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ON REAL QUADRATIC FIELDS OF CLASS NUMBER TWO

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ABSTRACT. It is the primary purpose of the paper to determine all real quadratic fields $Q(\sqrt{d})$ of class number $h(d) = 2$ when $k \le 24$ (with one possible exception). Here, k is the period length of the continued fraction expansion of either $\omega = \sqrt{d}$, in the case $d \equiv 2$ or $3 \pmod 4$, or of $\omega = (1+\sqrt{d})/2$, in the case $d \equiv 1 \pmod 4$.

1. INTRODUCTION

In [6] the authors found all real quadratic fields $Q(\sqrt{d})$ with $h(d) = 1$ and $k \le 24$, k as above, with one possible exception remaining. The result of [6] allowed a solution of several conjectures in the literature (see [3-6] for details). The techniques used there provide a basis for examining the class number 2 problem herein. We are able to improve upon them, and as a result, to reduce the computational workload for this paper. The new results (Theorem 2.1 and Lemmas 2.1 and 2.2) are of interest in their own right. In fact, Theorem 2.1 is a very useful means of immediately getting an explicit lower bound on the class number $h(d)$ in terms of k. The complete listings of our findings for $k \le 24$ and $h(d) = 2$ are in Tables 2.1 and 2.2 at the end of the paper. Moreover, our results vastly generalize the results of (and improve upon the techniques of) [9].

Throughout, d will be a positive square-free integer. For convenience sake we give the basic continued fraction notation which we will use in the paper. For ω as in the abstract let the continued fraction expansion of ω be denoted by $\omega = \langle a, \overline{a_1, \ldots, a_k}\rangle$. Then $a_0 = a = \lfloor \omega \rfloor$ and $a_i = \lfloor (P_i + \sqrt{d})/Q_i \rfloor$ for $i \ge 1$ (here $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x), where $(P_0, Q_0) = (\sigma - 1, \sigma)$, with $\sigma = 2$ if $d \equiv 1 \pmod 4$ and $\sigma = 1$ otherwise. Also, $P_{i+1} = a_i Q_i - P_i$ and $Q_{i+1} Q_i = d - P_{i+1}^2$ for $i \ge 0$. For more detailed information and connections with other topics, such as reduced ideals, the reader is referred to [2, 10].

2. CLASS NUMBER 2 FOR $k < 25$

In order to find the real quadratic fields $Q(\sqrt{d})$ with $k \le 24$ and $h(d) = 2$, we proceed in a fashion similar to that in Mollin and Williams [6]. However,

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the techniques here are different in that we can reduce the amount of work by first proving the following result. This result is in fact an improvement on the inequality $R < k \log \sqrt{\Delta}$ used in [6] (here, $R = \log \varepsilon$, where ε is the fundamental unit of $Q(\sqrt{d})$).

Theorem 2.1. *If R is the regulator of $Q(\sqrt{d})$, then $R < \lfloor 3(k+1)/4 \rfloor \log \sqrt{\Delta}$.*

Proof. By results noted in Stephens and Williams [7] we have

$$\varepsilon = \prod_{i=1}^{k} \varphi_i, \tag{2.1}$$

where $\varphi_i = (P_i + \sqrt{d})/Q_i$ and $0 < P_i < \sqrt{d}$. We can write (2.1) as

$$\varepsilon = \lambda \prod_{i=1}^{\lfloor k/2 \rfloor} \chi_i, \tag{2.2}$$

where $\chi_i = \varphi_i \varphi_{k-i+1}$ and

$$\lambda = \begin{cases} 1 & \text{when } 2|k, \\ \varphi_{(k+1)/2} & \text{when } 2 \nmid k. \end{cases}$$

By [7, Theorem 2.1] we have $Q_{k-i} = Q_i$ and $P_{k-i} = P_{i+1}$; hence,

$$\chi_i = ((P_i + \sqrt{d})/Q_i)((P_i + \sqrt{d})/Q_{i-1}) = (\sqrt{d} + P_i)/(\sqrt{d} - P_i)$$

from (2.2) of [7]. Furthermore, when k is odd we have $Q_{(k-1)/2} = Q_{(k+1)/2}$, which means that $d = P^2_{(k+1)/2} + Q^2_{(k+1)/2}$ and therefore

$$\lambda = \varphi_{(k+1)/2} < \sqrt{\Delta},$$

as $\sigma | Q_i$ for all $i \geq 0$.

If we define

$$\nu = \begin{cases} 0 & \text{when } 2|k, \\ 1 & \text{when } 2 \nmid k, \end{cases}$$

then it is an easy matter to show that

$$\lfloor k/2 \rfloor + \lfloor (k+2)/4 \rfloor + \nu \leq \lfloor 3(k+1)/4 \rfloor. \tag{2.3}$$

If $\sigma = 2$ and $2 | \lfloor \sqrt{d} \rfloor$, then, since $P_i \equiv 1 \pmod 2$ for all $i \geq 0$, we cannot have $P_i = \lfloor \sqrt{d} \rfloor$. Thus, in this case, $\chi_i < 2\sqrt{d}$ and $\varepsilon < \lambda(2\sqrt{\Delta})^{\lfloor k/2 \rfloor}$, so

$$\varepsilon < 2^{\lfloor k/2 \rfloor} (\sqrt{\Delta})^{\lfloor k/2 \rfloor + \nu}. \tag{2.4}$$

When $\Delta > 16$, we have

$$2^{\lfloor k/2 \rfloor} < (\sqrt{\Delta})^{\lfloor (k+2)/4 \rfloor},$$

hence by (2.3) we have our result in this case. For the remaining values of $\Delta < 16$ we see that $\sigma = 2$ and $\lfloor \sqrt{d} \rfloor$ even forces $d = 5$, for which the theorem is easily verified.

If $P_i \neq \lfloor \sqrt{d} \rfloor$ and $\lfloor \sqrt{d} \rfloor \equiv 1 \pmod{\sigma}$, then we must have $P_i \leq \lfloor \sqrt{d} \rfloor - \sigma$ and $\chi_i < (2/\sigma)\sqrt{d} = \sqrt{\Delta}$. Furthermore, if j is the least positive integer such that $P_j = P_{j+1}$, we must have $k = 2j$. Therefore, the case in which we have the largest possible number of P_i-values equal to $\lfloor \sqrt{d} \rfloor$ can only occur when

$$P_1 = P_2 = \cdots = P_{\lfloor (k+2)/4 \rfloor} = \lfloor \sqrt{d} \rfloor.$$

Thus, if n is the number of values of P_i with $P_i = \lfloor\sqrt{d}\rfloor$ for $i \le \lfloor k/2\rfloor$, then

$$\varepsilon < \lambda(\sqrt{\Delta})^{\lfloor k/2\rfloor - n}(\sqrt{d} + \lfloor\sqrt{d}\rfloor)/(\sqrt{d} - \lfloor\sqrt{d}\rfloor) \tag{2.5}$$

and $n \le \lfloor (k+2)/4\rfloor$.

Since $d - \lfloor\sqrt{d}\rfloor^2 \ge \sigma^2$, we have $(\sqrt{d}+\lfloor\sqrt{d}\rfloor)/(\sqrt{d}-\lfloor\sqrt{d}\rfloor) < (2\sqrt{d}/\sigma)^2 = \Delta$. Hence,

$$\varepsilon < \lambda(\sqrt{\Delta})^{\lfloor k/2\rfloor - n}\Delta^n = (\sqrt{\Delta})^{\lfloor k/2\rfloor + n + \nu}.$$

By (2.3) the result follows. □

We are now able to prove

Lemma 2.1. *If $k \le 24$ and $\Delta > 6\times 10^9$, then with at most one possible exception we must have $h(d) > 2$.*

Proof. By Tatuzawa [8], we have (with at most one exception) $L(1, \chi) > 0.655\eta\Delta^{-\eta}$ for $0 < \eta < \frac{1}{2}$ and $\Delta \ge \max(e^{1/\eta}, e^{11\cdot 2})$ (where $L(1, \chi) = \sum_{n=1}^{\infty}(\Delta/n)/n$ and $(\cdot/n)$ is the Kronecker symbol). Also, since $2Rh(d) = \sqrt{\Delta}L(1, \chi)$, by Theorem 2.1 we must have

$$2h(d) > 2\Delta^{1/2-\eta}(0.655\eta)/(\lfloor 3(k+1)/4\rfloor \log\Delta) > 4$$

when $\Delta > 6\times 10^9$, $\eta = 0.04442$, and $k \le 24$. □

In the case $\Delta = d \equiv 1 \pmod 4$ we can improve Lemma 2.1 somewhat.

Lemma 2.2. *If $d \equiv 1 \pmod 4$, $\Delta > 4.75\times 10^9$, and $k \le 24$, then with at most one possible exception we must have $h(d) > 2$.*

Proof. This result can be easily verified by using the methods of Theorem 2.1 and Lemma 2.1 in the case where $\lfloor\sqrt{d}\rfloor$ is even. Thus, we will assume that $\lfloor\sqrt{d}\rfloor$ is odd and write (2.5) as

$$\varepsilon < (\sqrt{\Delta})^{\lfloor (k+1)/2\rfloor - n}(\sqrt{d} + \lfloor\sqrt{d}\rfloor)^n\gamma^{-n},$$

where $\gamma = \sqrt{d} - \lfloor\sqrt{d}\rfloor$. We also note that the value of n (the number of values of $P_i = \lfloor\sqrt{d}\rfloor$ for $i \le \lfloor k/2\rfloor$) cannot exceed the number of divisors of $(d - \lfloor\sqrt{d}\rfloor^2)/4$. This is a fact because each Q_i associated with one of the P_i-values must be distinct from any other, must be even, and must be a divisor of $(d - \lfloor\sqrt{d}\rfloor^2)/2$.

Now,

$$\varepsilon < (\sqrt{\Delta})^{\lfloor (k+1)/2\rfloor - n}(2\sqrt{\Delta}/\gamma)^n = (\sqrt{\Delta})^{(k+1)/2}(2/\gamma)^n$$

and

$$R < \lfloor (k+1)/2\rfloor \log\sqrt{\Delta} + n\log(2/\gamma).$$

Hence, if $\Delta > 4.75\times 10^9$, $n\log(2/\gamma) < 52.6$, $\eta = 0.045$, and $k \le 24$, then

$$2h(d) > \Delta^{1/2-\eta}(0.655\eta)/(\lfloor (k+1)/2\rfloor \log\sqrt{\Delta} + n\log(2/\gamma)) > 4.$$

If $n\log(2/\gamma) \ge 52.6$, then, since $k \le \lfloor (k+2)/4\rfloor \le 6$, we have $-\log\gamma > 8.0$, $\gamma < 0.000335$, and $d - \lfloor\sqrt{d}\rfloor^2 < 2\sqrt{d}\gamma < 46.2$. It follows that in this case $(d - \lfloor\sqrt{d}\rfloor^2)/4 < 11$. However, the maximum value of the number of divisors of l for $1 \le l \le 11$ is 4, thus we must have $n \ge 4$. This now means that $-\log\gamma > 12.4$ and $d - \lfloor\sqrt{d}\rfloor^2 < 1$, which is impossible. □

Thus, to find all real quadratic fields with $k \leq 24$ and $h(d) = 2$ (with at most one more value remaining), we need only examine those with $d < 1.5 \times 10^9$ when $d \not\equiv 1 \pmod 4$ and those with $d < 4.75 \times 10^9$ when $d \equiv 1 \pmod 4$.

A computer search was run on all numbers of these forms up to the bounds given above to find all values of d such that $k \leq 24$. Once this had been done, we used the method in Mollin and Williams [1] to eliminate most of the values of d for which the corresponding field has $h(d) > 2$. The value of $h(d)$ was actually determined for those fields which remained and those for which $h(d) > 2$ were also eliminated, leaving only those for which $h(d) = 2$. Our results are summarized in Tables 2.1 and 2.2. There were a surprisingly large number of them, 1958 to be exact.

TABLE 2.1. $h(d) = 2$ for $k \leq 24$

k	d
1	10,26,85,122,362,365,533,629,965,1685,1853,2813
2	15,30,35,39,42,51,66,87,102,110,123,143,146,165,182,203,221,230,258,285, 327,357,402,447,635,645,678,741,843,902,957,1085,1245,1298,1517,1533,2037,2045, 2085,2397,2613,4245,4277,4773,5645,5957,6573,8333
3	65,185,458,485,1157,2117,2285,3077,3293,3365,12365
4	34,55,78,95,119,138,155,174,194,205,215,222,266,287,299,305,318,335,377,395,429, 482,527,623,755,782,861,885,1022,1055,1205,1405,1469,1965,2013,2093,2222,2301, 2373,2877,3005,3237,3597,3813,4893,5117,5397,5757,5885,6005,6285,6293,7157,7733, 7973,8357,9005,9077
5	74,218,493,565,1037,1565,1781,2138,2165,2173,3869,5165,5213,5837,6485,8021, 10397,14213
6	70,105,111,114,178,183,187,267,273,303,371,374,407,418,470,498,518,545,551,590, 602,618,642,803,805,822,923,1005,1007,1034,1118,1167,1173,1178,1202,1581,1605, 1623,1653,1707,1749,1790,2103,2109,2147,2245,2261,2445,2717,2723,2765,2845,3405, 3605,3638,3737,3893,4085,4301,4445,4605,5133,5453,7805,10237,10317,10653,11837, 12845,13253,13277,13445,14405,14573,15197,19445,21677,23693,25437
7	58,202,314,538,685,949,1165,1261,2885,3133,3277,3653,5429,5765,6437,7373,9197, 9509,12557,16757,17141,17261,18317,22301
8	91,238,282,638,695,707,710,854,866,942,1247,1403,1643,1655,1869,1883,1943,2238, 2390,2483,2685,2978,3205,3333,3765,4247,4565,5069,5141,5829,6341,6365,6693,6773, 6837,6965,7405,7469,8165,8853,9141,9453,9485,10013,10293,10373,10517,10797, 10805,11357,11501,15677,16805,17357,17853,19493,31533,37373,38213
9	106,698,1073,1189,1285,1385,1418,1865,2581,3233,4469,4553,4709,5597,8885,9365, 9773,9893,10229,10685,12053,12077,13565,14285,16733,23285,28757,29957
10	115,154,159,186,246,259,286,339,345,354,403,411,451,465,494,515,534,543,561,583, 591,598,665,671,682,687,703,705,762,779,830,938,978,1047,1102,1203,1263,1265, 1363,1383,1645,1671,1727,1742,2098,2123,2127,2485,2651,2658,2701,2747,2802,2829, 2867,2882,3157,3165,3218,3587,3685,3741,3743,3827,3867,4103,4619,4667,5057,5061, 5205,5253,5285,5405,5522,6149,6613,6789,7005,7845,8045,8445,8517,8533,8621,9085, 9093,9581,9701,9821,10365,10645,10877,11373,11557,11973,12117,12165,12837, 14773,14861,16037,16077,16205,17045,17741,17877,18093,18357,18717,19253,21405, 21749,21885,22413,22517,22781,23933,23997,24213,24845,25077,25133,26333,26477, 27173,28005,28853,30245,30693,33677,37565,39245,41477,47195

11	265,298,554,794,1322,1658,2218,2509,3242,4181,4682,4685,11413,11773,13085,14453, 15685,16085,18485,20285,20765,25565,28013,28685,31037,39797,40157,43733,46637 51917,56117
12	247,295,355,366,385,386,426,535,609,767,802,815,851,969,995,1027,1113,1162,1207, 1343,1353,1355,1358,1535,1538,1703,1717,1799,1910,1946,2018,2047,2054,2105,2231, 2318,2327,2334,2365,2438,2507,2735,2855,2987,3002,3263,3302,3563,3695,4322,4382, 4415,4453,4542,4717,4917,5447,6605,6853,6905,7365,7413,7797,9429,10262,11077, 12341,12453,12485,12605,12669,13837,15333,16365,18557,18805,22893,23253,24293, 24485,25397,25413,25517,27053,27389,27605,29141,29405,29861,30173,32357,36533, 40533,44117,44693,45485,45573,47157,52037,59213,59573,75677
13	746,778,1082,1241,1514,1649,2042,2426,3085,3338,3349,4058,4573,4589,4885,5389, 7418,7421,8765,9389,9965,10085,12965,14837,16277,17533,19357,21053,22373,25877, 30733,31373,31853,36965,38597,39437,40757,53477,69893,81413
14	190,319,406,430,471,474,611,667,670,699,742,745,806,807,1001,1043,1070,1115,1119, 1309,1315,1338,1347,1398,1542,1545,1562,1634,1670,1691,1826,1839,1874,2282,2294, 2315,2323,2337,2345,2427,2435,2463,2630,2714,2782,2821,3297,3378,3478,3621,3878, 4115,4154,4178,4307,4331,4381,4499,4506,4646,4835,5222,5246,5282,5442,5673,5781, 5917,6098,6213,6357,6443,6461,6477,6611,6645,7145,7285,7445,7619,7885,8205,8393, 8437,8483,8565,8733,8805,8877,8965,9285,9645,9717,9877,10149,10573,11051,11805, 12578,12621,12733,12869,12885,13197,13213,13973,14181,15861,15965,16541,17013, 17805,18845,18941,19205,19277,19365,19397,19677,21197,21245,21549,21909,21917, 22557,22965,23493,24069,24893,25109,25597,26285,26373,26885,27677,27845,28397, 28605,28797,28805,31349,31413,32973,34013,34133,35045,35477,35765,35789,36917, 38253,40445,42413,42933,43493,44957,45453,47253,51653,52013,54557,54677,55893, 64181,64253,71357,85973,98045
15	481,1417,1466,2858,3065,3589,3785,3977,4538,5317,5941,6641,6749,7082,11861, 12701,12833,13793,14909,16589,17153,18185,18365,18581,20885,24221,27989,29069, 32885,33365,44813,47165,51173,66197,67973,70493,78917
16	310,391,415,654,655,679,955,1038,1146,1166,1267,1282,1346,1391,1578,1662,1739, 1833,1858,1895,1902,2183,2195,2198,2407,2526,2553,2615,3227,3278,3374,3497,3565, 3611,3755,3818,3918,4043,4069,4087,4142,4233,4298,4405,4955,4958,5123,5198,5267, 5543,5558,5579,5726,5855,6062,6167,6254,6383,6501,6527,7322,7337,7355,7898,8029, 8078,8207,8378,8421,8493,8718,9107,9309,9373,10509,11303,11405,11517,11917, 12878,12957,13802,13943,15213,15365,15573,16685,17673,17997,19293,19389,21965, 22029,22173,24101,25885,26133,29685,30413,30581,31493,34989,35861,38309,38405, 38517,40685,41741,43053,43253,43805,45773,48965,49565,49685,50973,55013,55173, 57293,58253,58373,58973,59237,61277,63557,67133,67205,67997,69621,75413
17	1018,1994,2965,4285,5354,5498,5585,8917,9242,9665,10265,12085,13061,13957, 14677,15242,15613,16109,16565,16613,17173,17285,17429,17861,18037,18737, 18965,34037,34957,35285,36413,37949,40085,40501,41165,47813,48149,50357,53285, 57797,58853,61133,62957,63653,86957,146453
18	519,562,831,879,951,1185,1199,1209,1281,1310,1362,1379,1505,1506,1526,1606,1630, 1686,1698,1842,1903,1919,1923,1983,1991,2202,2219,2283,2363,2631,2697,2771,2985, 3183,3282,3414,3470,3642,3702,3707,3830,3839,4029,4287,4343,4430,4562,4697,4791, 4803,5027,5363,5705,5797,5845,5870,6177,6182,6278,6407,6470,6758,6767,6830,6842, 7358,7485,7802,7869,7958,8582,8589,8697,8843,9119,9269,9381,9383,9445,9470, 10245,10461,10502,10643,11397,12093,12162,12278,12549,12722,12749,12909,13019, 13237,13593,13605,13682,13821,14053,14309,14605,15485,15933,16845,17197,17381, 17733,18173,19229,19245,20253,20541,20645,20677,20917,21533,22085,22181,22245, 23309,24357,24365,24477,24837,25445,25485,25781,26781,27485,28461,28589,29309, 30317,30957,31733,32021,32669,32773,33789,33845,34405,34685,34853,35189,36669, 36813,39893,40613,41789,41837,41973,43581,43589,43797,44645,47333,47549,48245, 49805,50133,51557,51845,52853,54197,54845,57845,59637,61477,62405,64277,66549, 70133,71213,73253,74973,76037,80013,80117,92477,96701,128117,138773,139277, 146333,151373,168773,171797
19	922,1706,2186,2257,3386,8522,8714,9997,16781,17177,20513,20813,21509,24341, 26165,28453,29597,30365,31085,35333,35885,36173,37685,37757,38765,41765,43469, 46157,50453,52637,53765,57965,59765,62285,65501,70733,75197,79085,82277,84773, 107333,109757,139037,144317

TABLE 2.1. (continued)

20	511,559,606,790,1002,1065,1079,1182,1195,1374,1415,1510,1513,1537,1603,1687,1961,2193,2215,2455,2471,2627,2863,3155,3239,3295,3383,3647,3746,3857,4163,4295,4458,4535,4595,4727,4782,4847,4922,5038,5143,5195,5258,5678,5709,5759,5803,5822,5962,6107,6141,6338,6415,6467,6702,6914,6943,7189,7295,7343,7367,7787,7813,7895,8238,8258,8507,8567,8903,9527,9861,9885,10622,11949,13413,13461,14541,14565,15029,15203,16237,16463,16989,18821,19085,19221,19337,19509,19533,20517,20642,21093,21765,21962,23069,23133,25197,25557,26637,26697,28205,28965,29261,29365,32053,32997,34773,35085,36093,36165,36237,36573,38685,39093,40557,41285,42605,42749,43085,44765,45933,46277,48237,48453,49181,50213,52917,54893,57189,58493,58533,60205,61053,61773,61973,62237,62933,63213,65285,65813,67373,69293,70213,71405,74261,76677,77957,80693,80717,81317,81357,81437,82869,88413,93197,96773,106685,109997,113693,115037,240077
21	394,865,1769,1985,2561,2762,3098,4385,5465,5485,5965,6122,7141,7265,10565,11101,11485,11581,13285,13466,14381,14765,16442,21365,22565,28373,34493,35197,36221,44861,47477,48485,48941,51365,54317,57317,58133,58589,69365,75917,78053,78557,78773,80165,84173,85277,85949,89333,91013,92165,94877,97877,104837,120893,127613,130037,156917,167477,212357
22	466,763,771,834,1059,1194,1266,1334,1558,1563,1798,1835,1843,1905,1963,1986,2001,2082,2270,2274,2279,2406,2514,2519,2546,2585,2643,2778,2823,2859,2931,2937,2947,3063,3107,3131,3147,3182,3207,3310,3417,3506,3635,3657,3687,3938,4119,4145,4187,4202,4433,4630,4645,4814,4863,4883,4938,4965,5111,5163,5315,5345,5367,5603,5703,5718,5747,5862,5989,6023,6061,6378,6387,6403,6431,6585,6635,6738,6743,7026,7122,7143,7257,7334,7545,7553,7622,7701,7842,7982,8018,8027,8270,8365,8531,8630,8749,8897,8998,9113,9138,9158,9205,9263,9687,9709,10190,10298,10307,11507,11747,12257,12261,12305,12405,12662,12827,13067,13265,13405,13817,14205,14510,14845,15085,15113,15617,15765,16149,16797,17445,17549,18002,20301,21837,23777,24141,24405,25149,25653,25682,26197,27101,28673,29805,30165,31965,32469,33045,33429,33549,33645,33989,34005,34933,36581,37077,37317,37445,37677,39693,40605,42117,44373,45237,45605,46949,48045,48813,48885,49629,49677,50045,50861,51861,52085,52421,54645,56069,56357,56397,57749,57893,58197,58205,59285,60277,62733,63237,63437,63453,64085,65189,66557,67469,68933,71021,71885,72597,76053,76773,77357,77693,78869,80597,83765,85917,86933,89957,92573,94085,94317,95933,98813,99413,101213,101909,104357,106757,107645,112013,116045,116261,116597,120245,121733,126245,130613,135453,136037,136253,138437,145277,145613,148133,148973,149357,149957,150293,156053,157493,165557,168797,204245,248093
23	586,634,1585,2474,3578,4121,4141,5114,6074,6109,6362,6506,6602,7261,8042,8249 10673,12349,20557,22837,24869,26773,26869,33017,34165,34541,37661,37837,43693,51757,55565,56285,56381,58277,59293,63677,64349,67253,74693,76565,77453,86213,87485,90485,90557,90653,94973,99557,107117,107573,107957,113237,119477,139157,154853,160277,172133,176837,247397
24	826,871,1147,1255,1614,1711,1795,2051,2119,2154,2409,2414,2534,2594,2698,2743,2759,3009,3018,3110,3206,3633,3806,3854,3882,4031,4118,4310,4638,4665,4826,5322,5466,6155,6302,6455,6618,7091,7222,7278,7763,8302,8489,8927,8939,9002,9287,9347,9393,9469,9741,9785,10058,10142,10415,10442,10562,10823,11042,11262,11546,11714,11843,12311,12741,13118,13502,13958,14198,14987,15815,16502,16765,16859,17063,17447,17501,17531,18381,18501,18885,19685,20013,21047,21287,21565,21669,22523,23037,23927,25053,25322,26277,27069,27669,27789,27853,29877,30093,31485,34669,34709,35277,36453,37157,37293,37805,39957,40893,41109,42285,43301,43727,44133,44429,44717,45069,47613,48005,48765,49101,50165,51837,54389,54797,54885,56885,57813,59717,59853,60477,61269,61365,62493,63885,68285,70565,70685,71133,71165,72197,72917,76973,78837,83613,84413,85533,87989,88877,89549,91077,92405,92597,97277,102605,106413,109205,109805,112517,113813,118685,122405,125333,126965,132117,136205,141797,151205,154013,158933,165413,169133,172493,175637,197333,205805

TABLE 2.2

k	The number of $h(d) = 2$
1	12
2	48
3	11
4	58
5	18
6	79
7	24
8	59
9	28
10	135
11	31
12	102
13	40
14	169
15	37
16	130
17	46
18	187
19	44
20	161
21	59
22	245
23	59
24	176

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