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Supplement to

CONVERGENCE OF A FINITE ELEMENT METHOD FOR THE DRIFT-DIFFUSION SEMICONDUCTOR DEVICE EQUATIONS: THE ZERO DIFFUSION CASE

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3. PROOFS OF THEOREMS 2.1, 2.2 AND 2.5

In this section we prove Theorems 2.1, 2.2 and 2.5. In §3a we obtain a maximum principle for u_h, and in §3b a bound on the total variation in space. In §3c the important continuity result (with respect to the data) is obtained. In §3d we end the proof of Theorems 2.1 and 2.2. In §3e we prove several results that will allow us to prove Theorem 2.5 in §3f.

3a. A maximum principle for u_h. We begin by showing that under a suitable condition the approximate concentration u_h satisfies a maximum principle. We show in particular that $u_h \geq 0$ provided $u_i, u_0, u_1 \geq 0$, as we expect from the physical point of view. In this section we prove a maximum principle for the approximate solution u_h. We stress the fact that in the framework of conservation laws it is very well known that any maximum principle for explicit schemes is valid only under a certain Courant-Friedrichs-Levy (CFL) condition. Roughly speaking, this condition asks for the speed of the propagation of the information allowed by the numerical scheme to be bigger than the maximum speed of propagation of the information inherent in the conservation law. Since in our case the latter speed is approximated by the quantity $\| \beta_h \|_{L^\infty(0,1)}$ it is clear that in order to get a maximum principle for u_h, we must obtain an upper bound for that quantity. The next three lemmas are devoted to obtain the CFL condition. The bound on $\| \beta_h \|_{L^\infty(0,1)}$ is obtained in Proposition 3.4. The maximum principle is given in Proposition 3.5.

The following simple but important result is the essential link between the mixed finite element method (2.7) and the monotone scheme (2.6). It is a restatement of equation (2.7a) with $w_{\Delta x}(x) = \begin{cases} 1, & x \in I_i, \\ 0, & \text{otherwise.} \end{cases}$

LEMMA 3.1. *For $n = 0, \cdots, n_T - 1$ we have*

$$\beta^n_{i+\frac{1}{2}} = \beta^n_{i-\frac{1}{2}} - (1 - u^n_i)\Delta x_i, \quad i = 1, \dots, n_x.$$

The next result displays the CFL condition under which a maximum principle is satisfied.

LEMMA 3.2 (The CFL condition). *Let u^n_h be such that*

$$u^n_h(0-), u^n_h(x), u^n_h(1+) \in [0, u^*], \quad x \in (0,1).$$

Suppose that

$$1 - \frac{\Delta\tau^n}{\Delta x_i}(\beta^{n+}_{i+\frac{1}{2}} - \beta^{n-}_{i-\frac{1}{2}}) \geq \Delta\tau^n, \quad i = 1, \cdots, n_x. \tag{3.1}$$

Then

$$u^{n+1}_h(x) \in [0, u^*], \quad x \in (0,1).$$

The assumption $u^* \geq 1$, see (2.1e), is crucial; the result is simply not true if $u^* < 1$. This reflects the fact that $u = 1$ is an asymptotically stable equilibrium point for the equation satisfied by u along the characteristics, (1.5); see remark (iii) in §1.

Proof. Pick $i \in \{1, \cdots, n_x\}$. By (2.6),

$$\begin{aligned} u_i^{n+1} &= u_i^n - \frac{\Delta\tau^n}{\Delta x_i}(\beta_{i+\frac12}^{n-} u_{i+1}^n + \beta_{i+\frac12}^{n+} u_i^n - \beta_{i-\frac12}^{n-} u_i^n - \beta_{i-\frac12}^{n+} u_{i-1}^n) \\ &= D_i^n \,[A_{i+1}^n u_{i+1}^n + B_i^n u_i^n + C_{i-1}^n u_{i-1}^n], \end{aligned}$$

where

$$\begin{aligned} A_{i+1}^n &= -\frac{\Delta\tau^n}{\Delta x_i}\frac{\beta_{i+\frac12}^{n-}}{D_i^n}, \\ B_i^n &= (1 - \frac{\Delta\tau^n}{\Delta x_i}(\beta_{i+\frac12}^{n+} - \beta_{i-\frac12}^{n-}))/D_i^n, \\ C_{i-1}^n &= \frac{\Delta\tau^n}{\Delta x_i}\frac{\beta_{i-\frac12}^{n+}}{D_i^n}, \\ D_i^n &= 1 - \frac{\Delta\tau^n}{\Delta x_i}(\beta_{i+\frac12}^n - \beta_{i-\frac12}^n). \end{aligned}$$

Note that by definition and by (3.1),

(3.2a) $$D_i^n \geq 1 - \frac{\Delta\tau^n}{\Delta x_i}(\beta_{i+\frac12}^{n+} - \beta_{i-\frac12}^{n-}) \geq \Delta\tau^n,$$

(3.2b) $$A_{i+1}^n + B_i^n + C_{i-1}^n = 1,$$

(3.2c) $$A_{i+1}^n,\ B_i^n,\ C_{i-1}^n \geq 0.$$

It is thus clear that $u_i^{n+1} \geq 0$ by the hypothesis on u_h^n.

To prove that $u_i^{n+1} \leq u^*$, we proceed as follows. Since

$$\begin{aligned} u_i^{n+1} &= D_i^n\,[u^* - A_{i+1}^n (u^* - u_{i+1}^n) - B_i^n (u^* - u_i^n) - C_{i-1}^n (u^* - u_{i-1}^n)] \\ &= \{D_i^n\,[1 - A_{i+1}^n \frac{u^* - u_{i+1}^n}{u^*} - B_i^n \frac{u^* - u_i^n}{u^*} - C_{i-1}^n \frac{u^* - u_{i-1}^n}{u^*}]\}\, u^* \\ &= \{D_i^n (1 - E_i^n)\}\, u^* \\ &= \{(1 + \Delta\tau^n (1 - u_i^n))(1 - E_i^n)\}\, u^*, \quad \text{by Lemma 3.1,} \end{aligned}$$

where

$$E_i^n = A_{i+1}^n \frac{u^* - u_{i+1}^n}{u^*} + B_i^n \frac{u^* - u_i^n}{u^*} + C_{i-1}^n \frac{u^* - u_{i-1}^n}{u^*},$$

it is evident that $u_i^{n+1} \leq u^*$ if and only if $E_i^n \geq \Delta\tau^n (1 - u_i^n)/(1 + \Delta\tau^n (1 - u_i^n))$. (Notice that the denominator in the above fraction is nonnegative thanks to Lemma 3.1 and condition (3.1).) If $u_i > 1$, the above inequality is trivially verified. If $u_i \leq 1$, we have, by the hypothesis on u_h^n and (3.2),

$$\begin{aligned} E_i^n &\geq B_i^n \frac{u^* - u_i^n}{u^*} \\ &\geq B_i^n (1 - u_i^n), \quad \text{since } u^* \geq 1, \\ &= \frac{1 - \frac{\Delta\tau^n}{\Delta x_i}(\beta_{i+\frac12}^{n+} - \beta_{i-\frac12}^{n-})}{1 + \Delta\tau^n (1 - u_i^n)} (1 - u_i^n) \\ &\geq \frac{\Delta\tau^n (1 - u_i^n)}{1 + \Delta\tau^n (1 - u_i^n)}, \end{aligned}$$

provided condition (3 1) is satisfied. This completes the proof. □

We need now to rewrite condition (3.1) in terms of u_h^n and $\phi_1(\tau^n)$ only.

LEMMA 3.3. *Suppose that the following condition is satisfied:*

(3.3) $$\Delta\tau^n \leq \min\{1, \frac{1}{\| u_h^n \|_{L^\infty(0,1)}}, \frac{\Delta x_i}{\Delta x_i + \| \beta_h^n \|_{L^\infty(0,1)}}\}.$$

Then condition (3.1) *holds.*

Proof. We have

$$\begin{aligned} \Theta_i^n &:= 1 - \frac{\Delta\tau^n}{\Delta x_i}(\beta_{i+\frac12}^{n+} - \beta_{i-\frac12}^{n-}) \\ &= \begin{cases} 1 - \frac{\Delta\tau^n}{\Delta x_i}(\beta_{i+\frac12}^n - \beta_{i-\frac12}^n) & \text{if } \beta_{i+\frac12}^n \geq 0 \text{ and } \beta_{i-\frac12}^n \leq 0, \\ 1 - \frac{\Delta\tau^n}{\Delta x_i}|\beta_{i+\frac12}^n| & \text{if } \beta_{i+\frac12}^n \geq 0 \text{ and } \beta_{i-\frac12}^n \geq 0, \\ 1 - \frac{\Delta\tau^n}{\Delta x_i}|\beta_{i-\frac12}^n| & \text{if } \beta_{i+\frac12}^n \leq 0 \text{ and } \beta_{i-\frac12}^n \leq 0, \\ 1 & \text{otherwise} \end{cases} \\ &\geq \begin{cases} 1 + \Delta\tau^n (1 - u_i^n) & \text{if } \beta_{i+\frac12}^n \geq 0 \text{ and } \beta_{i-\frac12}^n \leq 0, \\ 1 - \frac{\Delta\tau^n}{\Delta x_i}\| \beta_h^n \|_{L^\infty(0,1)} & \text{if } \beta_{i+\frac12}^n \geq 0 \text{ and } \beta_{i-\frac12}^n \geq 0, \\ 1 - \frac{\Delta\tau^n}{\Delta x_i}\| \beta_h^n \|_{L^\infty(0,1)} & \text{if } \beta_{i+\frac12}^n \leq 0 \text{ and } \beta_{i-\frac12}^n \leq 0, \\ 1 & \text{otherwise,} \end{cases} \end{aligned}$$

where we have used Lemma 3.1 in the last line. Thus, $\Theta_i^n \geq \Delta\tau^n$ if

$$\Delta\tau^n \leq \begin{cases} \frac{1}{\| u_h^n \|_{L^\infty(0,1)}} & \text{if } \beta_{i+\frac12}^n \geq 0 \text{ and } \beta_{i-\frac12}^n \leq 0, \\ \frac{\Delta x_i}{\Delta x_i + \| \beta_h^n \|_{L^\infty(0,1)}} & \text{if } \beta_{i+\frac12}^n \geq 0 \text{ and } \beta_{i-\frac12}^n \geq 0, \\ \frac{\Delta x_i}{\Delta x_i + \| \beta_h^n \|_{L^\infty(0,1)}} & \text{if } \beta_{i+\frac12}^n \leq 0 \text{ and } \beta_{i-\frac12}^n \leq 0, \\ 1 & \text{otherwise.} \end{cases}$$

This completes the proof. □

Finally, we have to estimate $\| \beta_h^n \|_{L^\infty(0,1)}$.

PROPOSITION 3.4 (Estimate of $\| \beta_h \|_{L^\infty(0,1)}$). *For $n = 0, \cdots, n_T$ we have*

$$\| \beta_h^n - \phi_{1,\Delta\tau}(\tau^n) \|_{L^\infty(0,1)} \le \frac{1}{2} \| 1 - u_h^n \|_{L^\infty(0,1)}.$$

Proof. From equations (2.7) we easily obtain that

$$-(\beta_h^n - \phi_{1,\Delta\tau}(\tau^n))_x = 1 - u_h^n,$$
$$(\beta_h^n - \phi_{1,\Delta\tau}(\tau^n), 1) = 0.$$

The result follows immediately. □

We are now ready to state our maximum principle.

PROPOSITION 3.5 (A maximum principle for u_h). *Let u_0, u_1 and $u_\imath$ be such that*

$$u_0(\tau), u_1(\tau), u_\imath(x) \in [0, u^*], \quad \tau \in [0,T],\ x \in (0,1),$$

for some $u^* \ge 1$. Suppose that for $\imath = 1, \cdots, n_x$, and $n = 0, \cdots, n_T - 1$ we have

$$\Delta\tau^n \le \min\{\frac{1}{u^*}, \frac{\Delta x_\imath}{\Delta x_\imath + \phi_{1,\Delta\tau}(\tau^n) + \frac{1}{2}\max\{1, u^* - 1\}}\}. \tag{3.4}$$

Then the following maximum principle holds:

$$u_h^n(x) \in [0, u^*], \quad x \in (0,1),\ n = 0, \cdots, n_T.$$

Proof. We proceed by induction on n. The result is trivially true for $n = 0$ by (2.5). Let us assume that it is true for $n = m$, and let us prove it is also true for $n = m + 1$. By the inductive hypothesis and (2.5) we have that

$$u_h^m(0-), u_h^m(x), u_h^m(1+) \in [0, u^*], \quad x \in (0,1).$$

Thus, $\| u_h^m \|_{L^\infty(0,1)} \le u^*$ and $\| 1 - u_h^m \|_{L^\infty(0,1)} \le \max\{1, u^* - 1\}$. Since $\phi_{1,\Delta\tau}(\tau^m) \ge 0$, the latter inequality implies that $\| \beta_h^n \|_{L^\infty(0,1)} \le \phi_{1,\Delta\tau}(\tau^m) + \frac{1}{2}\max\{1, u^* - 1\}$ by Proposition 3.4. Hence, by (3.4),

$$\Delta\tau^m \le \min\{\frac{1}{u^*}, \frac{\Delta x_\imath}{\Delta x_\imath + \phi_{1,\Delta\tau}(\tau^m) + \frac{1}{2}\max\{1, u^* - 1\}}\}$$
$$\le \min\{1, \frac{1}{\| u_h^m \|_{L^\infty(0,1)}}, \frac{\Delta x_\imath}{\Delta x_\imath + \| \beta_h^m \|_{L^\infty(0,1)}}\},$$

and by Lemma 3.3, condition (3.1) is satisfied for $n = m$. By Lemma 3.2, we have that $u_h^{m+1}(x) \in [0, u^*]$, $\forall x \in (0,1)$. This completes the proof. □

3b. Bound on the total variation of u_h. In this section we show that under a suitable CFL condition the total variation of u_h^n in $(0,1)$ is uniformly bounded. Lemma 3.6 is a preliminary result. Corollary 3.7 is a monotonicity result for a special (but important) case. Lemma 3.8 relates the total variation of u_h^{n+1} to the total variation of u_h^n. Proposition 3.9 contains our total variation boundedness result.

LEMMA 3.6. *We have, for $\imath = 0$,*

$$u_1^{n+1} - u_0^{n+1} = -\frac{\Delta\tau^n}{\Delta x_1}\beta_{\frac{3}{2}}^{n-}(u_2^n - u_1^n)$$
$$+ (1 + \Delta\tau^n(1 - u_1^n - u_0^n) - \frac{\Delta\tau^n}{\Delta x_1}\beta_{\frac{1}{2}}^{n+})(u_1^n - u_0^n)$$
$$+ ((1 + \Delta\tau^n(1 - u_0^n))u_0^n - u_0^{n+1}),$$

for $\imath \in \{1, \cdots, n_x - 1\}$,

$$u_{\imath+1}^{n+1} - u_\imath^{n+1} = -\frac{\Delta\tau^n}{\Delta x_{\imath+1}}\beta_{\imath+\frac{3}{2}}^{n-}(u_{\imath+2}^n - u_{\imath+1}^n)$$
$$+ (1 + \Delta\tau^n(1 - u_{\imath+1}^n - u_\imath^n) - \frac{\Delta\tau^n}{\Delta x_{\imath+1}}\beta_{\imath+\frac{1}{2}}^{n+} + \frac{\Delta\tau^n}{\Delta x_\imath}\beta_{\imath+\frac{1}{2}}^{n-})(u_{\imath+1}^n - u_\imath^n)$$
$$+ \frac{\Delta\tau^n}{\Delta x_\imath}\beta_{\imath-\frac{1}{2}}^{n+}(u_\imath^n - u_{\imath-1}^n),$$

and for $\imath = n_x$

$$u_{n_x+1}^{n+1} - u_{n_x}^{n+1} = (u_{n_x+1}^{n+1} - (1 + \Delta\tau^n(1 - u_{n_x+1}^n))u_{n_x+1}^n)$$
$$+ (1 + \Delta\tau^n(1 - u_{n_x+1}^n - u_{n_x}^n) + \frac{\Delta\tau^n}{\Delta x_{n_x}}\beta_{n_x+\frac{1}{2}}^{n-})(u_{n_x}^n - u_{n_x+1}^n)$$
$$+ \frac{\Delta\tau^n}{\Delta x_{n_x}}\beta_{n_x-\frac{1}{2}}^{n+}(u_{n_x}^n - u_{n_x-1}^n).$$

Proof. By (2.6a) ,

$$u_{\imath+1}^{n+1} = -\frac{\Delta\tau^n}{\Delta x_{\imath+1}}\beta_{\imath+\frac{3}{2}}^{n-}u_{\imath+2}^n + \frac{\Delta\tau^n}{\Delta x_{\imath+1}}\beta_{\imath+\frac{1}{2}}^{n+}u_\imath^n$$
$$+ (1 - \frac{\Delta\tau^n}{\Delta x_{\imath+1}}(\beta_{\imath+\frac{3}{2}}^{n+} - \beta_{\imath+\frac{1}{2}}^{n-}))u_{\imath+1}^n$$
$$= -\frac{\Delta\tau^n}{\Delta x_{\imath+1}}\beta_{\imath+\frac{3}{2}}^{n-}(u_{\imath+2}^n - u_{\imath+1}^n) + \frac{\Delta\tau^n}{\Delta x_{\imath+1}}\beta_{\imath+\frac{1}{2}}^{n+}(u_\imath^n - u_{\imath+1}^n)$$
$$+ (1 - \frac{\Delta\tau^n}{\Delta x_{\imath+1}}(\beta_{\imath+\frac{3}{2}}^n - \beta_{\imath+\frac{1}{2}}^n))u_{\imath+1}^n$$
$$= -\frac{\Delta\tau^n}{\Delta x_{\imath+1}}\beta_{\imath+\frac{3}{2}}^{n-}(u_{\imath+2}^n - u_{\imath+1}^n) + \frac{\Delta\tau^n}{\Delta x_{\imath+1}}\beta_{\imath+\frac{1}{2}}^{n+}(u_\imath^n - u_{\imath+1}^n)$$
$$+ (1 + \Delta\tau^n(1 - u_{\imath+1}^n))u_{\imath+1}^n,$$

by Lemma 3.1. The result follows after a few very simple algebraic manipulations. □

An immediate consequence of this result is the following monotonicity result.

COROLLARY 3.7 (A monotonicity result). *Let u_i, u_0 and u_1 be such that*

$$u_i(x) \in [0,1], x \in (0,1), \text{ is a nondecreasing function,}$$
$$u_0(\tau) = 0, \tau \in (0,T),$$
$$u_1(\tau) = 1, \tau \in (0,T),$$

or such that

$$u_i(x) \in [0,1], x \in (0,1), \text{ is a nonincreasing function,}$$
$$u_0(\tau) = 1, \tau \in (0,T),$$
$$u_1(\tau) = 0, \tau \in (0,T).$$

Suppose that for $i = 1, \cdots, n_x$ and $n = 0, \cdots, n_T - 1$ we have

$$\Delta\tau^n \le \min\{1, \frac{\Delta x_i}{\Delta x_i + \phi_{1,\Delta\tau}(\tau^n) + \frac{1}{2}}\}. \tag{3.5}$$

Then $u_h(\tau^n, \cdot)$ is a monotone function for $n = 0, \cdots, n_T$, and

$$\| u_h^n \|_{BV[0,1]} = \sum_{i=0}^{n_x} | u_{i+1}^n - u_i^n | = 1, \quad n = 0, \cdots, n_T.$$

Proof. We proceed by induction on n. The result is trivially true for $n = 0$ by (2.5d). Now, assume that the result is true for $n = m$ and let us prove that it is true for $n = m+1$. Pick $i \in \{1, \cdots, n_x - 1\}$. Consider the coefficient

$$\begin{aligned} C^m_{i+\frac{1}{2}} &= 1 + \Delta\tau^m (1 - u_{i+1}^m - u_i^m) - \frac{\Delta\tau^m}{\Delta x_{i+1}}\beta^{m+}_{i+\frac{1}{2}} + \frac{\Delta\tau^m}{\Delta x_i}\beta^{m-}_{i+\frac{1}{2}} \\ &\ge 1 - \Delta\tau^m - \frac{\Delta\tau^m}{\Delta x_{i+1}}\beta^{m+}_{i+\frac{1}{2}} + \frac{\Delta\tau^m}{\Delta x_i}\beta^{m-}_{i+\frac{1}{2}}, \text{ by the inductive hypothesis,} \\ &\ge 1 - \Delta\tau^m - \frac{\Delta\tau^m}{\min\{\Delta x_i, \Delta x_{i+1}\}} \| \beta_h^m \|_{L^\infty(0,1)} \\ &\ge 1 - \Delta\tau^m - \frac{\Delta\tau^m}{\min\{\Delta x_i, \Delta x_{i+1}\}}(\phi_{1,\Delta\tau}(\tau^m) + \frac{1}{2}), \text{ by Lemma 3.4.} \end{aligned}$$

By (3.5), $C^m_{i+\frac{1}{2}} \ge 0$ for $i \in \{1, \cdots, n_x - 1\}$. In a similar way it can be proven that $C^m_{\frac{1}{2}}, C^m_{n_x+\frac{1}{2}} \ge 0$. Thus, all the coefficients of the terms of the form $(u_{j+1} - u_j)$ in Lemma 3.6 are nonnegative, and the result follows. □

Notice that this result implies the maximum principle $u_h \in [0,1]$. In fact, since in our case $u^* = 1$, condition (3.5) is nothing but condition (3.4), and hence Proposition 3.5 is valid. This result is the discrete version of the corresponding result for the continuous problem proven in [8].

LEMMA 3.8 (Local total variation boundedness). *Suppose that for $i = 1, \cdots, n_x$, we have*

$$\Delta\tau^n \le \min\{1, \frac{\Delta x_i}{\Delta x_i \max\{1, 2\| u_h^n \|_{L^\infty(0,1)} - 1\} + \phi_{1,\Delta\tau}(\tau^n) + \frac{1}{2}\max\{1, \| u_h^n \|_{L^\infty(0,1)} - 1\}}\}. \tag{3.6}$$

Then

$$\begin{aligned} \| u_h^{n+1} \|_{BV[0,1]} &= \sum_{i=0}^{n_x} | u_{i+1}^{n+1} - u_i^{n+1} | \\ &\le \Delta\tau^n \left| \frac{u_0^{n+1} - u_0^n}{\Delta\tau^n} - (1 - u_0^n) u_0^n \right| \\ &\quad + \sum_{i=0}^{n_x}(1 + \Delta\tau^n (1 - u_{i+1}^n - u_i^n)) | u_{i+1}^n - u_i^n | \\ &\quad + \Delta\tau^n \left| \frac{u_{n_x+1}^{n+1} - u_{n_x+1}^n}{\Delta\tau^n} - (1 - u_{n_x+1}^n) u_{n_x+1}^n \right|. \end{aligned}$$

Proof. First, notice that condition (3.6) ensures the nonnegativity of the coefficients of the terms of the form $(u_{j+1}^n - u_j^n)$ in the expressions of Lemma 3.6. Taking absolute values in the expressions of Lemma 3.6, we get, for $i = 0$,

$$\begin{aligned} | u_1^{n+1} - u_0^{n+1} | &\le -\frac{\Delta\tau^n}{\Delta x_1}\beta^{n-}_{\frac{3}{2}} | u_2^n - u_1^n | \\ &\quad + (1 + \Delta\tau^n(1 - u_1^n - u_0^n) - \frac{\Delta\tau^n}{\Delta x_1}\beta^{n+}_{\frac{1}{2}}) | u_1^n - u_0^n | \\ &\quad + | (1 + \Delta\tau^n(1 - u_0^n)) u_0^n - u_0^{n+1} |, \end{aligned}$$

for $i = 1, \cdots, n_x - 1$,

$$\begin{aligned} | u_{i+1}^{n+1} - u_i^{n+1} | &\le -\frac{\Delta\tau^n}{\Delta x_{i+1}}\beta^{n-}_{i+\frac{3}{2}} | u_{i+2}^n - u_{i+1}^n | \\ &\quad + (1 + \Delta\tau^n (1 - u_{i+1}^n - u_i^n) - \frac{\Delta\tau^n}{\Delta x_{i+1}}\beta^{n+}_{i+\frac{1}{2}} + \frac{\Delta\tau^n}{\Delta x_i}\beta^{n-}_{i+\frac{1}{2}}) | u_{i+1}^n - u_i^n | \\ &\quad + \frac{\Delta\tau^n}{\Delta x_i}\beta^{n+}_{i-\frac{1}{2}} | u_i^n - u_{i-1}^n |, \end{aligned}$$

and for $i = n_x$,

$$\begin{aligned} | u_{n_x+1}^{n+1} - u_{n_x}^{n+1} | &\le | u_{n_x+1}^{n+1} - (1 + \Delta\tau^n(1 - u_{n_x+1}^n)) u_{n_x+1}^n | \\ &\quad + (1 + \Delta\tau^n(1 - u_{n_x+1}^n - u_{n_x}^n) + \frac{\Delta\tau^n}{\Delta x_{n_x}}\beta^{n-}_{n_x+\frac{1}{2}}) | u_{n_x}^n - u_{n_x+1}^n | \\ &\quad + \frac{\Delta\tau^n}{\Delta x_{n_x}}\beta^{n+}_{n_x-\frac{1}{2}} | u_{n_x}^n - u_{n_x-1}^n |. \end{aligned}$$

Hence we obtain

$$\begin{aligned}\| u_h^{n+1} \|_{BV[0,1]} &= \sum_{i=0}^{n_x} | u_{i+1}^{n+1} - u_i^{n+1} | \\ &\le | (1 + \Delta\tau^n(1-u_0^n))\, u_0^n - u_0^{n+1} | \\ &\quad + (1 + \Delta\tau^n(1 - u_1^n - u_0^n) - \frac{\Delta\tau^n}{\Delta x_1}\beta_{\frac12}^{n+})\, | u_1^n - u_0^n | \\ &\quad - \sum_{i=1}^{n_x} \frac{\Delta\tau^n}{\Delta x_i}\beta_{i+\frac12}^{n-}\, | u_{i+1}^n - u_i^n | \\ &\quad + \sum_{i=1}^{n_x-1} (1 + \Delta\tau^n(1 - u_{i+1}^n - u_i^n) - \frac{\Delta\tau^n}{\Delta x_{i+1}}\beta_{i+\frac12}^{n+} + \frac{\Delta\tau^n}{\Delta x_i}\beta_{i+\frac12}^{n-})\, | u_{i+1}^n - u_i^n | \\ &\quad + \sum_{i=0}^{n_x-1} \frac{\Delta\tau^n}{\Delta x_{i+1}}\beta_{i+\frac12}^{n+}\, | u_{i+1}^n - u_i^n | \\ &\quad + (1 + \Delta\tau^n(1 - u_{n_x+1}^n - u_{n_x}^n) + \frac{\Delta\tau^n}{\Delta x_{n_x}}\beta_{n_x+\frac12}^{n-})\, | u_{n_x}^n - u_{n_x+1}^n | \\ &\quad + | u_{n_x+1}^{n+1} - (1 + \Delta\tau^n(1 - u_{n_x+1}^n))\, u_{n_x+1}^n |,\end{aligned}$$

and so,

$$\begin{aligned}\| u_h^{n+1} \|_{BV[0,1]} &\le | (1 + \Delta\tau^n(1-u_0^n))\, u_0^n - u_0^{n+1} | \\ &\quad + \sum_{i=0}^{n_x} (1 + \Delta\tau^n(1 - u_{i+1}^n - u_i^n))\, | u_{i+1}^n - u_i^n | \\ &\quad + | u_{n_x+1}^{n+1} - (1 + \Delta\tau^n(1 - u_{n_x+1}^n))\, u_{n_x+1}^n |,\end{aligned}$$

and the result follows □

Notice that if the segment $\{x = 0\} \times (\tau^n, \tau^{n+1})$ lies on a characteristic, then the expression

$$\frac{u_0^{n+1} - u_0^n}{\Delta\tau^n} - (1 - u_0^n)\, u_0^n$$

can be considered as a discretization of the equation (1.5). If the above quantity is equal to zero, then nothing flows across the boundary $x = 0$ during the time interval (τ^n, τ^{n+1}), and hence the influence of the boundary condition on the total variation of u_h should be none, as our result confirms. If $u_0 \equiv 0$, or if $u_0 \equiv 1$, this is precisely what happens. A similar remark can be made for the corresponding expression at the boundary $x = 1$.

Notice also that in the case of Corollary 3.7 it can be easily verified that this result reads as follows:

$$\| u_h^{n+1} \|_{BV[0,1]} \le \| u_h^n \|_{BV[0,1]},$$

as expected. However, this is a particular case that is far from being typical. By remark (iv) in §1 we expect the total variation of the exact solution to (locally) increase with a rate equal to e^t. This explains the factor e^T in our following result.

PROPOSITION 3.9 (Total variation boundedness). *Suppose that for $i = 1, \cdots, n_x$ and $n = 0, \cdots, n_T - 1$ we have*

$$\Delta\tau^n \le \min\{\frac{1}{u^*}, \frac{\Delta x_i}{(2u^* - 1)\,\Delta x_i + \phi_{1,\Delta\tau}(\tau^n) + \frac12 \max\{1, u^* - 1\}}\} \tag{3.7}$$

Then

$$\| u_h \|_{L^\infty(0,T,BV[0,1])} \le C_1,$$

where

$$\begin{aligned}C_1 = e^T \{ &\| u_0 \|_{BV[0,T]} + \| 1 - u_0 \|_{L^\infty(0,T)} \| u_0 \|_{L^1(0,T)} \\ &+ (\Delta\tau^0 \| u_0 \|_{BV[0,\Delta\tau^0]} + |u_0(0+) - u_i(0+)| \\ &+ \| u_i \|_{BV(0,1)} \\ &+ \Delta\tau^0 \| u_1 \|_{BV[0,\Delta\tau^0]} + |u_1(0+) - u_i(1-)|) \\ &+ \| u_1 \|_{BV[0,T]} + \| 1 - u_1 \|_{L^\infty(0,T)} \| u_1 \|_{L^1(0,T)} \}.\end{aligned}$$

Notice that u_0 and u_1 are the boundary data and that u_i is the initial condition.

Proof. Since condition (3.7) implies condition (3.4), Proposition 3.5 is satisfied. Since condition (3.7) is stronger than condition (3.6), Lemma 3.8 is also satisfied, and so

$$\begin{aligned}\| u_h^{n+1} \|_{BV[0,1]} &\le \Delta\tau^n \left| \frac{u_0^{n+1} - u_0^n}{\Delta\tau^n} - (1 - u_0^n)\, u_0^n \right| \\ &\quad + (1 + \Delta\tau^n) \| u_h^n \|_{BV[0,1]} \\ &\quad + \Delta\tau^n \left| \frac{u_{n_x+1}^{n+1} - u_{n_x+1}^n}{\Delta\tau^n} - (1 - u_{n_x+1}^n)\, u_{n_x+1}^n \right|.\end{aligned}$$

This implies that

$$\begin{aligned}\| u_h^{n+1} \|_{BV[0,1]} &\le \sum_{\ell=0}^{n} e^{\tau^{n+1} - \tau^\ell} \left| \frac{u_0^{\ell+1} - u_0^\ell}{\Delta\tau^\ell} - (1 - u_0^\ell)\, u_0^\ell \right| \Delta\tau^\ell \\ &\quad + e^{\tau^{n+1}} \| u_h^0 \|_{BV[0,1]} \\ &\quad + \sum_{\ell=0}^{n} e^{\tau^{n+1} - \tau^\ell} \left| \frac{u_{n_x+1}^{\ell+1} - u_{n_x+1}^\ell}{\Delta\tau^\ell} - (1 - u_{n_x+1}^\ell)\, u_{n_x+1}^\ell \right| \Delta\tau^\ell,\end{aligned}$$

and the result follows from (2.5). □

3c. Continuity with respect to the data. In this section we obtain a result concerning the continuous dependence of the approximate solution u_h with respect to the initial and boundary data. The Lemmas 3.10, 3.11 and 3.12 contain preliminary results. The continuity result is given in Proposition 3.13. Proposition 3.14 is an equicontinuity-in-time result which will be used in §4.1.

In what follows, we denote by (v_h, σ_h, ψ_h) the approximate solution (u_h, β_h, ϕ_h) of (1.3), (1.4) with u_i, u_0, u_1 and ϕ_1 replaced by v_i, v_0, v_1 and ψ_1, respectively.

LEMMA 3.10. *For $n = 0, \ldots, n_T - 1$ and $i = 1, \ldots, n_x$ we have the following identity:*

$$
\begin{aligned}
u_i^{n+1} - v_i^{n+1} = &\left\{ -\frac{1}{2}\frac{\Delta\tau^n}{\Delta x_i}(\beta^{n-}_{i+\frac12} + \sigma^{n-}_{i+\frac12}) \right\} (u^n_{i+1} - v^n_{i+1}) \\
&+ \left\{ 1 - \frac12 \Delta\tau^n (u_i^n + v_i^n) - \frac12 \frac{\Delta\tau^n}{\Delta x_i}((\beta^{n+}_{i+\frac12} + \sigma^{n+}_{i+\frac12}) - (\beta^{n-}_{i-\frac12} + \sigma^{n-}_{i-\frac12})) \right\} (u_i^n - v_i^n) \\
&+ \left\{ \frac12 \frac{\Delta\tau^n}{\Delta x_i}(\beta^{n+}_{i-\frac12} + \sigma^{n+}_{i-\frac12}) \right\} (u^n_{i-1} - v^n_{i-1}) \\
&+ \left(\frac12 \frac{\Delta\tau^n}{\Delta x_i}(-\beta^{n-}_{i+\frac12} + \sigma^{n-}_{i+\frac12}) \right) ((u^n_{i+1} - u^n_i) + (v^n_{i+1} - v^n_i)) \\
&+ \left(\frac12 \frac{\Delta\tau^n}{\Delta x_i}(-\beta^{n+}_{i-\frac12} + \sigma^{n+}_{i-\frac12}) \right) ((u^n_i - u^n_{i-1}) + (v^n_i - v^n_{i-1})) .
\end{aligned}
$$

The proof of this lemma is obtained by using (2.6), Lemma (3.1) and some straightforward algebraic manipulations.

LEMMA 3.11. *We have, for $n = 0, \ldots, n_T$,*

$$
\|\beta^n_h - \sigma^n_h\|_{L^\infty(0,1)} \le \|u^n_h - v^n_h\|_{L^1(0,1)} + |\phi^n_{1,\Delta\tau} - \psi^n_{1,\Delta\tau}|.
$$

Proof. From equations (2.7) we have

$$
\begin{aligned}
((\beta^n_h - \sigma^n_h) - (\phi^n_{1,\Delta\tau} - \psi^n_{1,\Delta\tau}), 1) &= 0, \\
(\beta^n_h - \sigma^n_h)_x &= u^n_h - v^n_h,
\end{aligned}
$$

and hence the result follows. □

LEMMA 3.12. *Suppose that the* CFL *condition* (3.7) *is satisfied* (*for both sets of data*). *Then*

$$
\begin{aligned}
\|u_h^{n+1} - v_h^{n+1}\|_{L^1(0,1)} \le &\frac12 \Delta\tau^n \left(\|\beta^n_h\|_{L^\infty(0,1)} + \|\sigma^n_h\|_{L^\infty(0,1)} \right) \left(|u^n_{n_x+1} - v^n_{n_x+1}| + |u^n_0 - v^n_0| \right) \\
&+ \frac12 \Delta\tau^n \left(\|u^n_h\|_{\mathrm{BV}(0,1)} + \|v^n_h\|_{\mathrm{BV}(0,1)} \right) |\phi^n_{1,\Delta\tau} - \psi^n_{1,\Delta\tau}| \\
&+ \left(1 + \Delta\tau^n \frac12 \left(\|u^n_h\|_{\mathrm{BV}(0,1)} + \|v^n_h\|_{\mathrm{BV}(0,1)} \right) \right) \|u^n_h - v^n_h\|_{L^1(0,1)}.
\end{aligned}
$$

Proof. Under the hypothesis of this lemma the terms between braces in Lemma 3.10 are nonnegative. Hence,

$$
\begin{aligned}
|u_i^{n+1} - v_i^{n+1}|\Delta x_i \le &-\frac12 \Delta\tau^n (\beta^{n-}_{i+\frac12} + \sigma^{n-}_{i+\frac12}) |u^n_{i+1} - v^n_{i+1}| \\
&- \frac12 \Delta\tau^n (\beta^{n+}_{i+\frac12} + \sigma^{n+}_{i+\frac12}) |u^n_i - v^n_i| \\
&+ (1 - \frac12 \Delta\tau^n (u^n_i + v^n_i)) |u^n_i - v^n_i| \Delta x_i \\
&+ \frac12 \Delta\tau^n (\beta^{n-}_{i-\frac12} + \sigma^{n-}_{i-\frac12}) |u^n_i - v^n_i| \\
&- \frac12 \Delta\tau^n (\beta^{n+}_{i-\frac12} + \sigma^{n+}_{i+\frac12}) |u^n_{i-1} - v^n_{i-1}| \\
&+ \frac12 \Delta\tau^n |\beta^{n-}_{i+\frac12} - \sigma^{n-}_{i+\frac12}| (|u^n_{i+1} - u^n_i| + |v^n_{i+1} - v^n_i|) \\
&+ \frac12 \Delta\tau^n |\beta^{n+}_{i-\frac12} - \sigma^{n+}_{i-\frac12}| (|u^n_i - u^n_{i-1}| + |v^n_i - v^n_{i-1}|).
\end{aligned}
$$

Summing over i and rearranging terms, we get

$$
\begin{aligned}
\|u_h^{n+1} - v_h^{n+1}\|_{L(0,1)} \le &-\frac12 \Delta\tau^n (\beta^{n-}_{n_x+\frac12} + \sigma^{n-}_{n_x+\frac12}) |u^n_{n_x+1} - v^n_{n_x+1}| \\
&+ \sum_{i=1}^{n_x} (1 - \frac12 \Delta\tau^n (u^n_i + v^n_i)) |u^n_i - v^n_i| \Delta x_i \\
&+ \frac12 \Delta\tau^n (\beta^{n+}_{\frac12} + \sigma^{n+}_{\frac12}) |u^n_0 - v^n_0| \\
&+ \frac12 \Delta\tau^n \sum_{i=0}^{n_x} (|\beta^{n-}_{i+\frac12} - \sigma^{n-}_{i+\frac12}| + |\beta^{n+}_{i+\frac12} - \sigma^{n+}_{i+\frac12}|)(|u^n_{i+1} - u^n_i| + |v^n_{i+1} - v^n_i|),
\end{aligned}
$$

and since $|a^- - b^-| + |a^+ - b^+| = |a - b|$, we obtain

$$
\begin{aligned}
\|u_h^{n+1} - v_h^{n+1}\|_{L(0,1)} \le &-\frac12 \Delta\tau^n (\beta^{n-}_{n_x+\frac12} + \sigma^{n-}_{n_x+\frac12}) |u^n_{n_x+1} - v^n_{n_x+1}| \\
&+ \sum_{i=1}^{n_x} (1 - \frac12 \Delta\tau^n (u^n_i + v^n_i)) |u^n_i - v^n_i| \Delta x_i \\
&+ \frac12 \Delta\tau^n (\beta^{n+}_{\frac12} + \sigma^{n+}_{\frac12}) |u^n_0 - v^n_0| \\
&+ \frac12 \Delta\tau^n \sum_{i=0}^{n_x} |\beta^n_{i+\frac12} - \sigma^n_{i+\frac12}| (|u^n_{i+1} - u^n_i| + |v^n_{i+1} - v^n_i|) \\
\le &\frac12 \Delta\tau^n \left(\|\beta^n_h\|_{L^\infty(0,1)} + \|\sigma^n_h\|_{L^\infty(0,1)} \right) |u^n_{n_x+1} - v^n_{n_x+1}| \\
&+ \sum_{i=1}^{n_x} (1 - \frac12 \Delta\tau^n (u^n_i + v^n_i)) |u^n_i - v^n_i| \Delta x_i \\
&+ \frac12 \Delta\tau^n \left(\|\beta^n_h\|_{L^\infty(0,1)} + \|\sigma^n_h\|_{L^\infty(0,1)} \right) |u^n_0 - v^n_0| \\
&+ \frac12 \Delta\tau^n \|\beta^n_h - \sigma^n_h\|_{L^\infty(0,1)} \left(\|u^n_h\|_{\mathrm{BV}(0,1)} + \|v^n_h\|_{\mathrm{BV}(0,1)} \right).
\end{aligned}
$$

Using Lemma 3.11, we obtain

$$\begin{aligned}\|u_h^{n+1} - v_h^{n+1}\|_{L^1(0,1)} \le{}& \frac{1}{2}\Delta\tau^n \left(\|\beta_h^n\|_{L^\infty(0,1)} + \|\sigma_h^n\|_{L^\infty(0,1)}\right) |u_{n_x+1}^n - v_{n_x+1}^n| \\ &+ \frac{1}{2}\Delta\tau^n \left(\|\beta_h^n\|_{L^\infty(0,1)} + \|\sigma_h^n\|_{L^\infty(0,1)}\right) |u_0^n - v_0^n| \\ &+ \frac{1}{2}\Delta\tau^n \left(\|u_h^n\|_{\mathrm{BV}(0,1)} + \|v_h^n\|_{\mathrm{BV}(0,1)}\right) |\phi_{1,\Delta\tau}^n - \psi_{1,\Delta\tau}^n| \\ &+ \sum_{i=1}^{n_x}(1 - \frac{1}{2}\Delta\tau^n(u_i^n + v_i^n))|u_i^n - v_i^n|\Delta x_i \\ &+ \frac{1}{2}\Delta\tau^n \left(\|u_h^n\|_{\mathrm{BV}(0,1)} + \|v_h^n\|_{\mathrm{BV}(0,1)}\right) \|u_h^n - v_h^n\|_{L^1(0,1)}.\end{aligned}$$

The result follows from the fact that $u_i^n, v_i^n \ge 0$, see Proposition 3.5. □

The following result is a direct consequence of the preceding lemma.

PROPOSITION 3.13 (Continuity with respect to the data). *Suppose that the* CFL *condition* (3.7) *is satisfied (for both sets of data). Then, for* $n = 0, \ldots, n_T$, *we have*

$$\begin{aligned}\|u_h^n - v_h^n\|_{L^1(0,1)} \le{}& e^{\nu\tau^n}\|u_i - v_i\|_{L^1(0,1)} \\ &+ \frac{1}{2}\left(\|\beta_h\|_{L^\infty(L^\infty)} + \|\sigma_h\|_{L^\infty(L^\infty)}\right) \cdot e^{\nu\tau^n} \\ &\cdot \left(\|u_0 - v_0\|_{L^1(0,\tau^n)} + \|u_1 - v_1\|_{L^1(0,\tau^n)}\right) \\ &+ 2\nu e^{\nu\tau^n}\|\phi_1 - \psi_1\|_{L^1(0,\tau^n)},\end{aligned}$$

where

$$\nu = \frac{1}{2}\left(\|u_h\|_{L^\infty(0,T,\mathrm{BV}(0,1))} + \|v_h\|_{L^\infty(0,T;\mathrm{BV}(0,1))}\right).$$

The following result will be needed in §3.d and §4.1.

PROPOSITION 3.14 (Equicontinuity in time). *Suppose that the* CFL *condition* (3.7) *is satisfied. Then, for* $n = 0, \ldots, n_T - 1$ *we have*

$$\|u_h^{n+1} - u_h^n\|_{L^1(0,1)} \le C_6\Delta\tau^n,$$

where

$$C_6 = C_1 (\| \phi_1 \|_{L^\infty(0,T)} + \frac{1}{2}u^*) + \max\{\frac{1}{4}(u^* - 1)u^*\}.$$

The constant C_1 is given in Proposition 3.9.

Proof. From (2.6) and Lemma 3.1, we easily obtain that

$$(u_i^{n+1} - u_i^n)\Delta x_i = -\left\{\beta_{i+1/2}^{n-}(u_{i+1}^n - u_i^n) - (1 - u_i^n)u_i^n\Delta x_i - \beta_{i-1/2}^{n+}(u_{i-1}^n - u_i^n)\right\}\Delta\tau^n.$$

Taking absolute values, summing over i and making some simple algebraic manipulations, we obtain

$$\begin{aligned}\|u_h^{n+1} - u_h^n\|_{L^1(0,1)} &\le \left\{\sum_{i=0}^{n_x} |\beta_{i+1/2}^n|\,|u_{i+1}^n - u_i^n| + |1 - u_i^n|\,|u_i^n\,\Delta x_i\right\}\Delta\tau^n \\ &\le \left\{C_1 \|\beta_h\|_{L^\infty(0,T;L^\infty(0,1))} + \max\{\frac{1}{4}, (u^* - 1)u^*\}\right\}\Delta\tau^n \\ &\le \left\{C_1 (\|\phi_1\|_{L^\infty(0,T)} + u^*/2) + \max\{\frac{1}{4}, (u^* - 1)u^*\}\right\}\Delta\tau^n,\end{aligned}$$

by Propositions 3.4, 3.5 and (2.5). This completes the proof. □

3d. Proofs of Theorems 2.1 and 2.2. First, consider Theorem 2.1. Since the CFL condition (2.9) implies condition (3.4), (2.10a) holds by Proposition 3.5. The inequality (2.10b) follows from (2.10a) and Proposition 3.4. The inequality (2.10c) follows from (2.10a) and equation (2.7a). To obtain (2.10d) and (2.10e), we first note that, by (2.7b),

$$\| \phi_h^n \|_{\mathrm{BV}[0,1]} \le \| \beta_h^n \|_{L^1(0,1)},$$
$$\| \phi_h^n \|_{L^\infty(0,1)} \le \| \beta_h^n \|_{L^1(0,1)}.$$

Thus, (2.10d) and (2.10e) follow from (2.5) and (2.10b). Finally, since the CFL condition (2.9) implies the condition (3.7), the uniform bound (2.10f) follows from Proposition 3.9. This completes the proof of Theorem 2.1.

Theorem 2.2 follows from Proposition 3.13 and Theorem 2.1.

3e. Bounds for $\nu_{x,0}^+$ and $\nu_{x,1}^-$. In this section we show that under the CFL condition (3.7), the quantities $\nu_{x,0}^+(\epsilon, u_h; \beta_h)$ and $\nu_{x,1}^-(\epsilon, u_h; \beta_h)$ can be estimated as stated in Theorem 2.5. We proceed in several steps.

First step: The case $\beta_h(x = 0) \ge \epsilon + \Delta x$. We begin by considering the case in which

$$\beta_{1/2}^n \ge \epsilon + \Delta x, \text{ for } n = 0, \cdots, n_T. \tag{3.8}$$

In what follows we let i_0 be such that $x_{i_0-1/2} \le \epsilon < x_{i_0+1/2}$. Condition (3.8) ensures that the flow of the electrons does not change sign in a neighborhood of $\{x = 0\} \times [0, T]$ uniformly with respect to Δx.

LEMMA 3.15. *Suppose that condition* (3.8) *is satisfied. Then for* $n = 0, \cdots, n_T$ *and* $i = 0, \cdots, i_0$,

$$\beta_{i+1/2}^n \ge 0.$$

Proof. For $i > 0$ we have

$$\begin{aligned}
\beta^n_{i+1/2} &= \beta^n_{1/2} - \sum_{j=1}^{i}(1-u^n_j)\,\Delta x_j, && \text{by Lemma 3.1,}\\
&= \beta^n_{1/2} - x_{i+1/2} + \sum_{j=1}^{i} u^n_j\,\Delta x_j\\
&\geq \beta^n_{1/2} - x_{i+1/2}, && \text{by Proposition 3.5,}\\
&\geq \epsilon - x_{i-1/2}, && \text{by condition (3.8),}\\
&\geq 0, && \text{for } i = 0,\cdots,i_0,
\end{aligned}$$

by the definition of i_0. This completes the proof. □

LEMMA 3.16. *Suppose that condition* (3.8) *is satisfied. Then*

$$\nu^+_{x,0}(\epsilon,u_h;\beta_h) \leq \left\{(u^*)^2\,T + \sup_{1\leq j\leq i_0}\sum_{n=0}^{n_T-1}|\,u^{n+1}_j - u^n_j\,|\right\}(\epsilon+\Delta x).$$

Proof. By definition of $\nu^+_{x,0}(\epsilon,u_h;\beta_h)$ and of i_0, we have

$$\begin{aligned}
\nu^+_{x,0}(\epsilon,u_h;\beta_h) &= \sup_{0\leq\Delta\leq\epsilon}\int_0^T|\,u_h(\tau,\Delta)-u_{0h}(\tau)\,|\,\beta^+_h(\tau,0)\,d\tau\\
&= \sup_{0\leq i\leq i_0}\sum_{n=0}^{n_T-1}|\,u^n_i - u^n_0\,|\,\beta^{n\;+}_{1/2}\,\Delta\tau^n\\
&\leq \sup_{0\leq i\leq i_0}\sum_{n=0}^{n_T-1}\left\{|\beta^n_{1/2}-\beta^n_{i+1/2}|\,u^n_i + |\,\beta^n_{i+1/2}\,u^n_i - \beta^n_{1/2}\,u^n_0\,|\right\}\Delta\tau^n\\
&\leq \sup_{0\leq i\leq i_0}\sum_{n=0}^{n_T-1}\left\{|\beta^n_{1/2}-\beta^n_{i+1/2}|\,u^n_i + \sum_{j=1}^{i}|\,\beta^n_{j+1/2}\,u^n_j - \beta^n_{j-1/2}\,u^n_{j-1}\,|\right\}\Delta\tau^n\\
&\leq (u^*)^2\,T\,(\epsilon+\Delta x) + \sum_{n=0}^{n_T-1}\sum_{j=1}^{i_0}|\,\beta^n_{j+1/2}\,u^n_j - \beta^n_{j-1/2}\,u^n_{j-1}\,|\,\Delta\tau^n,
\end{aligned}$$

by Lemma 3.1 and Proposition 3.5. By Lemma 3.15, we can rewrite the scheme (2.6), for $j \leq i_0$, as follows

$$(\beta^n_{j+1/2}\,u^n_j - \beta^n_{j-1/2}\,u^n_{j-1})\,\Delta\tau^n = -(u^{n+1}_j - u^n_j)\,\Delta x_j,$$

and hence

$$\begin{aligned}
\nu^+_{x,0}(\epsilon,u_h;\beta_h) &\leq (u^*)^2\,T\,(\epsilon+\Delta x) + \sum_{n=0}^{n_T-1}\sum_{j=1}^{i_0}|u^{n+1}_j - u^n_j|\,\Delta x_i\\
&\leq (u^*)^2\,T\,(\epsilon+\Delta x) + \sup_{1\leq j\leq i_0}\left\{\sum_{n=0}^{n_T-1}|u^{n+1}_j - u^n_j|\right\}x_{i_0+1/2}\\
&\leq (u^*)^2\,T\,(\epsilon+\Delta x) + \sup_{1\leq j\leq i_0}\left\{\sum_{n=0}^{n_T-1}|u^{n+1}_j - u^n_j|\right\}(\epsilon+\Delta x),
\end{aligned}$$

by definition of i_0. This completes the proof. □

We now turn to estimate the variation in time of $u_h(x_j)$, $\sum_{n=0}^{n_T-1}|u^{n+1}_j - u^n_j|$. We first obtain a basic estimate from which the wanted estimate will follow.

LEMMA 3.17 (Basic Estimate). *Suppose that the condition* (3.8) *and the* CFL *condition* (3.7) *are satisfied. Then, for* $i = 1,\ldots,i_0$ (*and* $n_T \geq 2$),

$$\begin{aligned}
\sum_{n=0}^{n_T-1}|u^{n+1}_i - u^n_i - (1-u^n_i)u^n_i\Delta\tau^n| &\leq |u^0_i - u^0_{i-1}|\\
&\quad + \sum_{n=0}^{n_T-2}|u^{n+1}_{i-1} - u^n_{i-1} - (1-u^n_{i-1})u^n_{i-1}\Delta\tau^n|\\
&\quad - \sum_{n=0}^{n_T-1}(1-u^n_i-u^n_{i-1})|u^n_i - u^n_{i-1}|\Delta\tau^n.
\end{aligned}$$

Proof. Using Lemma 3.15, we can rewrite (2.6) as follows:

$$u^{n+1}_i = \Big(1-\frac{\Delta\tau^n}{\Delta x_i}\beta^n_{i+\frac12}\Big)\,u^n_i + \Big(\frac{\Delta\tau^n}{\Delta x_i}\beta^n_{i-\frac12}\Big)\,u^n_{i-1}, \qquad i = 1,\ldots,i_0.$$

If we use Lemma 3.1 to write $\beta^n_{i-\frac12}$ in terms of $\beta^n_{i+\frac12}$, and set

$$a^n_i = \frac{\Delta\tau^n}{\Delta x_i}\beta^n_{i+\frac12} + \Delta\tau^n, \tag{3.9a}$$

$$\Gamma^n_i = 1 + (1-u^n_{i-1})\Delta\tau^n, \tag{3.9b}$$

we can rewrite the equation for the numerical scheme as follows:

$$u^{n+1}_i = (\Gamma^n_i - a^n_i)\,u^n_i + a^n_i\,u^n_{i-1}, \qquad i = 1,\cdots,i_0.$$

If we set

$$c^{n,j}_i = \begin{cases} 1 & \text{for } j = n+1,\\ \Pi^n_{\ell=j}(\Gamma^\ell_i - a^\ell_i) & \text{for } j \leq n,\end{cases} \tag{3.9c}$$

a simple computation leads us to the following expression:

$$u_i^{n+1} = c_i^{n,0} u_i^0 + \sum_{j=0}^{n} c_i^{n,j+1} a_i^j u_{i-1}^j .$$

Since by definition

$$c_i^{n,j+1} a_i^j = \Gamma_i^j c_i^{n,j+1} - c_i^{n,j}, \qquad j = 0, \ldots, n,$$

we have

$$\begin{aligned} u_i^{n+1} &= c_i^{n,0} u_i^0 + \sum_{j=0}^{n} (\Gamma_i^j c_i^{n,j+1} - c_i^{n,j}) u_{i-1}^j \\ &= c_i^{n,0}(u_i^0 - u_{i-1}^0) + \sum_{j=0}^{n-1} c_i^{n,j+1} (\Gamma_i^j u_{i-1}^j - u_{i-1}^{j+1}) + \Gamma_i^n u_{i-1}^n, \end{aligned}$$

and hence, for $n \geq 1$,

$$\begin{aligned} u_i^{n+1} - u_i^n = {} & (c_i^{n,0} - c_i^{n-1,0})(u_i^0 - u_{i-1}^0) \\ & - \sum_{j=0}^{n-2} (c_i^{n,j+1} - c_i^{n-1,j+1})(u_{i-1}^{j+1} - \Gamma_i^j u_{i-1}^j) \\ & - c_i^{n,n}(u_{i-1}^n - \Gamma_i^{n-1} u_{i-1}^{n-1}) + \Gamma_i^n u_{i-1}^n - \Gamma_i^{n-1} u_{i-1}^{n-1} \\ = {} & (c_i^{n,0} - c_i^{n-1,0})(u_i^0 - u_{i-1}^0) \\ & - \sum_{j=0}^{n-1} (c_i^{n,j+1} - c_i^{n-1,j+1})(u_{i-1}^{j+1} - \Gamma_i^j u_{i-1}^j) \\ & - (u_{i-1}^n - \Gamma_i^{n-1} u_{i-1}^{n-1}) + \Gamma_i^n u_{i-1}^n - \Gamma_i^{n-1} u_{i-1}^{n-1} . \end{aligned}$$

The form of this expression suggests to consider the expression $(u_i^{n+1} - \Gamma_{i+1}^n u_i^n)$ instead of the expression $(u_i^{n+1} - u_i^n)$. Thus,

$$\begin{aligned} (u_i^{n+1} - \Gamma_{i+1}^n u_i^n) = {} & (c_i^{n,0} - c_i^{n-1,0})(u_i^0 - u_{i-1}^0) \\ & - \sum_{j=0}^{n-1} (c_i^{n,j+1} - c_i^{n-1,j+1})(u_{i-1}^{j+1} - \Gamma_i^j u_{i-1}^j) \\ & + (1 - \Gamma_{i+1}^n) u_i^n - (1 - \Gamma_i^n) u_{i-1}^n \\ = {} & (c_i^{n,0} - c_i^{n-1,0})(u_i^0 - u_{i-1}^0) \\ & - \sum_{j=0}^{n-1} (c_i^{n,j+1} - c_i^{n-1,j+1})(u_{i-1}^{j+1} - \Gamma_i^j u_{i-1}^j) \\ & - \Delta\tau^n (1 - u_i^n - u_{i-1}^n)(u_i^n - u_{i-1}^n). \end{aligned}$$

Notice that this formula holds for $n = 0$, if we define $c_i^{-1,0} = 1$. Notice also that by the CFL condition (3.7) and the equations (3.9) we have, for $j = 0, \ldots, n$,

$$\text{(3.10a)} \qquad c_i^{n,j} \in [0,1],$$

$$\text{(3.10b)} \qquad \begin{aligned} c_i^{n,j} - c_i^{n-1,j} &= (\Gamma_i^n - a_i^n - 1)\, c_i^{n-1,j} \\ &= -\left(\frac{\Delta\tau^n}{\Delta x_i} \beta_{i+\frac{1}{2}}^n + \Delta\tau^n u_{i-1}^n \right) c_i^{n-1,j} \\ &\leq 0. \end{aligned}$$

We can thus take absolute values and obtain, by (3.10b),

$$\begin{aligned} |u_i^{n+1} - \Gamma_{i+1}^n u_i^n + \Delta\tau^n (1 - u_i^n - u_{i-1}^n)(u_i^n - u_{i-1}^n)| \leq {} & (c_i^{n-1,0} - c_i^{n,0}) |u_i^0 - u_{i-1}^0| \\ & + \sum_{j=0}^{n-1} (c_i^{n-1,j+1} - c_i^{n,j+1}) |u_{i-1}^{j+1} - \Gamma_i^j u_{i-1}^j| . \end{aligned}$$

Let us consider the term in the left-hand side. Since

$$\begin{aligned} u_i^{n+1} - \Gamma_{i+1}^n u_i^n &= -\left\{ \frac{\Delta\tau^n}{\Delta x_i} \beta_{i+\frac{1}{2}}^n + \Delta\tau^n (1 - u_i^n) \right\} (u_i^n - u_{i-1}^n) \\ &= -\left\{ \frac{\Delta\tau^n}{\Delta x_i} \beta_{i-\frac{1}{2}}^n \right\} (u_i^n - u_{i-1}^n), \text{ by Lemma 3.1,} \end{aligned}$$

and

$$\begin{aligned} (u_i^{n+1} - \Gamma_{i+1}^n u_i^n) + \left\{ \Delta\tau^n (1 - u_i^n - u_{i-1}^n)(u_i^n - u_{i-1}^n) \right\} \\ = -\left\{ \frac{\Delta\tau^n}{\Delta x_i} \beta_{i+\frac{1}{2}}^n + \Delta\tau^n u_{i-1}^n \right\} (u_i^n - u_{i-1}^n), \end{aligned}$$

and since, by the CFL condition (3.7), the terms between braces are nonnegative, it is clear that

$$\begin{aligned} |(u_i^{n+1} - \Gamma_{i+1}^n u_i^n) + \Delta\tau^n (1 - u_i^n - u_{i-1}^n)(u_i^n - u_{i-1}^n)| \\ = |u_i^{n+1} - \Gamma_{i+1}^n u_i^n| + \Delta\tau^n (1 - u_i^n - u_{i-1}^n) |u_i^n - u_{i-1}^n|, \end{aligned}$$

and thus

$$\begin{aligned} |u_i^{n+1} - \Gamma_{i+1}^n u_i^n| \leq {} & (c_i^{n-1,0} - c_i^{n,0}) |u_i^0 - u_{i-1}^0| \\ & + \sum_{j=0}^{n-1} (c_i^{n-1,j+1} - c_i^{n,j+1}) |u_{i-1}^{j+1} - \Gamma_i^j u_{i-1}^j| \\ & - \Delta\tau^n (1 - u_i^n - u_{i-1}^n) |u_i^n - u_{i-1}^n| . \end{aligned}$$

Summing over n, we get

$$\begin{aligned} \sum_{n=0}^{n_T - 1} |u_i^{n+1} - \Gamma_{i+1}^n u_i^n| \leq {} & (c_i^{-1,0} - c_i^{n_T - 1,0}) |u_i^0 - u_{i-1}^0| \\ & + \sum_{n=1}^{n_T - 1} \sum_{j=1}^{n} (c_i^{n-1,j} - c_i^{n,j}) |u_{i-1}^j - \Gamma_i^{j-1} u_{i-1}^{j-1}| \\ & - \sum_{n=0}^{n_T - 1} \Delta\tau^n (1 - u_i^n - u_{i-1}^n) |u_i^n - u_{i-1}^n| . \end{aligned}$$

Rearranging, we get

$$\sum_{n=0}^{n_T-1} |u_i^{n+1} - \Gamma_{i+1}^n u_i^n| \le \left(c_i^{-1,0} - c_i^{n_T-1,0}\right) |u_i^0 - u_{i-1}^0|$$
$$+ \sum_{j=1}^{n_T-1} \Big[\sum_{n=j}^{n_T-1} (c_i^{n-1,j} - c_i^{n,j}) \Big] |u_{i-1}^j - \Gamma_i^{j-1} u_{i-1}^{j-1}|$$
$$- \sum_{n=0}^{n_T-1} \Delta\tau^n (1 - u_i^n - u_{i-1}^n) |u_i^n - u_{i-1}^n|$$
$$\le \left(c_i^{-1,0} - c_i^{n_T-1,0}\right) |u_i^0 - u_{i-1}^0|$$
$$+ \sum_{j=1}^{n_T-1} \left(c_i^{j-1,j} - c_i^{n_T-1,j}\right) |u_{i-1}^j - \Gamma_i^{j-1} u_{i-1}^{j-1}|$$
$$- \sum_{n=0}^{n_T-1} \Delta\tau^n (1 - u_i^n - u_{i-1}^n) |u_i^n - u_{i-1}^n|.$$

Finally, using (3 10a), we get

$$\sum_{n=0}^{n_T-1} |u_i^{n+1} - \Gamma_{i+1}^n u_i^n| \le |u_i^0 - u_{i-1}^0| + \sum_{n=0}^{n_T-2} |u_{i-1}^{n+1} - \Gamma_i^n u_{i-1}^n|$$
$$- \sum_{n=0}^{n_T-1} \Delta\tau^n (1 - u_i^n - u_{i-1}^n) |u_i^n - u_{i-1}^n|.$$

This completes the proof. □

When the solution has the monotonicity property of Corollary 3.7, a sharp estimate on the total variation in time can be obtained.

COROLLARY 3.18. *Let u_i, u_0 and u_1 be such that*

$$u_i(x) \in [0,1], x \in (0,1), \text{ is a nondecreasing function},$$
$$u_0(\tau) = 0, \tau \in (0,T),$$
$$u_1(\tau) = 1, \tau \in (0,T),$$

and suppose that the CFL *condition* (3.7) *is satisfied Then, for* $\Delta x \le \inf_{\tau\in[0,T]} \phi_1(\tau)/2$, *we have*

$$\sum_{n=0}^{n_T-1} |u_i^{n+1} - u_i^n| \le 1, \quad \text{for } x_i \le \inf_{\tau\in[0,T]} \phi_1(\tau)/4.$$

Proof It is easy to verify that under the hypotheses on the data, condition (3.8) is satisfied for ϵ and Δx such that $\inf_{\tau\in[0,T]} \phi_1(\tau) \ge \Delta x + \epsilon$. Set $\epsilon = \inf_{\tau\in[0,T]} \phi_1(\tau)/2$. Then the condition (3.8) is satisfied for $\Delta x \le \inf_{\tau\in[0,T]} \phi_1(\tau)/2$ and, by Lemma 3 17, we have, for $i = 1, \cdots, i_0$,

$$\sum_{n=0}^{n_T-1} |u_i^{n+1} - u_i^n| \le \sum_{n=0}^{n_T-1} |u_i^{n+1} - u_i^n - (1-u_i^n) u_i^n \Delta\tau^n| + \sum_{n=0}^{n_T-1} |(1-u_i^n) u_i^n| \Delta\tau^n$$
$$\le \sum_{j=1}^{i} |u_j^0 - u_{j-1}^0| + \sum_{n=0}^{n_T-1} |u_0^{n+1} - u_0^n - (1-u_0^n) u_0^n \Delta\tau^n|$$
$$- \sum_{n=0}^{n_T-1} \sum_{j=1}^{i} (1 - u_j^n - u_{j-1}^n) |u_j^n - u_{j-1}^n| \Delta\tau^n$$
$$+ \sum_{n=0}^{n_T-1} |(1-u_i^n) u_i^n| \Delta\tau^n.$$

By (2.5) and the hypothesis on the initial data, $\sum_{j=1}^{i} |u_j^0 - u_{j-1}^0| \le 1$. Since $u_0^n = 0$, the second term of the right-hand side is zero; also, by Corollary 3.7, u_h^n is nondecreasing, and we have

$$\sum_{n=0}^{n_T-1} \sum_{j=1}^{i} (1 - u_j^n - u_{j-1}^n) |u_j^n - u_{j-1}^n| \Delta\tau^n = \sum_{n=0}^{n_T-1} \sum_{j=1}^{i} (1 - u_j^n - u_{j-1}^n)(u_j^n - u_{j-1}^n) \Delta\tau^n$$
$$= \sum_{n=0}^{n_T-1} \sum_{j=1}^{i} (u_j^n - (u_j^n)^2) \Delta\tau^n$$
$$= \sum_{n=0}^{n_T-1} (1 - u_i^n) u_i^n \Delta\tau^n$$
$$= \sum_{n=0}^{n_T-1} |(1 - u_i^n) u_i^n| \Delta\tau^n.$$

This proves the inequality for $x_i \le x_{i_0}$. Finally, if $x_i \le \inf_{\tau\in[0,T]} \phi_1(\tau)/4 = \epsilon/2$, then $x_{i+1/2} \le \epsilon/2 + \Delta x/2 = \epsilon \le x_{i_0+1/2}$, and so $x_i \le x_{i_0}$. This completes the proof. □

In the general case the estimate of the total variation in time takes a more complicated form.

LEMMA 3.19 (Estimate of the total variation in time). *Suppose that the condition* (3.8) *and the* CFL *condition* (3.7) *are satisfied. Then, for* $i = 0, \ldots, i_0$,

$$\sum_{n=0}^{n_T-1} |u_i^{n+1} - u_i^n| \le C_1 + C_7(0,T),$$

where

$$C_7(\tau_1,\tau_2) = \|u_0\|_{BV(\tau_1,\tau_2)} + \|1-u_0\|_{L^\infty(\tau_1,\tau_2)} \|u_0\|_{L^1(\tau_1,\tau_2)}$$
$$+ (2u^* - 1)(\tau_2 - \tau_1) C_1 + (\tau_2 - \tau_1) \max\{1/4, (u^* - 1) u^*\}.$$

Proof. From Lemma 3.17 and Proposition 3.5 we get

$$\begin{aligned}\sum_{n=0}^{n_T-1} |u_i^{n+1} - u_i^n| \le \|u_h(\tau = 0)\|_{BV(0,1)} \\ + \sum_{n=0}^{n_T-1} |u_0^{n+1} - u_0^n - (1-u_0^n)\, u_0^n\, \Delta\tau^n| \\ + (2u^* - 1)\, T\, \|u_h\|_{L^\infty(0,T,BV(0,1))} \\ + T \max\left\{\frac{1}{4}, (u^*-1)\, u^*\right\}.\end{aligned}$$

The result follows from (2.5), Proposition 3.9, and (2.5c). □

From Lemmas 3.16 and 3.19 we immediately obtain the following result.

COROLLARY 3.20. *Suppose that condition* (3.8) *and the* CFL *condition* (3.7) *are satisfied. Then*

$$\nu_{x,0}^+(\epsilon, u_h; \beta_h) \le ((u^*)^2\, T + C_1 + C_7(0,T))\, (\epsilon + \Delta x).$$

Second step: The general case. To treat the general case, we divide the interval $[0,T)$ into intervals on which we have information about the size of the flux $\beta_h(x=0)$. We need to introduce some notation. For a given nonnegative number δ and a given partition of the computational domain we introduce the following sets:

$$\begin{aligned}\mathcal{J}_h(\delta) &= \{n : \beta_{1/2}^n \le \delta\},\\ \mathcal{L}_h(\delta) &= \cup_{m\in\mathcal{J}_h(\delta)} [\tau^m, \tau^{m+1})\\ &= \cup_{j=1}^{N_{h,\delta,T}-1} [\tau_{h,\delta}^{n_{2j-1}}, \tau_{h,\delta}^{n_{2j}}),\\ [0,T)\setminus\mathcal{L}_h(\delta) &= \cup_{j=1}^{N_{h,\delta,T}} [\tau_{h,\delta}^{n_{2j-2}}, \tau_{h,\delta}^{n_{2j-1}}).\end{aligned}$$

Notice that $\beta(\tau, x=0) \le \delta$ if and only if $\tau \in \mathcal{L}_h(\delta)$. The complement of $\mathcal{L}_h(\delta)$ is made of $N_{h,\delta,T}$ disjoint intervals on which $\beta(x=0) > \delta$.

LEMMA 3.21. *Suppose that the* CFL *condition* (3.7) *is satisfied. Then, for every* $\delta \ge \epsilon + \Delta x$,

$$\nu_{x,0}^+(\epsilon, u_h; \beta_h) \le C_8\, \delta + N_{h,\delta,T}\, C_1\, (\epsilon + \Delta x),$$

where $C_8 = u^*\, T + (u^*)^2\, T + C_7(0,T)$.

Proof By definition of $\nu_{x,0}^+(\epsilon, u_h; \beta_h)$ we have

$$\nu_{x,0}^+(\epsilon, u_h; \beta_h) = \sup_{0\le\Delta\le\epsilon} \{\Theta_1(\Delta) + \Theta_2(\Delta)\},$$

where

$$\Theta_1(\Delta) = \int_{\mathcal{L}_h(\delta)} |u_h(\tau,\Delta) - u_{0,\Delta x}(\tau)|\, |\beta_h^+(\tau,0)\, d\tau,$$

$$\Theta_2(\Delta) = \int_{(0,T)\setminus\mathcal{L}_h(\delta)} |u_h(\tau,\Delta) - u_{0,\Delta x}(\tau)|\, |\beta_h^+(\tau,0)\, d\tau.$$

Since, by construction, $\beta_h(\tau, x=0) \le \delta$, for $\tau^n \in \mathcal{L}_h(\delta)$, we have that

$$\Theta_1(\Delta) \le u^*\, T\, \delta.$$

On the other hand, for $\tau \in (0,T)\setminus\mathcal{L}_h(\delta)$, $\beta_h(\tau, x=0) > \delta \ge (\epsilon + \Delta x)$, and we have, by Corollary 3.20,

$$\begin{aligned}\Theta_2(\Delta) &= \int_{(0,T)\setminus\mathcal{L}_h(\delta)} |u_h(\tau,\Delta) - u_{0,\Delta x}(\tau)|\, |\beta_h^+(\tau,0)\, d\tau\\ &= \sum_{j=1}^{N_{h,\delta,T}} \int_{\tau_{h,\delta}^{n_{2j-2}}}^{\tau_{h,\delta}^{n_{2j-1}}} |u_h(\tau,\Delta) - u_{0,\Delta x}(\tau)|\, |\beta_h^+(\tau,0)\, d\tau\\ &\le \left\{ \sum_{j=1}^{N_{h,\delta,T}} ((u^*)^2\, |\tau_{h,\delta}^{n_{2j-1}} - \tau_{h,\delta}^{n_{2j-2}}| + C_1 + C_7(\tau_{h,\delta}^{n_{2j-2}}, \tau_{h,\delta}^{n_{2j-1}})) \right\} (\epsilon + \Delta x).\end{aligned}$$

Finally,

$$\Theta_2(\Delta) \le \left\{ (u^*)^2\, T + C_7(0,T) + N_{h,\delta,T}\, C_1 \right\} (\epsilon + \Delta x),$$

by the definition of C_7 in Lemma 3.19. This completes the proof. □

Third step: the case (2.12a).

PROPOSITION 3.22 (Estimate of $\nu_{x,0}^+(\epsilon, u_h; \beta_h)$). *Suppose that the* CFL *condition* (3.7) *is satisfied. If the hypothesis* (2.12a) *is satisfied, then,*

$$\nu_{x,0}^+(\epsilon, u_h; \beta_h) \le C_4\, (\epsilon + \Delta x),$$

where the constant $C_4 = C_8 + C_1 \sup_{h>0, 0<\delta\le\epsilon+\Delta x} N_{h,\delta,T}$ *is independent of* ϵ *and* Δx.

Proof. From Lemma 3.21, it is clear that we only have to prove that the number $N_{h,\delta,T}$ is uniformly bounded. Since the number of intervals on which the boundary data is constant, N, is finite, it is enough to prove the result for the data being constant on the whole interval $(0,T)$.

From (2.7), we have

$$\begin{aligned}\beta_h^n(0) &= \phi_1 + \int_0^1 (u_h^n(s) - 1)(s-1)\,ds \\ &\geq \phi_1,\end{aligned}$$

by the hypothesis (2.12a) and the maximum principle (2.10a) of Theorem 2.1. Thus, $N_{h,\delta,T} \leq 1$ provided $\phi_1 \neq 0$.

To analyze the case $\phi_1 = 0$, we have to use a different argument. For the sake of clarity, let us consider the continuous equations (1.3) and (1.4) instead of the discrete equations (2.6) and (2.7). From (1.3) and (1.4) we easily obtain the following equations:

$$\begin{aligned}\frac{d\beta(0)}{dt} &= \eta^2/2 + \eta\,\beta(0), \\ \frac{d\eta}{dt} &= -u_1\,(\eta + \beta(0)),\end{aligned}$$

where $\eta = \int_0^1 (u(s) - 1)\,ds$. Notice that to obtain these equations, we have to use the fact that $u_0 = 0$ and that

$$\beta(1) = \int_0^1 (u_h^n(s) - 1)\,s\,ds \leq 0,$$

since $u^* = 1$. A simple analysis of the above dynamical system allows us to conclude that the number of times $\frac{\beta(0)}{dt} = 0$ is at most one. Thus, $\beta(0) = \epsilon > 0$ at most twice for ϵ small enough. The discrete equations (2.6) and (2.7) lead to a similar dynamical system which, for Δx small enough, displays the same property. Thus $N_{h,\delta,T}$ can be uniformly bounded. □

Fourth step: the case (2.12b). Next we prove that when ϕ_1 satisfies the hypothesis (2.12b), the number $N_{h,\delta,T}$ in Lemma 3.22 can be estimated in terms of $(\epsilon + \Delta x)$ only. It is enough to obtain the bound assuming $N = 1$.

We use the following auxiliary result that carries the information of the negative electric field β_h being a uniformly Lipschitz continuous function in time.

We recall that, by definition, the set $\mathcal{L}_h(\delta)$ is the union of intervals $[\tau_{h,\delta}^{n_{2j}-1}, \tau_{h,\delta}^{n_{2j}})$ on which $\beta_{1/2} \leq \delta$. Let us consider a subset $\tilde{\mathcal{L}}_h(\delta)$ of $\mathcal{L}_h(\delta)$ which is the union of those intervals $[\tau_{h,\delta}^{n_{2j}-1}, \tau_{h,\delta}^{n_{2j}})$ on which the minimum of $\beta_{1/2}$ is smaller than, or equal to, $\delta/2$. Let us denote by $\tilde{N}_{h,\delta,T} - 1$ the number of those intervals.

LEMMA 3.23. *We have that*

$$\tilde{N}_{h,\delta,T} \leq 2\,C_9\,T/\delta,$$

where $C_9 = (C_6 + \|(\phi_1)_\tau\|_{L^\infty(0,T)})$.

Proof. By Lemma 3.11 and Proposition 3.14, we have that

$$\|\,\beta_h(\tau^m) - \beta_h(\tau^n)\,\|_{L^\infty(0,1)} \leq C_9\,|\tau^m - \tau^n|.$$

Let $\tau^n \in \tilde{\mathcal{L}}_h(\delta)$ be such that $\beta_{1/2}(\tau^n) \leq \delta/2$, and let τ^m be in $[0,T) \setminus \tilde{\mathcal{L}}_h(\delta)$, i.e., such that $\beta_{1/2}(\tau^m) > \delta$. Then

$$\begin{aligned}|\tau^m - \tau^n| &\geq \|\,\beta_h(\tau^m) - \beta_h(\tau^n)\,\|_{L^\infty(0,1)}/C_9 \\ &\geq (\,|\,\beta_{1/2}(\tau^m)\,| - |\,\beta_{1/2}(\tau^n)\,|\,)/C_9 \\ &\geq \delta/2\,C_9.\end{aligned}$$

Hence, $T \geq \tilde{N}_{h,\delta,T}\,\delta/2\,C_9$. This completes the proof. □

COROLLARY 3.24. *We have that*

$$\nu_{x,0}^+(\epsilon, u_h; \beta_h) \leq \hat{C}_4\,(\epsilon + \Delta x)^{1/2},$$

where $\hat{C}_4 = 2\,\{2\,C_1\,C_8\,C_9\,T\}^{1/2}$, *provided that* $(\epsilon + \Delta x) \leq 2\,C_1\,C_9\,T/C_8$.

Proof. If we proceed as in Lemma 3.21 with $\tilde{\mathcal{L}}_h(\delta)$ instead of $\mathcal{L}_h(\delta)$, we easily obtain that, for $\delta \geq (\epsilon + \Delta x)$,

$$\begin{aligned}\nu_{x,0}^+(\epsilon, u_h; \beta_h) &\leq C_8\,\delta + \tilde{N}_{h,\delta,T}\,C_1\,(\epsilon + \Delta x) \\ &\leq C_8\,\delta + 2\,C7\,C_1\,T\,(\epsilon + \Delta x)/\delta,\end{aligned}$$

by Lemma 3.23. Minimizing over δ, we get

$$\nu_{x,0}^+(\epsilon, u_h; \beta_h) \leq \{8\,C_1\,C_8\,C_9\,T\}^{1/2}\,(\epsilon + \Delta x)^{1/2},$$

provided that $\delta_{\min} = \{2\,C_1\,C_9\,T/C_8\}^{1/2}\,(\epsilon + \Delta x)^{1/2} \geq (\epsilon + \Delta x)$, i.e., provided that $(\epsilon + \Delta x) \leq 2\,C_1\,C_9\,T/C_8$. This completes the proof. □

3f. Proof of Theorem 2.5. The inequality (2.13a) follows directly from Proposition 3.22 and the inequality (2.14a) from Corollary 3.24. The inequalities (2.13b) and (2.14b) can be obtained by completely similar arguments. This ends the proof of Theorem 2.5.

4. PROOFS OF THEOREMS 2.3 AND 2.4

In this section we prove Theorems 2.3 and 2.4. In §4a, we prove that there is a subsequence $\{\,(u_{h'}, \beta_{h'}, \phi_{h'})\,\}_{h'>0}$ converging to a limit (u, β, ϕ) satisfying the weak equations

(2.3). In §4b, §4c, and §4d, we obtain results that allow us to prove that (u, β) satisfies the weak equations (2.2). In §4b, we prove that for $\varphi \in \mathcal{C}^1([0,T) \times [0,1])$ and $c \in \mathbf{R}$

$$\lim_{h'\to 0} \Theta(u_{h'}, c; \beta_{h'}; \varphi) = \Theta(u, c; \beta; \varphi),$$

provided that

$$\lim_{h'\to 0} \Theta(u_{h'}, c; \beta_{h'}; \varphi) \leq 0.$$

In §4c and §4d we obtain a bound of the form

$$\Theta(u_{h'}, c; \beta_{h'}; \varphi) \leq L\, C_2\, (\Delta x/\epsilon + \Delta\tau/\epsilon_0) + M\, C_3\, \Delta\tau.$$

With these results, Theorems 2.3 and 2.4 can be easily proven. In §4e we show how to do that.

A delicate point in this procedure is how to take the limit in the boundary terms of the form Θ. This was done in [25] in the framework of classical conservation laws. Another point we want to stress is the necessity of using entropies U which are smoother than the classical entropy $U(u) = |u|$; see [13]. Notice how the second derivative of the entropy function U is needed in the second inequality of Theorem 2.4.

4a. Existence of a converging subsequence. First, we reduce the problem of finding a converging subsequence $\{(u_{h'}, \beta_{h'}, \phi_{h'})\}_{h'>0}$ to the problem of finding a converging subsequence $\{u_{h'}\}_{h'>0}$.

LEMMA 4.1. *Suppose there is a subsequence $\{u_{h'}\}_{h'>0}$ converging in $L^\infty(0,T; L^1(0,1))$ to a limit $u \in L^\infty(0,T; \mathrm{BV}(0,1)) \cap \mathcal{C}^0(0,T; L^1(0,1))$. Then, the subsequence $\{(u_{h'}, \beta_{h'}, \phi_{h'})\}_{h'>0}$ converges in*

$$L^\infty(0,T; L^1(0,1)) \times L^1(0,T; W^{1,1}(0,1)) \times L^1(0,T; BV(0,1))$$

to a limit (u, β, ϕ) satisfying the equations (2.3).

Proof. By (2.7a), and by (2.7b) with $v_h \equiv 1$, we have

$$(\beta_{h'_1} - \beta_{h'_2})_x = u_{h'_1} - u_{h'_2},$$

$$\int_0^1 (\beta_{h'_1} - \beta_{h'_2}) = \mathbf{P}_{\Delta\tau'_1}\phi_1 - \mathbf{P}_{\Delta\tau'_2}\phi_1.$$

This implies that $\{\beta_{h'}\}_{h'>0}$ is a Cauchy sequence in $L^1(0,T; W^{1,1}(0,1))$. (Notice that since ϕ_1 belongs to $\mathrm{BV}(0,T)$, ϕ_1 can have discontinuities. This forces us to consider convergence in $\mathrm{L}^1(0,T)$ instead of in $\mathrm{L}^\infty(0,T)$.) Denote by β the limit of the above sequence. By (2.7a), and by (2.7b) with $v_h \equiv 1$, we easily see that β satisfies (2.3a) and (2.3b) with $v \equiv 1$ a.e. in $(0,T)$.

By using (2.7b) and a similar argument, we can see that $\{\phi_{h'}\}_{h'>0}$ is a Cauchy sequence in $L^1(0,T; BV(0,1))$ and that its limit ϕ satisfies (2.3b) a.e. in $(0,T)$. □

Now we prove that a converging subsequence $\{u_{h'}\}_{h'>0}$ exists.

LEMMA 4.2. *Suppose that the* CFL *condition* (2.9) *is satisfied. Then there exists a subsequence $\{u_{h'}\}_{h'>0}$ converging in $L^\infty(0,T; L^1(0,1))$ to a function u in $L^\infty(0,T; BV(0,1)) \cap \mathcal{C}^0(0,T; L^1(0,1))$.*

Proof. To prove this result, we only have to follow Crandall and Majda [15], who used a discrete version of the Ascoli-Arzelà Theorem. The equicontinuity in time is provided by Proposition 3.19 and the compactness of the range is given by Proposition 3.9. □

Thus we have the following immediate result.

COROLLARY 4.3. *Suppose that the* CFL *condition* (2.9) *is satisfied. Then there exists a subsequence $\{(u_{h'}, \beta_{h'}, \phi_{h'})\}_{h'>0}$ converging as indicated in Theorem 2.3 to a limit (u, β, ϕ) satisfying* (2.3). *Moreover, $u \in L^\infty(0,T; BV(0,1)) \cap \mathcal{C}^0(0,T; L^1(0,1))$.*

4b. Passing to the limit in $\Theta(u_{h'}, c; \beta_{h'}; \varphi)$. In this section we prove the following result.

LEMMA 4.4. *Suppose that for every $c \in \mathbf{R}$ and for every nonnegative $\varphi \in \mathcal{C}^1([0,T] \times [0,1])$,*

$$\lim_{h'\to 0} \Theta(u_{h'}, c; \beta_{h'}; \varphi) \leq 0.$$

Then

$$\lim_{h'\to 0} \Theta(u_{h'}, c; \beta_{h'}; \varphi) = \Theta(u, c; \beta; \varphi).$$

Proof. By a standard argument, we have that for $c \in \mathbf{R}$ and for every nonnegative $\varphi \in \mathcal{C}^1_0([0,T] \times (0,1))$,

$$\lim_{h'\to 0} \Theta(u_{h'}, c; \beta_{h'}; \varphi) = \Theta(u, c; \beta; \varphi).$$

Now, consider the case in which $\varphi \in \mathcal{C}^1_0((0,T) \times [0,1))$; it is enough to consider $\varphi(\tau, x)$ of the form $\omega(\tau)\eta(x)$. Notice that since, by Lemma 4.1, $\{\beta_{h'}\}_{h'>0}$ converges to β in $L^1(0,T; W^{(1,1)}(0,1))$, $\{\beta_{h'}(\cdot,0)\}_{h'>0}$ converges to $\beta(\cdot,0)$ in $L^1(0,T)$. Also, notice that since, by Theorem 2.1, the sequence $\{u_{h'}(\cdot,0+)\}_{h'>0}$ is included in $L^\infty(0,T)$, there is a subsequence $\{u_{h''}(\cdot,0+)\}_{h''>0}$ converging in $L^\infty(0,T)$ weak* to a limit $\tilde{u}$. Let γ_τ be the Young measure associated with $\tilde{u}$. Thus,

$$\begin{aligned}\lim_{h''\to 0} \Theta(u_{h''}, c; \beta_{h''}; \varphi) = &- \int_0^T \int_0^1 U(u(\tau,x) - c)\omega'(\tau)\eta(x)\, dx d\tau \\ &- \int_0^1 g(x)\eta'(x) dx - g_0\eta(0) \\ &- \int_0^T \int_0^1 (\beta_x) V(u(\tau,x), c)\omega(\tau)\eta(x)\, dx d\tau,\end{aligned}$$

where

$$g(x) = \int_0^T U(u(\tau,x)-c)\beta(\tau,x)\omega(\tau)d\tau, \tag{4.1a}$$

$$g_0 = \int_0^T \{\beta^+(\tau,0)U(u_0(\tau)-c)+\beta^-(\tau,0)v(\tau)\}\,\omega(\tau)d\tau, \tag{4.1b}$$

and

$$v(\tau) = \int_0^{u^*} U(\lambda - c)\,d\gamma_\tau(\lambda).$$

To prove

$$\lim_{h''\to 0} \Theta(u_{h''}, c; \beta_{h''}; \varphi) = \Theta(u, c; \beta; \varphi),$$

we only have to prove that $g_0 = g_0^\star$, where

$$g_0^\star = \int_0^T \{\beta^+(\tau,0)U(u_0(\tau)-c)+\beta^-(\tau,0)U(u(\tau,0+)-c)\}\,\omega(\tau)d\tau. \tag{4.2}$$

Take η such that its support is contained in $[0,\varepsilon]$. By our hypothesis and Theorem 2.1,

$$-\int_0^1 g(x)\eta'(x)dx - g_0\,\eta(0) \le C\,\varepsilon\,\|\,\eta\,\|_{L^1(0,1)}.$$

Since ε is arbitrary, this implies that

$$g(0+) - g_0 \le 0. \tag{4.1c}$$

Notice that since $g \in BV(0,1)$, by (4.1a) and Theorem 2.1, the limit $g(0+)$ exits. Next, take U and c such that $U(u-c) = \alpha\,(u-c)$ for $u \in [0,u^*]$ and $\alpha \in \mathbf{R}$. Then

$$v(\tau) = \alpha\,(\tilde{u} - c),$$

and so, by (4.1a,b), the last inequality reads as follows:

$$\alpha\left\{\int_0^T (u(\tau,0+)-u_0(\tau))\beta^+(\tau,0)\omega(\tau)d\tau + \int_0^T (u(\tau,0+)-\tilde{u}(\tau))\beta^-(\tau,0)\omega(\tau)d\tau\right\} \le 0.$$

Since the sign of α is arbitrary and this holds for every nonnegative $\omega \in C_0^1(0,T)$, this implies that for τ a.e. in $[0,T]$,

$$\beta^+(\tau,0)\big(u(\tau,0+)-u_0(\tau)\big)= 0, \tag{4.3a}$$

$$\beta^-(\tau,0)\big(u(\tau,0+)-\tilde{u}(\tau)\big)= 0 \tag{4.3b}$$

Thus, we have

$$\begin{aligned}
0 &\ge g(0+) - g_0, \quad \text{by (4.1c)},\\
&= g_0^\star - g_0, \quad \text{by (4.1a), (4.2), and (4.3a)},\\
&= \int_0^T \big[U(u(\tau,0+)-c) - v(\tau)\big]\,\beta^-(\tau,0)\,\omega(\tau)\,d\tau, \quad \text{by (4.1b), and (4.2)},\\
&= \int_0^T \big[U(\tilde{u}(\tau)-c) - v(\tau)\big]\,\beta^-(\tau,0)\,\omega(\tau)\,d\tau, \quad \text{by (4.3b)},\\
&= \int_0^T \big[U(\int_0^{u*}(\lambda-c)\,d\gamma_\tau(\lambda)) - v(\tau)\big]\,\beta^-(\tau,0)\,\omega(\tau)\,d\tau\\
&\ge \int_0^T \big[\int_0^{u*} U(\lambda-c)\,d\gamma_\tau(\lambda) - v(\tau)\big]\,\beta^-(\tau,0)\,\omega(\tau)\,d\tau, \quad \text{by Jensen's inequality},\\
&= 0, \quad \text{by the definition of } v.
\end{aligned}$$

This proves that $g_0^\star = g_0$, and ends the proof for the case $\varphi \in C_0^1([0,T]\times[0,1))$.

A similar argument shows that the result is also true for $\varphi \in C^1([0,T]\times(0,1])$. This completes the proof. □

4c. A discrete entropy inequality. In this section we obtain a discrete entropy inequality, which is crucial for obtaining an upper bound for $\Theta(u_{h'}, c; \beta_{h'}; \varphi)$.

PROPOSITION 4.5 (A discrete entropy inequality). *Suppose that the* CFL *condition* (2.9) *is satisfied. Then for every* $c \in \mathbf{R}$,

$$U(u_i^{n+1}-c) - U(u_i^n - c) + \frac{\Delta\tau^n}{\Delta x_i}(G^n_{i+\frac12} - G^n_{i-\frac12}) - ((\beta_h)_x)_i^n V(u_i^{n+1}, c)\Delta\tau^n \le 0,$$

where

$$\begin{aligned}
G^n_{i+\frac12} &= \beta^{n+}_{i+\frac12} U(u_i^n - c) + \beta^{n-}_{i+\frac12} U(u^n_{i+1} - c),\\
V(u_i^{n+1}, c) &= U(u_i^{n+1}-c) - u_i^{n+1} U'(u_i^{n+1}-c).
\end{aligned}$$

Proof. By the definition of the scheme (2.6), and by the fact that β_h^n is piecewise linear, we have, for $c \in \mathbf{R}$,

$$\begin{aligned}
u_i^{n+1} - c = \{-\frac{\Delta\tau^n}{\Delta x_i}\beta^{n-}_{i+\frac12}\}(u^n_{i+1}-c) + \{1 - \frac{\Delta\tau^n}{\Delta x_i}(\beta^{n+}_{i+\frac12} - \beta^{n-}_{i-\frac12})\}(u_i^n - c)\\
+ \{\frac{\Delta\tau^n}{\Delta x_i}\beta^{n+}_{i-\frac12}\}(u^n_{i-1} - c) - ((\beta_h)_x)_i^n \Delta\tau^n c.
\end{aligned}$$

Since, by the CFL condition (2.9), the terms between braces are nonnegative, we obtain, after multiplying by $\operatorname{sgn}(u_i^{n+1} - c)$,

$$|u_i^{n+1} - c| \le |u_i^n - c| - \frac{\Delta\tau^n}{\Delta x_i}(F^n_{i+\frac12} - F_{i-\frac12}) + ((\beta_h)_x)_i^n W(u_i^{n+1}, c)\,\Delta\tau^n,$$

where

$$F^n_{i+\frac12} = \beta^{n+}_{i+\frac12}\,|u^n_i - c| + \beta^{n^-}_{i+\frac12}\,|u^n_{i+1} - c|,$$
$$W(u^{n+1}_i, c) = -\mathrm{sgn}(u^{n+1}_i - c)\, c.$$

This proves the result in the case in which $U(\omega) = |\omega|$. To prove the result for a general even entropy function U, with Lipschitz second-order derivative having support in $[-\alpha, \alpha]$ and such that $U(0) = 0$, we use the following representation of U:

$$U(u) = \int_{-\alpha}^{\alpha} j(s)\{|u - s| - |s|\}ds,$$

where $j = \frac12 U''$. We replace c by $c + s$ in the above inequality, multiply by $j(s) \geq 0$ (since U is convex), and integrate over $[-\alpha, \alpha]$. The result follows from the above representation result and from the fact that

$$V(u^{n+1}_i, c) = \int_{-\alpha}^{\alpha} j(s)\{-\mathrm{sgn}(u^{n+1}_i - (c+s))(c+s) - |s|\}ds. \quad \square$$

In §4d we shall need a result concerning our estimate of the Lipschitz constant of $u \mapsto V(u, c)$.

LEMMA 4.6. *We have*

$$|V(u_1, c) - V(u_2, c)| \leq M \max\{|u_1|, |u_2|\}|u_1 - u_2|,$$

where $M = \sup_{u\in\mathbf{R}} |U''(u)|$.

Proof. By hypothesis, $U(0) = 0$, and since U is even and $C^1, U'(0) = 0$. Hence, it is easy to see that

$$V(u, c) = U(u - c) - uU'(u - c)$$
$$= -\int_0^{u-c} (c + s)U''(s)ds.$$

Hence,

$$V(u_1, c) - V(u_2, c) = -\int_{u_2-c}^{u_1-c} (c + s)U''(s)ds,$$

and the result follows. $\square$

4d. An upper bound for $\Theta(u_{h'}, c; \beta_{h'}; \varphi)$. To obtain an upper bound for $\Theta(u_{h'}, c; \beta_{h'}; \varphi)$, we decompose $\Theta(u_{h'}, c; \beta_{h'}; \varphi)$ as the sum $\Theta_{\mathrm{ent}}(u_{h'}, c; \beta_{h'}; \varphi) + \Theta_{\mathrm{comp}}(u_{h'}, c; \beta_{h'}; \varphi)$; this is done in Lemma 4.7. The upper bound for $\Theta_{\mathrm{ent}}(u_{h'}, c; \beta_{h'}; \varphi)$ is obtained by using the discrete entropy inequality of Proposition 4.5 and the upper bound for $\Theta_{\mathrm{comp}}(u_{h'}, c; \beta_{h'}; \varphi)$ by using the compactness properties obtained in §3; this is done in Lemma 4.8.

LEMMA 4.7 (Decomposition of $\Theta(u_{h'}, c; \beta_{h'}; \varphi)$). *We have*

$$\Theta(u_{h'}, c; \beta_{h'}; \varphi) = \Theta_{\mathrm{ent}}(u_{h'}, c; \beta_{h'}; \varphi) + \Theta_{\mathrm{comp}}(u_{h'}, c; \beta_{h'}; \varphi),$$

where

$$\Theta_{\mathrm{ent}}(u_{h'}, c; \beta_{h'}; \varphi) = \sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}\{U(u^{n+1}_i - c) - U(u^n_i - c) + \frac{\Delta\tau^n}{\Delta x_i}(G^n_{i+\frac12} - G^n_{i-\frac12}) - ((\beta_{h'})_x)^n_i V(u^{n+1}_i, c)\Delta\tau^n\}\varphi^{n+1}_i \Delta x_i,$$

and

$$\begin{aligned}\Theta_{\mathrm{comp}}(u_{h'}, c; \beta_{h'}; \varphi) = \sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}&\{[U(u^n_{i+1} - c) - U(u^n_i - c)](-\beta^{n^-}_{i+\frac12})[\varphi^{n+1}_i - \varphi^{n+\frac12}_{i+\frac12}]\\ &+ [U(u^n_{i-1} - c) - U(u^n_i - c)](\beta^{+}_{i-\frac12})[\varphi^{n+1}_i - \varphi^{n+\frac12}_{i-\frac12}]\}\Delta\tau^n\\ &- \sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)^n_i u^n_i U'(u^n_i - c)\,[\varphi^{n+1}_i - \varphi^{n+\frac12}_i]\Delta x_i\Delta\tau^n\\ &+ \sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)^n_i\,(V(u^{n+1}_i, c) - V(u^n_i, c))\,\varphi^{n+1}_i\Delta x_i\Delta\tau^n,\end{aligned}$$

where

$$\varphi^{n+1}_i = \frac{1}{\Delta x_i}\int_{x_{i-\frac12}}^{x_{i+\frac12}} \varphi(\tau^{n+1}, x)\,dx,$$
$$\varphi^{n+\frac12}_{i+\frac12} = \frac{1}{\Delta\tau^n}\int_{\tau^n}^{\tau^{n+1}} \varphi(\tau, x_{i+\frac12})\,d\tau,$$
$$\varphi^{n+\frac12}_i = \frac{1}{\Delta\tau^n}\int_{\tau^n}^{\tau^{n+1}} \frac{1}{\Delta x_i}\int_{x_{i-\frac12}}^{x_{i+\frac12}} \varphi(\tau, x)\,dx\,d\tau.$$

Proof. From the definition of the form $\Theta(u_h, c; \beta_h; \varphi)$, (2.11b), and the fact that $u_{h'}$ and $(\beta_{h'})_x$ are piecewise constant functions, we get

$$\Theta(u_{h'}, c; \beta_{h'}; \varphi) = \Psi_t(u_{h'}, c; \beta_{h'}; \varphi) + \Psi_x(u_{h'}, c; \beta_{h'}; \varphi) + \Psi(u_{h'}, c; \beta_{h'}; \varphi),$$

where

$$\Psi_t(u_{h'},c;\beta_{h'};\varphi) = -\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x} U(u_i^n - c)(\varphi_i^{n+1} - \varphi_i^n)\Delta x_i$$
$$+\sum_{i=1}^{n_x} U(u_i^{n_T} - c)\varphi_i^{n_T}\Delta x_i - \sum_{i=1}^{n_x} U(u_i^0 - c)\varphi_i^0 \Delta x_i,$$
$$\Psi_x(u_{h'},c;\beta_{h'};\varphi) = -\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x} U(u_i^n - c)(\beta^n_{i+\frac12}\varphi_{i+\frac12}^{n+\frac12} - \beta^n_{i-\frac12}\varphi_{i-\frac12}^{n+\frac12})\,\Delta\tau^n$$
$$+\sum_{n=0}^{n_T-1} G^n_{n_x+\frac12}\,\varphi^{n+\frac12}_{n_x+\frac12}\,\Delta\tau^n - \sum_{n=0}^{n_T-1} G^n_{\frac12}\,\varphi^{n+\frac12}_{\frac12}\,\Delta\tau^n,$$
$$\Psi(u_{h'},c;\beta_{h'};\varphi) = \sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)_i^n\, u_i^n\, U'(u_i^n - c)\,\varphi_i^{n+\frac12}\,\Delta x_i\,\Delta\tau^n.$$

Consider Ψ_t. After a simple integration by parts, we get

$$\Psi_t(u_{h'},c;\beta_{h'};\varphi) = \sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}\big(U(u_i^{n+1} - c) - U(u_i^n - c)\big)\,\varphi_i^{n+1}\Delta x_i,$$

and by the definition of $\Theta_{\text{ent}}(u_{h'},c;\beta_{h'};\varphi)$,

$$\Psi_t(u_{h'},c;\beta_{h'};\varphi) = \Theta_{\text{ent}}(u_{h'},c;\beta_{h'};\varphi)$$
$$-\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}(G^n_{i+\frac12} - G^n_{i-\frac12})\,\varphi_i^{n+1}\,\Delta\tau^n$$
$$+\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)_i^n\, V(u_i^{n+1},c)\,\varphi_i^{n+1}\,\Delta\tau^n\Delta x_i.$$

Since β_h is piecewise linear in space, $((\beta_{h'})_x)_i^n = (\beta^n_{i+1/2} - \beta^n_{i-1/2})/\Delta x_i$, and so

$$\Psi_t(u_{h'},c;\beta_{h'};\varphi) = \Theta_{\text{ent}}(u_{h'},c;\beta_{h'};\varphi)$$
$$-\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}\{[G^n_{i+\frac12} - U(u_i^n - c)\,\beta^n_{i+\frac12}]\,\varphi_i^{n+1}$$
$$-[G^n_{i-\frac12} - U(u_i^n - c)\,\beta^n_{i-\frac12}]\,\varphi_i^{n+1}\}\,\Delta\tau^n$$
$$-\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)_i^n\, U(u_i^n - c)\,\varphi_i^{n+1}\,\Delta\tau^n\Delta x_i$$
$$+\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)_i^n\, V(u_i^{n+1},c)\,\varphi_i^{n+1}\,\Delta\tau^n\Delta x_i.$$

Since $G^n_{i+1/2} = U(u^n_{i+1} - c)\,\beta^-_{i+1/2} + U(u_i^n - c)\,\beta^+_{i+1/2}$, by (2.11c), and $U(u_i^n - c) = V(u_i^n,c) + u_i^n U'(u_i^n - c)$, by (2.11d), we get

$$\Psi_t(u_{h'},c;\beta_{h'};\varphi) = \Theta_{\text{ent}}(u_{h'},c;\beta_{h'};\varphi)$$
$$+\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}\{[U(u^n_{i+1} - c) - U(u_i^n - c)](-\beta^{n^-}_{i+\frac12})\,\varphi_i^{n+1}$$
$$+[U(u^n_{i-1} - c) - U(u_i^n - c)](\beta^{n^+}_{i-\frac12})\,\varphi_i^{n+1}\}\,\Delta\tau^n$$
$$-\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)_i^n\, u_i^n\, U'(u_i^n - c)\,\varphi_i^{n+1}\Delta x_i\Delta\tau^n$$
$$+\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)_i^n(V(u_i^{n+1},c) - V(u_i^n,c))\,\varphi_i^{n+1}\Delta x_i\Delta\tau^n.$$

By the definition of $\Psi(u_{h'},c;\beta_{h'};\varphi)$, we have

$$\Psi_t(u_{h'},c;\beta_{h'};\varphi) = \Theta_{\text{ent}}(u_{h'},c;\beta_{h'};\varphi) - \Psi(u_{h'},c;\beta_{h'};\varphi)$$
$$+\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}\{[U(u^n_{i+1} - c) - U(u_i^n - c)]\,(-\beta^{n^-}_{i+\frac12})\,\varphi_i^{n+1}$$
$$+[U(u^n_{i-1} - c) - U(u_i^n - c)]\,(\beta^{n^+}_{i+\frac12})\,\varphi_i^{n+1}\}\,\Delta\tau^n$$
$$-\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)_i^n\, u_i^n\, U'(u_i^n - c)\,[\varphi_i^{n+1} - \varphi_i^{n+\frac12}]\,\Delta x_i\Delta\tau^n$$
$$+\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}((\beta_{h'})_x)_i^n\,(V(u_i^{n+1},c) - V(u_i^n,c))\,\varphi_i^{n+1}\,\Delta x_i\Delta\tau^n.$$

Finally, by the definition of $\Theta_{\text{comp}}(u_{h'},c;\beta_{h'};\varphi)$, we have

$$\Psi_t(u_{h'},c;\beta_{h'};\varphi) = \Theta_{\text{ent}}(u_{h'},c;\beta_{h'};\varphi) + \Theta_{\text{comp}}(u_{h'},c;\beta_{h'};\varphi) - \Psi(u_{h'},c;\beta_{h'};\varphi)$$
$$+\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}\{[U(u^n_{i+1} - c) - U(u_i^n - c)]\,(-\beta^{n^-}_{i+\frac12})\,\varphi^{n+\frac12}_{i+\frac12}$$
$$+[U(u^n_{i-1} - c) - U(u_i^n - c)]\,(\beta^{n^+}_{i-\frac12})\,\varphi^{n+\frac12}_{i-\frac12}\}\Delta\tau^n$$
$$= \Theta_{\text{ent}}(u_{h'},c;\beta_{h'};\varphi) + \Theta_{\text{comp}}(u_{h'},c;\beta_{h'};\varphi) - \Psi(u_{h'},c;\beta_{h'};\varphi)$$
$$-\sum_{n=0}^{n_T-1}\sum_{i=1}^{n_x}\{[G^n_{i+\frac12} - U(u_i^n - c)\,\beta^n_{i+\frac12}]\,\varphi^{n+\frac12}_{i+\frac12}$$
$$-[G^n_{i-\frac12} - U(u_i^n - c)\,\beta^n_{i-\frac12}]\,\varphi^{n+\frac12}_{i-\frac12}\}\Delta\tau^n$$

$$\begin{aligned}
&= \Theta_{\rm ent}(u_{h'}, c; \beta_{h'}; \varphi) + \Theta_{\rm comp}(u_{h'}, c; \beta_{h'}; \varphi) - \Psi(u_{h'}, c; \beta_{h'}; \varphi) \\
&\quad + \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} U(u_i^n - c)\,(\beta^n_{i+\frac12}\, \varphi^{n+\frac12}_{i+\frac12} - \beta^n_{i-\frac12}\, \varphi^{n+\frac12}_{i-\frac12})\, \Delta\tau^n \\
&\quad - \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} (G^n_{i+\frac12}\, \varphi^{n+\frac12}_{i+\frac12} - G^n_{i-\frac12}\, \varphi^{n+\frac12}_{i-\frac12})\, \Delta\tau^n \\
&= \Theta_{\rm ent}(u_{h'}, c; \beta_{h'}; \varphi) + \Theta_{\rm comp}(u_{h'}, c; \beta_{h'}; \varphi) \\
&\quad - \Psi(u_{h'}, c; \beta_{h'}; \varphi) - \Psi_x(u_{h'}, c; \beta_{h'}; \varphi),
\end{aligned}$$

by the definition of $\Psi_x(u_{h'}, c; \beta_{h'}; \varphi)$. This completes the proof. ∎

We are now ready to obtain the upper bounds we were looking for.

LEMMA 4.8 (Upper bounds for $\Theta_{\rm ent}(u_{h'}, c; \beta_{h'}; \varphi)$ and $\Theta_{\rm comp}(u_{h'}, c; \beta_{h'}; \varphi)$). *Suppose that the CFL condition (2.9) is satisfied. Then*

$$\begin{aligned}
\Theta_{\rm ent}(u_{h'}, c, \beta_{h'}; \varphi) &\le 0, \\
\Theta_{\rm comp}(u_{h'}, c; \beta_{h'}; \varphi) &\le L\, \hat C_2\, \left(\Delta x\, \|\varphi_x\|_{L^\infty(0,T,L^\infty(0,1))} + \Delta\tau\, \|\varphi_\tau\|_{L^\infty(0,T,L^\infty(0,1))}\right) \\
&\quad + M\, \hat C_3\, \Delta\tau \|\varphi\|_{L^\infty((0,T)\times(0,1))},
\end{aligned}$$

where $L = \sup_{u\in\mathbf{R}} |U'(u)|$, $M = \sup_{u\in\mathbf{R}} |U''(u)|$, *and*

$$\begin{aligned}
\hat C_2 &= \{(\phi_1^* + \frac12 \max\{1, u^* - 1\})C_1 + \frac12 u^* \max\{1, u^* - 1\}\}\, T, \\
\hat C_3 &= u^* \max\{1, u^* - 1\}\, T\, C_6.
\end{aligned}$$

The constant C_6 is given in Proposition 3.14.

Proof. The first inequality follows immediately from Lemma 4.7 and the discrete entropy inequality of Proposition 4.5.

To estimate $\Theta_{\rm comp}(u_{h'}, c; \beta_{h'}; \varphi)$, we proceed as follows. By Lemmas 4.7 and 4 6,

(4.4)

$$\begin{aligned}
\Theta_{\rm comp}(u_{h'}, c; \beta_{h'}; \varphi) &\le L\, \|\beta_{h'}\|_{L^\infty(0,T\, L^\infty(0,1))} \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} \{|u^n_{i+1} - u^n_i|\,|\varphi_i^{n+1} - \varphi^{n+\frac12}_{i+\frac12}| + \\
&\qquad |u_i^n - u^n_{i-1}|\,|\varphi_i^{n+1} - \varphi^{n+\frac12}_{i-\frac12}|\}\Delta\tau^n \\
&\quad + L\, \|(\beta_{h'})_x\|_{L^\infty(0,T,L^\infty(0,1))} \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} u_i^n\, |\varphi_i^{n+1} - \varphi_i^{n+\frac12}|\Delta x_i \Delta\tau^n \\
&\quad + M\, \|(\beta_{h'})_x\|_{L^\infty(0,T,L^\infty(0,1))}\, u^* \sum_{n=0}^{n_T-1} \sum_{i=1}^{n_x} |u_i^{n+1} - u_i^n|\, \varphi_i^{n+1} \Delta x_i \Delta\tau^n,
\end{aligned}$$

where $L = \sup_{u\in\mathbf{R}} |U'(u)|$ and $M = \sup_{u\in\mathbf{R}} |U''(u)|$.

Since (see Lemma 4.7)

$$\begin{aligned}
|\varphi_i^{n+1} - \varphi^{n+\frac12}_{i+\frac12}| &\le \frac12 \Delta x_i\, \|\varphi_x\|_{L^\infty(0,T,L^\infty(0,1))} + \frac12 \Delta\tau^n\, \|\varphi_\tau\|_{L^\infty(0,T,L^\infty(0,1))}, \\
|\varphi_i^{n+1} - \varphi_i^{n+\frac12}| &\le \frac12 \Delta\tau^n\, \|\varphi_\tau\|_{L^\infty(0,T,L^\infty(0,1))},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\Theta_{\rm comp}(u_{h'}, c; \beta_{h'}; \varphi) \le \\
&\quad L\, \|\beta_{h'}\|_{L^\infty(0,T,L^\infty(0,1))}\, \|u_{h'}\|_{L^\infty(0,T;{\rm BV}(0,1))}\, T \\
&\qquad \cdot \left(\Delta x \|\varphi_x\|_{L^\infty(0,T;L^\infty(0,1))} + \Delta\tau\, \|\varphi_\tau\|_{L^\infty(0,T,L^\infty(0,1))}\right) \\
&\quad + \frac12 L\, \|(\beta_{h'})_x\|_{L^\infty(0,T,L^\infty(0,1))}\, \|u_{h'}\|_{L^\infty(0,T,L^1(0,1))}\, T\, \Delta\tau\, \|\varphi_\tau\|_{L^\infty(0,T,L^\infty(0,1))} \\
&\quad + M \|(\beta_{h'})_x\|_{L^\infty(0,T,L^\infty(0,1))}\, u^*\, T \sup_{0\le n\le n_T-1} \|u^{n+1}_{h'} - u^n_{h'}\|_{L^1(0,1)}\, \|\varphi\|_{L^\infty((0,T)\times(0,1))},
\end{aligned}$$

and the result follows by using Theorem 2.1 and Proposition 3.14. ∎

We end this section with the following result.

COROLLARY 4.9 (An upper bound for $E^{\epsilon,\epsilon_0}(u_{h'}, v; \beta_{h'})$). *Suppose that the CFL condition (2.9) is satisfied. Then*

$$E^{\epsilon,\epsilon_0}(u_{h'}, v; \beta_{h'}) \le L\, C_2\, \left(\Delta x/\epsilon + \Delta\tau/\epsilon_0\right) + M\, C_3\, \Delta\tau,$$

where $L = \sup_{u\in\mathbf{R}} |U'(u)|$, $M = \sup_{u\in\mathbf{R}} |U''(u)|$, $C_2 = C_0\, \hat C_2$, $C_3 = C_0\, \hat C_3$, *and* C_0 *depends solely on the function* w *used to define the form* E^{ϵ,ϵ_0}.

Proof. In the definition of E^{ϵ,ϵ_0}, (2.11), the function φ is a function of four variables:

$$\varphi(\tau, x; \tau', x') = \frac{1}{\epsilon_0}\, w(\frac{\tau - \tau'}{\epsilon_0})\, \frac{1}{\epsilon}\, w(\frac{x - x'}{\epsilon}),$$

two of which are ‘frozen’ in the form Θ; see (2.11). Taking this into consideration, we can easily obtain that

$$\begin{aligned}
\int_0^T \int_0^1 |\varphi_i^{n+1} - \varphi^{n+\frac12}_{i+\frac12}| &\le \frac12 C_0 \Delta x_i/\epsilon + \frac12 C_0 \Delta\tau^n/\epsilon_0, \\
\int_0^T \int_0^1 |\varphi_i^{n+1} - \varphi_i^{n+\frac12}| &\le \frac12 C_0 \Delta\tau^n/\epsilon_0, \\
\int_0^T \int_0^1 \varphi_i^{n+1} &\le 1,
\end{aligned}$$

where C_0 depends solely on the function w.

The result follows easily from the above inequalities, (4.4), (2.11a) and the proof of Lemma 4.8. ∎

4e. Proof of Theorems 2.3 and 2.4. First, let us consider Theorem 2.3. By Corollary 4.3, and Lemmas 4.4 and 4.8, there is a subsequence $\{ (u_{h'}, \beta_{h'}, \phi_{h'}) \}_{h'>0}$ converging (in the topology indicated by Theorem 2.3) to a limit (u, β, ϕ) satisfying (2.3) a.e. in $(0,T)$ such that for $c \in \mathbf{R}$ and $\varphi \in \mathcal{C}^1([0,T] \times [0,1])$,

$$\Theta(u, c; \beta; \varphi) = \lim_{h' \to 0} \Theta(u_{h'}, c; \beta_{h'}; \varphi) \leq 0.$$

This implies that (u, β, ϕ) is the unique solution of (2.2), (2.3). As a consequence, the whole sequence $\{ (u_h, \beta_h, \phi_h) \}_{h>0}$ converges to (u, β, ϕ), and thus Theorem 2.3 is proved.

The first inequality of Theorem 2.4 follows from the above inequality and (2.11). The second inequality follows from Corollary 4.9.