamma\}) = -2$, then $\Gamma\backslash\{\gamma\}$ is a minimal graph with this property and (i) holds. Otherwise, $\Gamma\backslash\{\delta\}$ is represented by $v - 1$ linearly independent roots generating $E_{v-1}$, and $\Gamma$ satisfies (ii). □

In particular, a comparison with Figure 5 yields:

2.3. **Corollary.** *Let* $\Gamma$ *be a graph in* $\mathscr{G}^{\#}_{-2}$ *with* $v$ *vertices. If* $v \geq 5$, *then* $\Gamma$ *contains no quadrangle; if* $v \geq 6$, *then* $\Gamma$ *contains no subgraph of the form* $M_i$ $(i \leq 3)$; *and if* $v \geq 7$, *then* $\Gamma$ *contains no subgraph of the form* $M_i$ $(i \leq 7)$.

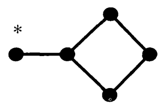
Theorem 2.2 and the corollary now allow a reasonably fast determination of a complete list of forbidden subgraphs for $\mathscr{G}_{-2}$ by computer. We have already seen that a graph $\Gamma \in \mathscr{G}^{\#}_{-2}$ with at most six vertices is a minimal forbidden subgraph and hence one of the 11 graphs in Figure 6. For graphs with more than six vertices, the fact that $\Gamma$ contains no quadrangle drastically restricts the possibilities for extending subgraphs of $\Gamma$ so that a systematic extension process together with checks on the smallest eigenvalues of $\Gamma$ and the $\Gamma\backslash\{\gamma\}$ yields a complete list in a reasonable time. (Several earlier trials to get a complete list turned out to be much too time consuming. The breakthrough was when Aart Blokhuis noticed that no minimal forbidden subgraph with six or seven vertices contained a quadrangle. After further experiments, this finally led to the corollary and then to the above theorem.)

The result of the computer search was that a complete list of minimal forbidden subgraphs for the class $\mathscr{G}_{-2}$ of graphs with smallest eigenvalue $\geq -2$ consists of 1812 graphs; cf. the following statistics (# = number of graphs in $\mathscr{G}^{\#}_{-2}$ with $v$ vertices).

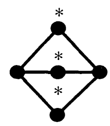
| $v$ | 5 | 6 | 7 | 8 | 9 | 10 | total |
|---|---|---|---|---|---|---|---|
| # | 3 | 8 | 14 | 67 | 315 | 1405 | 1812 |

The adjacency matrices and smallest eigenvalues of the 1812 graphs in $\mathscr{G}^{\#}_{-2}$ are listed in Table 4. The graphs in $\mathscr{G}^{\#}_{-2}$ with up to seven vertices are drawn in Figures 6 and 7.

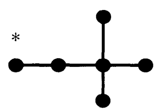
4.1
-2.135779



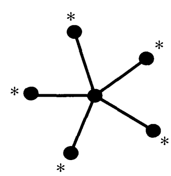
4.2
-2.449490



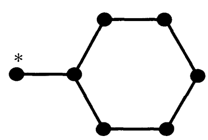
4.4
-2.074313



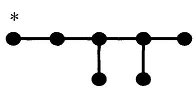
4.6
-2.236068



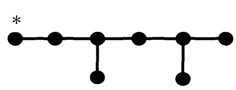
4.12
-2.101003



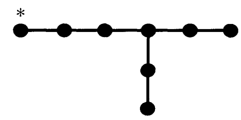
4.13
-2.052881



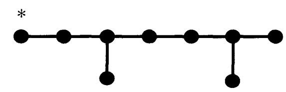
4.26
-2.042079



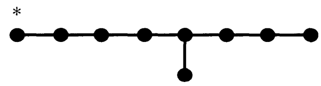
4.27
-2.028523



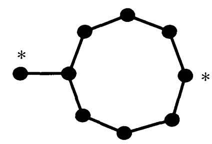
4.93
-2.035648



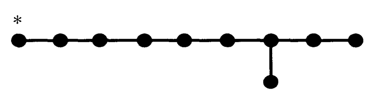
4.94
-2.015316



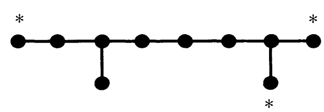
4.95
-2.083968



4.408
-2.006594



4.409
-2.031442



4.410
-2.010173

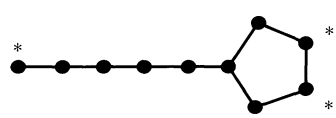


FIGURE 8. Minimal triangle-free graphs with smallest eigenvalue $< -2$ and their smallest eigenvalue. A vertex is starred when its deletion leaves a graph with smallest eigenvalue $-2$. (4.$i$ is graph number $i$ in Table 4.)

An inspection of Table 4 shows that there are only 14 graphs in $\mathcal{G}^{\#}_{-2}$ without triangles; they are drawn in Figure 8. The completeness of the list of triangle-free graphs in $\mathcal{G}^{\#}_{-2}$ can be established easily by hand on the basis of Theorem 2.2 and its corollary.

2.4. **Theorem** (Doob [20]). *Let* $\Gamma$ *be a graph with* $\lambda_{\min}(\Gamma) < -2$. *Then* $\Gamma$ *contains a minimal graph with smallest eigenvalue* $-2$ *and at most nine vertices.*

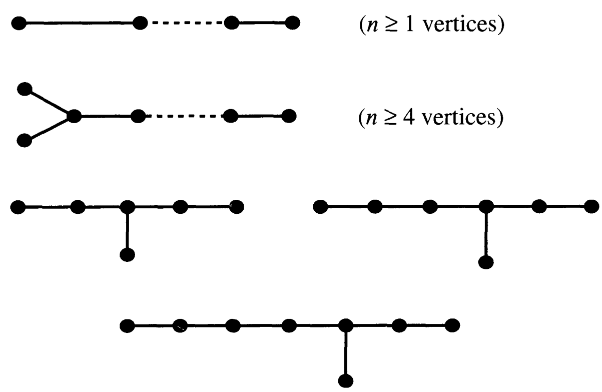



FIGURE 9. The graphs with largest eigenvalue $< 2$.

*Proof.* The graph $\Gamma$ contains a subgraph isomorphic to a graph in $\mathcal{G}^{\#}_{-2}$. Inspection of the list of Table 4 shows that each such graph contains a proper subgraph with smallest eigenvalue $-2$. □

Note that Doob [20] proved the theorem in a slightly different way, relying on computer calculations of Brendan McKay. It would be very interesting to have a computer-free proof of this result. By Theorems 2.2 and 2.4, the graphs in $\mathcal{G}^{\#}_{-2}$ can be characterized as the graphs obtained by adding to a minimal graph with smallest eigenvalue $-2$ and $\leq 9$ vertices a new vertex $\infty$ and edges containing $\infty$ in such a way that the eigenvector coefficients (with respect to the eigenvalue $-2$) of the neighbors of $\infty$ do *not* sum up to zero. This follows from a similar argument as in the second part of the proof of Proposition 2.1.

Let us digress for a moment and consider some related work on the *largest* eigenvalue of a graph. Denote by $\mathcal{G}_m$ $(m \geq 1)$ the class of graphs $\Gamma$ with largest eigenvalue $\lambda_{\max}(\Gamma) \leq m$. The graphs in $\mathcal{G}_2$, listed in Figures 9 and 10, have been determined by Smith [38] (cf. also Lemmens and Seidel [31]); they are precisely the spherical and affine Dynkin diagrams for so-called simply-laced root systems (cf. Hiller [24]).

A complete list of minimal forbidden subgraphs for $\mathcal{G}_2$ is easily established and can be deduced from the list of minimal hyberbolic Dynkin diagrams given in Chein [13] and Koszul [29]. The list $\mathcal{G}_2^{\#}$ consists of 18 graphs, namely the 13 bipartite graphs of Figure 8 (4.410 is not bipartite; $\lambda_{\min}(\Gamma) = -\lambda_{\max}(\Gamma)$ if $\Gamma$ is bipartite) and five further graphs drawn in Figure 11 which are not bipartite. The maximal number of vertices of graphs in $\mathcal{G}_2^{\#}$ is ten. One can read off from Figures 8–11 that every graph not containing a graph with largest eigenvalue 2 is contained in such a graph (cf. Doob [20]).

For $\hat{m} = \sqrt{2+\sqrt{5}} \doteq 2.058171$ $(= \tau^{3/2} = \tau^{1/2} + \tau^{-1/2}$, where $\tau = (1+\sqrt{5})/2)$, $\mathcal{G}_{\hat{m}}$ has been determined by Cvetković, Doob, and Gutman [16] and Brouwer and Neumaier [7]. $\mathcal{G}_{\hat{m}}$ consists of the paths, polygons, the trees $Y_{ij1}$, $Y_{i22}$, $Y_{332}$ (where $Y_{ijk}$ is the $Y$-shaped tree with a unique vertex of valency 3, the deletion of which leaves three disjoint paths with $i$, $j$, and $k$ vertices), and the trees $\Pi_{ijk}$ ($\pi$-shaped, consisting of a path with $i+j+k-1$ vertices and two further vertices of valency 1 adjacent to the $i$th and $(i+j)$th vertex of the path), where $\tau^j \geq (\tau^i-2)(\tau^k-2)$ (i.e., $j \geq i+k-\varepsilon_{ik}$, where $\varepsilon_{23} = \varepsilon_{32} = 4$,

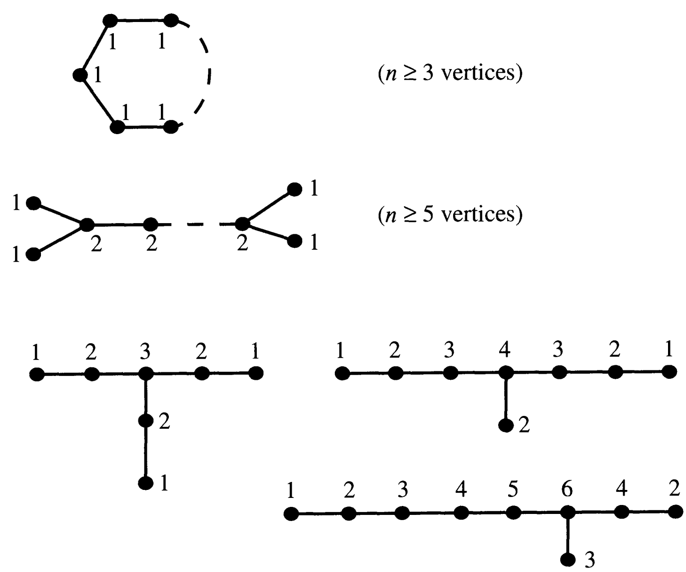



FIGURE 10. The graphs with largest eigenvalue 2, with a corresponding eigenvector.

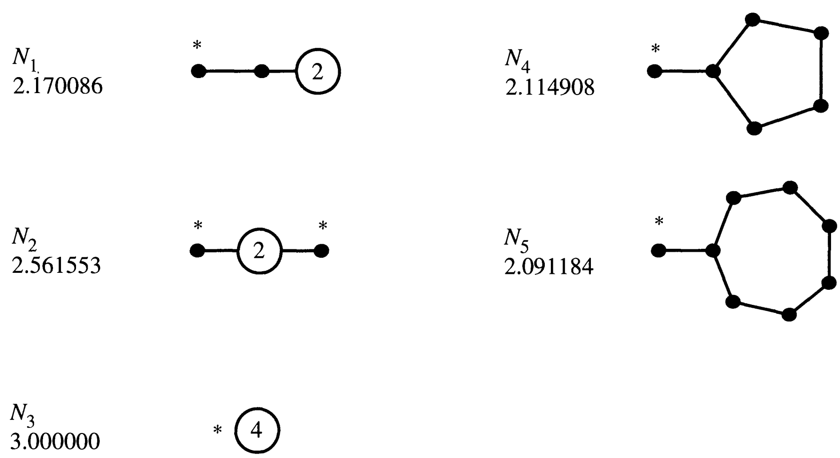



FIGURE 11. The minimal graphs with largest eigenvalue $> 2$ which are not bipartite, and their largest eigenvalue. A vertex is starred when its deletion leaves a graph with largest eigenvalue $-2$.

$\varepsilon_{2i} = \varepsilon_{i2} = 3$ for $i > 3$, $\varepsilon_{33} = \varepsilon_{34} = \varepsilon_{43} = 2$, $\varepsilon_{3i} = \varepsilon_{i3} = \varepsilon_{44} = \varepsilon_{45} = \varepsilon_{54} = 1$, and $\varepsilon_{ik} = 0$ otherwise). It is remarkable that $\hat{m} = \sup\{\lambda_{\max}(\Gamma) \mid \Gamma \in \mathscr{G}_{\hat{m}}\}$ although no graph with $\lambda_{\max}(\Gamma) = \hat{m}$ exists; in particular, this shows that the set of maximal eigenvalues of graphs is not closed. As observed by Hoffman [27], $\mathscr{G}_{\hat{m}}^{\#}$ is infinite, since it contains all subgraphs obtained by adding a vertex of valency 1 to the vertices of a polygon (and $\hat{m}$ is maximal with this property). It would be interesting to know the set of numbers $m$, $-m$ such that $\mathscr{G}_m^{\#}$ or $\mathscr{G}_{-m}^{\#}$ are finite; however, these seem to be very difficult problems.

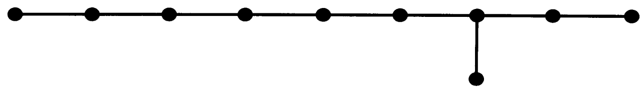


FIGURE 12. The extremal graph $Y_{621}$.

The fact that $\mathcal{G}^{\#}_{+2}$ and $\mathcal{G}^{\#}_{-2}$ are finite implies the existence of "eigenvalue gaps" at $\pm 2$ in the following sense (cf. [23] for $\lambda_{\max}$).

2.5. **Theorem.** *Let $\rho \doteq 2.006594$ be the largest solution of the equation $(\rho^3 - \rho)^2(\rho^2 - 3)(\rho^2 - 4) = 1$. Then there is no graph $\Gamma$ such that $-\rho < \lambda_{\min}(\Gamma) < -2$ or $2 < \lambda_{\max}(\Gamma) < \rho$. Moreover, every graph with $\lambda_{\min}(\Gamma) = -\rho$ or $\lambda_{\max}(\Gamma) = \rho$ is isomorphic to the graph $Y_{621}$.*

*Proof.* Inspection of Table 5 shows that the only graph in $\mathcal{G}^{\#}_{-2}$ with $\lambda_{\min}(\Gamma) \geq -\rho$ is $Y_{621}$, which has $\lambda_{\min}(\Gamma) = -\rho$. Now every graph with $\lambda_{\min}(\Gamma) < -2$ not in $\mathcal{G}^{\#}_{-2}$ contains a proper subgraph $\Gamma' \in \mathcal{G}^{\#}_{2}$; hence, $\lambda_{\min}(\Gamma) \leq \lambda_{\min}(\Gamma') \leq -\rho$. If equality holds, then $\Gamma' = Y_{621}$, and all subgraphs of $\Gamma$ strictly containing $\Gamma'$ have $-\rho$ as a multiple eigenvalue. In particular, the subgraph obtained by adjoining to $\Gamma'$ one further vertex of $\Gamma$ and deleting one of the vertices of $\Gamma'$ again has $-\rho$ as smallest eigenvalue, and hence must be isomorphic to $Y_{621}$. But no graph with 11 vertices has this property. Hence, $\lambda_{\min}(\Gamma) = -\rho$ implies $\Gamma \cong Y_{621}$. The statement about $\lambda_{\max}$ follows immediately, since $Y_{621}$ is bipartite and the nonbipartite minimal graphs with largest eigenvalue $> 2$ (Figure 11) have largest eigenvalue $> \rho$. □

For graphs with large minimum valency, the eigenvalue gap at $-2$ is considerably larger. The following highly nontrivial result was proved by Hoffman [26] using Ramsey's theorem.

2.6. **Theorem.** *Let $\hat{\lambda}_k = \sup\{\lambda_{\min}(\Gamma) \mid k_{\min}(\Gamma) \geq k,\ \lambda_{\min}(\Gamma) < -2\}$. Then $\hat{\lambda}_k$ is a monotonic decreasing sequence with limit $-1 - \sqrt{2} \doteq 2.414214$.*

Theorem 2.5 implies the value $\hat{\lambda}_1 = -\rho \doteq -2.006594$. Lower bounds for the values of $\hat{\lambda}_i$ can be obtained from particular graphs with minimal valency $k$. In particular, we get

$$\hat{\lambda}_2 \geq \tilde{\lambda}_2 = \frac{1 - 2\sqrt{13}}{3} \doteq -2.070368,$$

since the clique extension of an $n$-cycle, where a single vertex is replaced by a 2-clique, provides a sequence of graphs $\Gamma_i$ with minimal valency 2 and $\lambda_{\min}(\Gamma_i) \to \tilde{\lambda}_2$ for $i \to \infty$. The graph of Figure 13 gives the lower bound

$$\hat{\lambda}_k \geq \tilde{\lambda}_k \doteq -(1 + \sqrt{2})(1 - k^{-1} + O(k^{-2})) \quad \text{for } k \geq 3,$$

where $\tilde{\lambda}_k$ is the smallest solution of the equation

$$(x+1)^2(x+2)(x+3) + k(2x^3 + 9x^2 + 10x + 1) + k^2(x^2 + 2x - 1) = 0.$$

The reader is challenged to provide more extreme examples or to prove that the examples given are extremal. An explicit upper bound for $\hat{\lambda}_k$ which tends to $-1-\sqrt{2}$ for $k \to \infty$ would also be of considerable interest. In particular, for

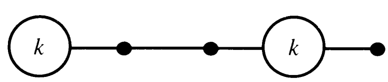



FIGURE 13. A graph with minimal valency $k$.

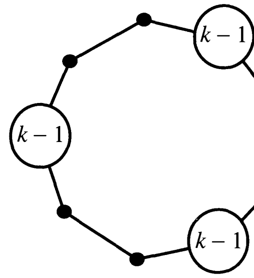



FIGURE 14. A regular graph with valency $k$.

applications to distance-regular graphs (see §3), we would like to know whether

$$\hat{\lambda}_k < -2.4 \quad \text{if } k \geq 64;$$

note that $\tilde{\lambda}_k < -2.4$ if $k > 29.4$.

If we require that $\Gamma$ is regular of large valency, then $-1-\sqrt{2}$ seems no longer to be the right limit. Among the regular graphs of valency $k$, the largest value of $\lambda_{\min}(\Gamma)$ observed in a limited number of test cases occurred for the graph in Figure 14, where $\lambda_{\min}(\Gamma) = -1-\alpha_k$ with the largest zero $\alpha_k$ of the equation $x^3+2x^2+x-3-k(x^2+2x-2)=0$, and $\lim_{k\to\infty}(-1-\alpha_k) = -1-\sqrt{3}$.

The result corresponding to Theorem 2.5 for the largest eigenvalue is trivial for minimal valency $k>2$, since (by Perron-Frobenius theory) the largest eigenvalue of a graph $\Gamma$ with minimal valency $k$ is at least $k$, with equality if and only if the graph is regular of valency $k$. The more relevant value

$$\begin{aligned}&\inf\{\lambda_{\max}(\Gamma) \mid k_{\min}(\Gamma) \geq k,\ \lambda_{\max}(\Gamma) > k\}\\ &\quad = \inf\{\lambda_{\max}(\Gamma) \mid \Gamma \text{ not regular, } k_{\min}(\Gamma) \geq k\}\end{aligned}$$

is not known, not even for $k=2$.

## 3. APPLICATIONS TO DISTANCE-REGULAR GRAPHS

In this section we apply the preceding results to a characterization problem in the theory of distance-regular graphs. A connected graph $\Gamma$ is called *distance-regular* if for any two vertices $\gamma$ and $\delta$ at distance $i = d(\gamma,\delta)$, there are precisely $c_i$ neighbors of $\delta$ at distance $i-1$ from $\gamma$, and $b_i$ neighbors of $\delta$ at distance $i+1$ from $\gamma$ (see Biggs [4], Bannai and Ito [2]). The sequence

$$\iota(\Gamma) = \{b_0, b_1, \dots, b_{d-1}; c_1, c_2, \dots, c_d\}, \tag{1}$$

where $d$ is the diameter of $\Gamma_2$, is called the *intersection array* of $\Gamma$. A fundamental problem in the theory of distance-regular graphs is the characterization of known graphs by their intersection array. Recently, Paul Terwilliger and the second author achieved a breakthrough in this direction by utilizing the classification of graphs with smallest eigenvalue $-2$ to settle this problem for a large class of intersection arrays containing those for the Hamming graphs, the Johnson graphs, and the half-cubes (Terwilliger [41], Neumaier [34]). Here we show that knowledge of the eigenvalue gap in Theorem 2.4 allows the characterization of distance-regular graphs for another class of intersection arrays, at least for large diameter. We begin by summarizing the background needed.

The adjacency matrix $A$ of a distance-regular graph $\Gamma$ of diameter $d$ has precisely $d+1$ distinct eigenvalues $\theta_0 > \theta_1 > \cdots > \theta_d$; the largest eigenvalue $\theta_0$ is the valency $k = b_0$ of $\Gamma$ and has multiplicity 1. To each eigenvalue $\theta$ of $\Gamma$ there corresponds a unique idempotent matrix $E$ in the algebra of polynomials in $A$ satisfying the equation $AE = \theta E$, and the rank $f$ of $E$ agrees with the multiplicity of $\theta$. The $(\gamma, \delta)$-entries $E_{\gamma\delta}$ of $E$ depend only on the distance of $\gamma$ and $\delta$,

$$E_{\gamma\delta} = u_i \quad \text{if } d(\gamma, \delta) = i, \tag{2}$$

and the $u_i$ satisfy the recurrence relations

$$\begin{aligned} &u_0 = 1, \qquad u_1 = \theta/k, \\ &c_i u_{i-1} + a_i u_i + b_i u_{i+1} = \theta u_i \qquad (i = 1, \dots, d-1), \\ &c_d u_{d-1} + a_d u_d = \theta u_d, \end{aligned} \tag{3}$$

where $a_i = k - b_i - c_i$. Conversely, if (3) holds, then $\theta$ is an eigenvalue of $A$ and (2) defines the corresponding idempotents. These facts can be found, e.g., in [2, 4, 6]. Since an idempotent symmetric matrix is positive semidefinite, $E$ can be considered as the Gram matrix of a set of vectors in $\mathbb{R}^f$; hence, there is a mapping $-: \Gamma \to \mathbb{R}^f$ such that the images $\bar{\gamma}$, $\bar{\delta}$ of two vertices $\gamma, \delta \in \Gamma$ have inner product

$$(\bar{\gamma}, \bar{\delta}) = u_i \quad \text{if } d(\gamma, \delta) = i.$$

From this graph representation in $\mathbb{R}^f$ it is possible to deduce the following result concerning the smallest eigenvalues of the local subgraphs $\Gamma(\gamma)$:

3.1. **Proposition** (Terwilliger [43]). *Let $\Gamma$ be a distance-regular graph with intersection array* (1), *and suppose that $\theta$ is an eigenvalue of $\Gamma$ with multiplicity $f$. If $-1 < \theta < k$, then*

$$\lambda_{\min}(\Gamma(\gamma)) \ge -b_1/(\theta+1) \quad \textit{for all } \gamma \in \Gamma. \tag{4}$$

*Moreover, if $f < k$, then* (4) *holds with equality, $\theta$ is the second-largest eigenvalue of $\Gamma$, and either $\theta + 1$ is an integer dividing $b_1$, or $\theta + 1$ and $b_1/(\theta+1)$ are irrational quadratic algebraic integers.*

*Proof.* Inequality (4) is essentially Theorem 1(2) in [43]. The second assertion is part of Theorem 5 in [43], apart from the statement about equality in (4), which derives from the proof of that theorem. □

The following result of Terwilliger is also relevant in the present context.

3.2. **Proposition** (Terwilliger [40]). *Let $\Gamma$ be a distance-regular graph with intersection array* (1).

(i) *If $\Gamma$ contains a quadrangle, then*

$$c_i - b_i \ge c_{i-1} - b_{i-1} + a_1 + 2 \qquad (i = 1, \dots, d). \tag{5}$$

(ii) *If $c_2 - b_2 = c_1 - b_1 + a_1 + 1$, then every 2-claw of $\Gamma$ is in at most one quadrangle.*

*Proof.* (i) is in Terwilliger [40], and (ii) is a simple consequence of the simplified proof of (i) in Terwilliger [42]. □

Terwilliger classified in [42] the distance-regular graphs satisfying (5) with equality for all $i$. Here we consider a class of intersection arrays which have equality in (5) for all $i \neq d$, namely the arrays (1) defined by

$$(6) \qquad \begin{aligned} b_i &= \mu\binom{m-i}{2} - (m-i)(m-i-2) \qquad && (i = 0, \ldots, d-1), \\ c_i &= \mu\binom{i}{2} - i(i-2) && (i = 1, \ldots, d-1), \\ c_d &= \gamma\left(\mu\binom{d}{2} - d(d-2)\right), \end{aligned}$$

where

$$(d, \gamma) \in \left\{\left(\frac{m}{2}, 2\right), \left(\frac{m-1}{2}, 1\right)\right\}, \quad d > 1.$$

Note that $\mu = c_2$ and $m \in \{2d, 2d+1\}$ are positive integers, $m \geq 4$. There are three families of distance-regular graphs realizing these arrays:

(i) The *folded $m$-cube* with $v = \frac{1}{2}2^m$ vertices is the graph obtained by identifying antipodal vertices in the $m$-cube, and realizes (6) with $\mu = 2$. It can be described as the graph whose vertices are the partitions of an $m$-set into two sets, where two such partitions are adjacent whenever their common refinement contains two singletons.

(ii) The *folded Johnson graph* with $v = \frac{1}{2}\binom{2m}{m}$ vertices is the graph whose vertices are the partitions of a $2m$-set into two $m$-sets, with adjacency defined as before. Its intersection array realizes (6) with $\mu = 4$.

(iii) The *folded half $2m$-cube* with $v = \frac{1}{2}2^{2m-1}$ vertices is the graph whose vertices are the partitions of a $2m$-set into two sets of even size, where two such partitions are adjacent whenever their common refinement contains two sets of size 2. Its intersection array realizes (6) with $\mu = 6$.

In view of these examples, we call a distance-regular graph with intersection array (6) a *pseudopartition graph*. We shall prove the following characterization theorem.

3.3. **Theorem.** *Let $\Gamma$ be a pseudopartition graph with diameter $d$.*

(i) *If $\mu = 2$, then either $\Gamma$ is a folded cube, or $d = 3$ and $\Gamma$ is the incidence graph of a $(16, 6, 2)$-biplane.*

(ii) *If $d \geq 3$, then $\mu \in \{2, 4, 6\}$.*

(iii) *If $d \geq 154$, then $\Gamma$ is a folded cube, a folded Johnson graph, or a folded half-cube.*

*Proof.* We proceed in several steps.

*Step* 1. $\Gamma$ has an eigenvalue $\theta = m - 4 + (\mu - 2)\binom{m-2}{2}$ with multiplicity

$$(7) \qquad f = \frac{m(m-1)(2 + (\mu-2)(m-1))(4 + (\mu-2)(2m-5))}{(4 + (\mu-2)(m-2))(4 + (\mu-2)(m-3))}.$$

To show this, we note that the intersection array belongs to the family of $Q$-polynomial intersection arrays of type II discussed in Bannai and Ito [2], with parameters (in the notation of [2])

$$r_1 = -\frac{m+1}{2}, \quad r_2 = -\frac{m+2}{2}, \quad r_3 = -m - \frac{2}{\mu - 2},$$
$$h = 2\mu - 4, \quad s = -m - \frac{1}{2} - \frac{2}{\mu-2}, \quad s^* = -m - 1.$$

Therefore, the eigenvalues of $\Gamma$ are given by

$$\theta_i = k - 4i + (\mu - 2)i(2i + 1 - 2m) \qquad (i = 0, \dots, d),$$

and their multiplicity is

$$f_i = \prod_{j=1}^{i} q_j, \tag{8}$$

where

$$q_j = \frac{b^*_{j-1}}{c^*_j} = \frac{(j+s)(2j+1+s)}{j(2j-1+s)} \cdot \frac{(j+r_1)(j+r_2)(j+r_3)}{(j+s-r_1)(j+s-r_2)(j+s-r_3)}.$$

In particular, since $k = b_0 = \mu\binom{m}{2} - m(m-2) = \frac{m}{2}(2 + (\mu - 2)(m-1))$, we get for $i = 1$ by simplification the above values $\theta$ for $\theta_1$ and $f$ for $f_1$.

*Step* 2. If $d \geq 3$, then $\mu \neq 1, 3, 5$.

$c_3 > 0$ excludes $\mu = 1$. For $\mu = 3$, (7) reduces to

$$f = \frac{m(m-1)(2m-1)}{(m+2)},$$

so that $m + 2 \mid 30$. For $d \geq 3$ $(m \geq 6)$ this leaves the cases $m = 8, 13, 28$, and (8) yields a nonintegral $f_3$, $f_4$, $f_3$ in the respective cases. And for $\mu = 5$, (7) becomes

$$f = \frac{m(m-1)(3m-1)(6m-11)}{(3m-2)(3m-5)},$$

which is nonintegral for all $m > 3$, hence for $d \geq 2$.

*Step* 3. If $d \geq 3$, then $\mu \in \{2, 4, 6\}$.

To get this, we apply Proposition 3.1; note that $\theta < k$ and $\theta + 1 = \frac{m-3}{m-1} b_1 > 0$. Since $m \geq 2d \geq 6$, $\theta + 1$ is no divisor of $b_2$, and since $\theta$ is rational, we must have $f \geq k = \frac{m}{2}(2 + (\mu - 2)(m-1))$. This implies

$$2(m-1)(4 + (\mu-2)(2m-5)) \leq (4 + (\mu-2)(m-2))(4 + (\mu-2)(m-3)),$$

which simplifies to $(\mu - 6)(m-3)(2 + (\mu-2)(m-2)) \leq 0$. Therefore, $\mu \leq 6$ and thus $\mu \in \{2, 4, 6\}$ by Step 2.

*Step* 4. If $\mu = 2$, then the conclusion of (i) holds.

For $m \geq 7$ this follows from Egawa [22]. For $m = 6$, $\Gamma$ has $v = 32$ vertices and intersection array $\{6, 5, 4; 1, 2, 6\}$; hence, $\Gamma$ is bipartite of diameter 3 and must be the incidence graph of a $2 - (\frac{v}{2}, k, \mu)$-design, i.e., of a $(16, 6, 2)$-biplane. For $m = 4, 5$, the graph $\Gamma$ is easily seen to be $K_4$ and $K_{4,4}$, respectively, and hence a folded $m$-cube.

*Step* 5. For any two nonadjacent vertices $\alpha, \beta \in \Gamma(\gamma)$, the number $\mu(\alpha, \beta)$ of common neighbors of $\alpha$ and $\beta$ in $\Gamma(\gamma)$ is $\mu - 1$ or $\mu - 2$.

For, if $\mu(\alpha, \beta) \le \mu - 3$, then there are two distinct vertices $\delta, \delta' \in \Gamma_2(\gamma)$ adjacent with $\alpha$ and $\beta$ so that the 2-claw $\alpha\gamma\beta$ is in two distinct quadrangles, contradicting Proposition 3.2(ii). Since $\mu(\alpha, \beta) \le \mu - 1$, this forces $\mu(\alpha, \beta) \in \{\mu - 1, \mu - 2\}$.

*Step* 6. Each neighborhood $\Gamma(\gamma)$ has smallest eigenvalue $\ge -2 - \frac{2}{m-3}$.
This follows from Proposition 3.1 since

$$\frac{b_1}{\theta + 1} = \frac{m-1}{m-3} = 1 + \frac{2}{m-3}.$$

*Step* 7. If $d \ge 154$, then each neighborhood $\Gamma(\gamma)$ is a line graph.
In this case, $m \ge 308$ so that $\lambda_{\min}(\Gamma(\gamma)) \ge -2 - \frac{2}{305} > -2.00656 > -\rho$, and by Theorem 2.5, $\lambda_{\min}(\Gamma(\gamma)) \ge -2$. Now $\Gamma(\gamma)$ is a regular graph with $k = \frac{m}{2}(2 + (\mu - 2)(m - 1)) \ge m^2 > 28$ (since we may assume $\mu \ge 4$ by Steps 3 and 4) vertices and valency $a_1 = k - 1 - b_1 < k - 2$, and by Theorem 1.2, each $\Gamma(\gamma)$ must be a line graph.

*Step* 8. If $\mu = 4$, then each neighborhood $\Gamma(\gamma)$ which is a line graph is in fact an $(m \times m)$-grid.
By Step 5, the hypothesis of Proposition 5 of Neumaier [34] is satisfied with $c = 2$ for $\Gamma(\gamma) = L(\Delta)$, and part (iii) of that proposition, together with the fact that $\Gamma(\gamma)$ contains $k = m^2$ vertices and is regular of valency $a_1 = k - 1 - b_1 = 2(m - 1)$, only leaves the case $\Delta = K_{m,m}$. Therefore, $\Gamma(\gamma) = L(K_{m,m})$ is an $(m \times m)$-grid.

*Step* 9. If $d \ge 154$ and $\mu = 4$, then $\Gamma$ is a folded Johnson graph.
For $\Gamma$ is locally an $(m \times m)$-grid by Steps 7 and 8, and has the same intersection array, hence the same number $\frac{1}{2}\binom{2m}{m}$ of vertices as a folded Johnson graph. By Blokhuis and Brouwer [5], $\Gamma$ is a quotient of a Johnson graph $J(2m, m)$, and distance regularity forces that antipodal vertices (corresponding to complementary $m$-sets) must be identified, so that $\Gamma$ is a folded Johnson graph.

*Step* 10. If $\mu = 6$, then each neighborhood $\Gamma(\gamma)$ which is a line graph is in fact a triangular graph $T(2m)$.
By Step 5, the hypothesis of Proposition 5 of Neumaier [34] is satisfied with $c = 4$ for $\Gamma(\gamma) \cong L(\Delta)$, and part (i) of that proposition, together with the fact that $\Gamma(\gamma)$ contains $k = m(2m - 1)$ vertices, implies $\Delta = K_{2m}$ and $\Gamma(\gamma) \cong T(2m)$.

*Step* 11. If $d \ge 154$ and $\mu = 6$, then $\Gamma$ is a folded half-cube.
For $\Gamma$ is locally $T(2m)$ by Steps 7 and 10, and has the same intersection array, hence the same number $\frac{1}{2}2^{2m-1}$ of vertices as the folded half $m$-cube. Since $d \ge 3$ and $m \ge 6$, the vertices and $m$-cliques of $\Gamma$ form a semibiplane, i.e., distinct vertices are in precisely zero or two blocks ($m$-cliques), and distinct blocks intersect in zero or two vertices. The incidence graph $\Gamma^*$ of this semibiplane is an amply regular graph with $\lambda = 0$ and $\mu = 2$. Application of Egawa [22] shows that $\Gamma^*$ is a folded $2m$-cube, so that $\Gamma$ is a folded half-cube. □

Together with the results of Terwilliger [41] (which inspired the first three steps of the preceding proof), this implies that for large diameters ($d \ge 154$), *all* $Q$-polynomial distance-regular graphs of type II are known.

The diameter bound seems much too pessimistic, and there should be no exceptions for $d \geq 4$. In order to obtain assertion 3.3(iii) for $d \geq 4$ in place of $d \geq 154$, the argument of Step 7 has to be improved; since $m \geq 8$ for $d \geq 4$, we would need a result like

$$\Gamma \text{ regular of valency } k \geq 64 \Rightarrow \lambda_{\min}(\Gamma) < -2.4 \text{ or } \lambda_{\min}(\Gamma) \geq -2.$$

For $d \leq 3$, there are many exceptions: Husain [28] shows that there are precisely three $(16, 6, 2)$-biplanes, and since each of them is self-dual, their incidence graphs give three nonisomorphic distance-regular graphs with intersection array $\{6, 5, 4; 1, 2, 6\}$, one of which is the folded 5-cube. Bussemaker et al. [9] show that there are at least 1853 strongly regular graphs with the same intersection array $\{16, 9; 1, 4\}$ as the folded Johnson graph with $v = \frac{1}{2}\binom{8}{4} = 35$ vertices. Any pair of orthogonal Latin squares of order 8 gives a Latin square graph $LS_4(8)$ with the same intersection array $\{28, 25; 1, 6\}$ as the folded half 8-cube. Any Latin square of order 16 gives a Latin square graph $LS_3(16)$ with the same intersection array $\{45, 28; 1, 6\}$ as the folded half 10-cube.

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