ANALYSIS OF A ROBUST FINITE ELEMENT APPROXIMATION FOR A PARABOLIC EQUATION WITH ROUGH BOUNDARY DATA

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ABSTRACT. The approximation of parabolic equations with nonhomogeneous Dirichlet boundary data by a numerical method that consists of finite elements for the space discretization and the backward Euler time discretization is studied. The boundary values are assumed in a least squares sense. It is shown that this method achieves an optimal rate of convergence for rough (only L^2) boundary data and for smooth data as well. The results of numerical computations which confirm the robust theoretical error estimates are also presented.

1. INTRODUCTION

Consider the initial boundary value problem

(1)
$$\begin{aligned} y_t + Ay &= 0 \quad \text{in } \Omega \times [0, T], \\ y &= g \quad \text{on } \Gamma \times [0, T], \\ y(\cdot, 0) &= v \quad \text{on } \Omega, \end{aligned}$$

where Ω is an open bounded convex polygonal domain in $\mathbf{R}^2$ with boundary Γ. We assume the elliptic operator

$$A = - \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i} \left(a_{i,j}(x) \frac{\partial}{\partial x_j} \right)$$

has smooth coefficients, say $C^2(\Omega)$, and the 2×2 symmetric matrix with entries $a_{i,j}$ is uniformly positive definite on Ω.

In this paper we are primarily concerned with rough boundary data g which belong to either the space $L^\infty(0, T; L^2(\Gamma))$ or $L^2(0, T; L^2(\Gamma))$. This is typical of certain problems in control theory where the control g has the bang-bang property (see [17]). As such our scheme for approximating the solution of (1) is a building block for solving these control problems.

For Neumann boundary control problems of parabolic type the finite element approximation has been analyzed by Winther [22]. For related time-optimal control problems see Knowles [16].

Ground-breaking work on parabolic control problems with boundary Dirichlet control was done by Lasiecka [17, 18, 19]. Approximation by finite element methods of problem (1) was also considered by Lasiecka [17, 18] and Choudury [6]. One such method that is analyzed in [18, 6] uses piecewise linear elements in space which vanish on Γ. Optimal-order convergence is proved for the continuous time method in [17] and for a fully discrete scheme in [6]. This nonstandard approach, while optimal for rough g, is suboptimal for smooth g.

The scheme we propose and analyze is optimal for both rough and smooth boundary data. Our approximation u consists of piecewise constants in time and finite elements in space, and assumes the boundary values in a least squares sense. Specifically, let V_k denote the space of piecewise constant functions on a partition $0 = t_0 < t_1 < \cdots < t_N = T$, where $t_n = nk$ and $k > 0$ is the time step. Then $\psi \in V_k$ if $\psi = \sum_{j=1}^{N} \psi_j \chi_{I_j}$, where χ_{I_j} is the characteristic function of $I_j = (t_{j-1}, t_j]$. We denote by V_h a finite element space on Ω with parameter h. In our estimates with rough boundary data we require

$$(2) \qquad\qquad k = ch^2;$$

however, this restriction is not necessary in our analysis for smooth boundary data.

To define our scheme, we need to introduce some L^2 projections. Let

$$V_h^0 = \{\chi \in V_h : \chi = 0 \text{ on } \Gamma\}$$

and $V_h(\Gamma)$ denote the restriction of V_h to Γ. Define the L^2 projections $Q_h : L^2(\Gamma) \to V_h(\Gamma)$ and $P_k : L^2(0, T) \to V_k$, and set $Q = Q_h P_k = P_k Q_h$. We denote on I_n

$$P_k w = P_k^n w = \frac{1}{k} \int_{I_n} w(t) \, dt.$$

Our numerical method is as follows: Find $u \in V_h \otimes V_k$ such that

$$(3) \qquad \sum_{n=1}^{N} [(u^n - u^{n-1}, \phi) + ka(u^n, \phi)] = 0, \qquad \forall \phi \in V_h^0 \otimes V_k,$$

with $u = Qg$ on $\Sigma = \Gamma \times (0, T]$, $u^0 = \mathscr{P}_h^0 v$, where u^n is the restriction of u to $\Omega \times I_n$, $\mathscr{P}_h^0 : L^2(\Omega) \to V_h$ is the L^2 projection,

$$a(w, z) = \int_\Omega \left(\sum_{i,j=1}^{2} a_{i,j}(x) \frac{\partial w}{\partial x_i} \frac{\partial z}{\partial x_j} \right) dx,$$

and $(w, z) = \int_\Omega w z \, dx$. In Theorem 1 we will consider $v \in H^{-1/2}(\Omega)$, bounded linear functionals on $H^{1/2}(\Omega)$, and define $\mathscr{P}_h^0 : H^{-1/2}(\Omega) \to V_h$ by

$$v(\phi) = (\mathscr{P}_h^0 v, \phi), \qquad \phi \in V_h.$$

Subsequently we denote the duality pairing for $v \in H^{-1/2}(\Omega)$ and $\phi \in H^{1/2}(\Omega)$ by (v, ϕ).

It is straightforward to demonstrate that (3) is uniquely solvable since for each $n = 1, 2, \ldots, N$

$$(u^n - u^{n-1}, \phi) + ka(u^n, \phi) = 0$$

for all $\phi \in V_h^0$, and on Γ

$$u^n = \frac{1}{k} \int_{I_n} Q_h g(\cdot, t)\, dt.$$

This method is equivalent to backward Euler and is the simplest discontinuous Galerkin method (see, for example, [14]). For more general discontinuous Galerkin methods, see [8, 9, 15]. Observe that defining the approximate boundary data through interpolation would be inappropriate, since the data, g, is not continuous.

We will prove the following error estimates.

Theorem 1. *There exists a constant C independent of h, u, and y such that if* (2) *holds then*

(4) $$\|y - u\|_{L^2(0, T; L^2(\Omega))} \leq Ch^{1/2}(\|g\|_{L^2(\Sigma)} + \|v\|_{H^{-1/2}(\Omega)})$$

for $v \in H^{-1/2}(\Omega)$ *and* $g \in L^2(\Sigma)$.

Theorem 2. *For any $\varepsilon > 0$ there exists a constant C independent of h, u, and y such that if* (2) *holds then*

(5) $$\max_{0 \leq n \leq N} \|(y - u)(\cdot, t_n)\|_{L^2(\Omega)} \leq Ch^{1/2-\varepsilon}(\|g\|_{L^\infty(0, T; L^2(\Gamma))} + \|v\|_{H^{1/2}(\Omega)}),$$

where $v \in H^{1/2}(\Omega)$ *and* $g \in L^\infty(0, T; L^2(\Gamma))$.

We denote by C a positive generic constant that is independent of h, k, and the data pair (v, g).

A key ingredient in the proofs of these error estimates is the orthogonal decomposition of V_h: $V_h = V_h^0 \oplus (V_h^0)^\perp$, where

$$(V_h^0)^\perp = \{\phi \in V_h : (\phi, \chi) = 0, \ \chi \in V_h^0\}$$

or

$$(V_h^0)^\perp = \{\phi \in V_h : a(\phi, \chi) = 0, \ \chi \in V_h^0\}.$$

Following Bramble, Pasciak, and Schatz [3], we refer to the latter choice of $(V_h^0)^\perp$ as discrete A-harmonic functions.

The outline of the remainder of the paper is as follows. In §2 we discuss problem (1) in our setting with Ω a convex polygonal domain. We present a weak formulation for this problem that is suitable for our analysis and obtain a priori estimates in $L^2(0, T; L^2(\Omega))$ and $L^\infty(0, T; L^2(\Omega))$. In §3 we state the approximation-theoretic and inverse properties of V_h needed in the proofs of the error estimates. We also derive certain useful estimates for functions in $(V_h^0)^\perp$. In §4 we establish a stability estimate for the method and prove the L^2 error estimate. Section 5 is devoted to the L^∞ error analysis. In §6 we sketch the proofs for optimal L^2 and L^∞ error estimates when y is smooth. Finally, in the last section we present some numerical experiments.

2. REGULARITY

Lasiecka proved in [17] that there exists a unique solution to (1) on a domain Ω with smooth boundary Γ which satisfies an a priori L^∞ in time estimate (see (21)). Lions and Magenes [20] proved similar results which were L^2 in time but still require a smooth boundary. We shall need both L^∞ and L^2 in

time estimates in the case where Ω is a convex polygon. Although we suspect such results are known to specialists in partial differential equations we could not find them in the literature. Therefore, we sketch the proofs of these and related results for use in later sections.

We denote by $H^m(\Omega)$ the usual Sobolev space of integer order $m \geq 0$ with norm $\|\cdot\|_m$. Note that $H^0(\Omega) = L^2(\Omega)$. Similarly, $H^r(\Gamma)$ denotes the Sobolev space of integer order $r \geq 0$ on Γ with norm $|\cdot|_r$, and on $H^0(\Gamma) = L^2(\Gamma)$ the inner product is given by

$$\langle w, z \rangle = \int_\Gamma wz\, d\sigma.$$

As usual, the Sobolev space of order one with functions that have trace equal to zero on Γ is denoted $H_0^1(\Omega)$. Also, $H^{-1}(\Gamma)$ is the dual of $H^1(\Gamma)$, and $H^{-1}(\Omega)$ is the dual of $H^1(\Omega)$. For real s the spaces $H^s(\Omega)$ and $H^s(\Gamma)$ are defined by interpolation.

We will have occasion to use the following norm interpolation inequality (see Proposition 2.3 in Chapter 1, §2 of [20] or [5, Theorem 3.2.3, p. 180])

$$(6) \qquad \|w\|_{\theta r + (1-\theta)s} \leq C\|w\|_r^\theta \|w\|_s^{1-\theta},$$

where $0 \leq \theta \leq 1$ and $0 \leq r, s \leq 2$. Also, from Grisvard [13, Theorem 1.5.10] we have for $\varepsilon > 0$ and $z \in H^1(\Omega)$ the inequality

$$(7) \qquad |z|_0^2 \leq \frac{C}{\varepsilon}\|z\|_1^2 + \varepsilon\|z\|_0^2.$$

From this it follows, for $v \in H^2(\Omega)$, that

$$\left|\frac{\partial v}{\partial \nu_A}\right|_0^2 \leq \frac{C}{\varepsilon}\|v\|_2^2 + \varepsilon\|v\|_1^2,$$

where

$$\frac{\partial v}{\partial \nu_A} = \sum_{i,j=1}^2 a_{i,j}(x)\frac{\partial v}{\partial x_j}\nu_i$$

and $\nu = (\nu_1, \nu_2)$ is the unit outward normal to Γ. Applying (6) to the second term on the right side and then the arithmetic-geometric mean inequality, we obtain

$$(8) \qquad \left|\frac{\partial v}{\partial \nu_A}\right|_0^2 \leq \frac{C}{\varepsilon}\|v\|_2^2 + \varepsilon^3\|v\|_0^2.$$

On $\mathscr{Q} = \Omega \times [0, T]$ let $H^{s,r}(\mathscr{Q}) = L^2(0, T; H^s(\Omega)) \cap H^r(0, T; L^2(\Omega))$ with norm

$$\|w\|_{s,r} = \left(\int_0^T \|w(\cdot, t)\|_s^2\, dt + \int_\Omega \|w(x, \cdot)\|_{r,[0,T]}^2\, dx\right)^{1/2},$$

where $\|\cdot\|_{r,[0,T]}$ denotes the norm on $H^r([0, T])$. Similarly, $H^{s,r}(\Sigma) = L^2(0, T; H^s(\Gamma)) \cap H^r(0, T; L^2(\Gamma))$, and the norm on $H^{s,r}(\Sigma)$ will be denoted by $|\cdot|_{s,r}$.

The elliptic operator A defined by $a(u, v) = (Au, v)$ for $v \in H_0^1(\Omega)$ satisfies (see Grisvard [13, Chapter 3])

$$(9) \qquad \|u\|_2 \leq C\|Au\|_0,$$

where $\mathscr{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$. There is a continuous extension of A to $L^2(\Omega)$ which we also denote by A and which is defined by $(Au, v) = (u, Av)$ for $v \in H_0^1(\Omega)$. This operator has an orthonormal in $L^2(\Omega)$ basis of eigenfunctions $\{\phi_m\}_{m=1}^\infty \subset \mathscr{D}(A)$ and real eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$ such that

$$a(\phi_m, v) = \lambda_m(\phi_m, v) \quad \forall v \in H_0^1(\Omega)$$

with

$$a(\phi_m, \phi_l) = \lambda_m(\phi_m, \phi_l) = \lambda_m \delta_{ml},$$

where δ_{ml} denotes the Kronecker data (see Babuška and Osborn [1]).

Following Bramble and Thomée [4], we denote by $\dot{H}^s(\Omega)$ the subspace of $L^2(\Omega)$ consisting of functions v such that

$$\|v\|_{\dot{H}^s(\Omega)} = \left(\sum_{m=1}^\infty \lambda_m^s v_m^2 \right)^{1/2} < \infty,$$

where $v_m = (v, \phi_m)$. It follows, for $0 \leq s \leq 2$, that

$$\|v\|_{\dot{H}^s(\Omega)} = \|A^{s/2}v\|_0,$$

where A^γ is defined by $A^\gamma u = \sum_{m=1}^\infty \lambda_m^\gamma u_m \phi_m$. We note that $\dot{H}^2(\Omega) = \mathscr{D}(A)$, $\dot{H}^1(\Omega) = H_0^1(\Omega) = \mathscr{D}(A^{1/2})$, $\dot{H}^0(\Omega) = L^2(\Omega)$, and $\dot{H}^s(\Omega) = H^s(\Omega)$ for $0 \leq s < \frac{1}{2}$. Also by the K-method of interpolation introduced by Lions and Peetre (see Butzer and Berens [5, p. 166]) one gets $\dot{H}^s(\Omega) = \mathscr{D}(A^{s/2}) = [H_0^1(\Omega), L^2(\Omega)]_s$ for $0 \leq s \leq 1$.

For $g = 0$ the solution of (1) is given by

$$(10) \qquad E(t)v = \sum_{m=1}^\infty e^{-\lambda_m t} v_m \phi_m.$$

For smooth Γ one has the well-known smoothing property for $t > 0$ and $0 \leq l \leq s$:

$$(11) \qquad \|E(t)v\|_{\dot{H}^s(\Omega)} \leq C t^{-(s-l)/2} \|v\|_{\dot{H}^l(\Omega)}, \qquad v \in \dot{H}^l(\Omega).$$

For a convex polygon Ω the solution $E(\cdot)v$ is only guaranteed to lie in $\mathscr{D}(A) = \dot{H}^2(\Omega)$. It follows that (11) is valid for $0 \leq l \leq s \leq 2$, by the same proof as in [4].

To establish the solvability of (1), we need the Dirichlet map $D: L^2(\Gamma) \to H^{1/2}(\Omega)$ defined by

$$(12) \qquad (Dg, A\phi) = -\left\langle g, \frac{\partial \phi}{\partial \nu_A} \right\rangle \quad \forall \phi \in \mathscr{D}(A).$$

It is well known that D is a bounded mapping.

We denote the solution to (1) with inhomogeneous right side f, $v = 0$, and $g = 0$ by

$$Bf = \int_0^t E(t-s)f(\cdot, s)\,ds.$$

Using the bounded mapping $T: L^2(\Omega) \to \mathscr{D}(A)$, defined by

$$a(Th, \phi) = (h, \phi) \quad \forall \phi \in H_0^1(\Omega),$$

we can show by energy arguments that

$$\|Bf\|_{1,0} \leq C\|Tf\|_{1,0} \quad \text{and} \quad \|Bf\|_{0,0} \leq C\|Tf\|_{0,0},$$

as well as for the E operator

$$\|E(\cdot)v\|_{1,0} \leq C\|v\|_0 \quad \text{and} \quad \|E(\cdot)v\|_{0,0} \leq C\|v\|_{-1}.$$

Taking $f = A\eta$ in the previous inequalities for B, we obtain

$$\|BA\eta\|_{1,0} \leq C\|\eta\|_{1,0} \quad \text{and} \quad \|BA\eta\|_{0,0} \leq C\|\eta\|_{0,0}.$$

By interpolation (see Theorem 5.1 in Chapter 1 of [20]) it follows that BA: $H^{s,0}(\mathscr{Q}) \to H^{s,0}(\mathscr{Q})$ and $E(\cdot)\colon H^{s-1}(\Omega) \to H^{s,0}(\mathscr{Q})$ are bounded maps for $0 \leq s \leq 1$.

We now turn to defining y as a solution of the following very weak formulation of (1): Find y defined on $\mathscr{Q}$ such that

$$(13) \qquad \int_0^T (y, w_t - Aw)\, dt = \int_0^T \left\langle g, \frac{\partial w}{\partial \nu_A} \right\rangle dt - (v, w(\cdot, 0))$$

for all $w \in H^{2,1}(\mathscr{Q}) \cap H_0^1(\Omega)$ with $w(\cdot, T) = 0$, where v and g are given. We will specify appropriate spaces for v, g, and the solution y in what follows. We note that uniqueness holds since the only solution for zero data is $y = 0$ (choose $w_t - Aw = y$). Moreover, formulation (13) is essentially the transposition procedure of Lions and Magenes [20, Chapter 4, §8] and will be the starting point for the error analysis of our method. We will obtain L^∞ and L^2 (in time) a priori estimates in terms of the data.

Let $\{\eta^n\}_{n=1}^\infty$ be a sequence of infinitely differentiable functions which have compact support in Ω for all t, and let $\{v^n\}_{n=1}^\infty \subset C^\infty(\overline{\Omega})$. Take z^n to be the solution of

$$z_t^n + Az^n = -\eta_t^n \quad \text{on } \mathscr{Q}$$

with $z^n = 0$ on Γ and $z^n(\cdot, 0) = v^n - \eta^n(\cdot, 0)$. Through integration by parts in both t and x it is easy to show for $w \in H^{2,1}(\mathscr{Q}) \cap H_0^1(\Omega)$, $w(\cdot, T) = 0$,

$$\int_0^T (z^n, w_t - Aw)\, dt = -\int_0^T (\eta^n, w_t)\, dt + (v^n, w(\cdot, 0)).$$

Setting $y^n = z^n + \eta^n$, we find

$$(14) \qquad \int_0^T (y^n, w_t - Aw)\, dt = \int_0^T (\eta^n, Aw)\, dt + (v^n, w(\cdot, 0)).$$

Moreover, z^n is given by

$$(15) \qquad z^n = E(\cdot)(v^n - \eta^n(\cdot, 0)) - B\eta_t^n,$$

and using integration by parts on the second term, we obtain

$$(16) \qquad y^n = E(\cdot)v^n + AB\eta^n.$$

Note that A and B commute on L^2. From the boundedness of the mappings $E(\cdot)$ and BA it follows that

$$(17) \qquad \|y^n\|_{1/2,0} \leq C(\|v^n\|_{-1/2} + \|\eta^n\|_{1/2,0}).$$

Assuming $v^n \to v$ in $H^{-1/2}(\Omega)$ and $\eta^n \to Dg$ in $H^{1/2,0}(\mathcal{Q})$, we have $y^n \to y$, and from (14) it follows that y satisfies

$$(18) \qquad \int_0^T (y, w_t - Aw)\, dt = \int_0^T (Dg, Aw)\, dt + (v, w(\cdot, 0))$$

for all $w \in H^{2,1}(\mathcal{Q}) \cap H_0^1(\Omega)$ with $w(\cdot, T) = 0$. Also, from (16) we have

$$(19) \qquad y = E(\cdot)v + ABDg.$$

By the definition of Dg, y satisfies (13) and

$$(20) \qquad \|y\|_{1/2,0} \le C(\|v\|_{-1/2} + |g|_{0,0}).$$

With the same argument as in [17] it follows that for $v \in H^{1/2}(\Omega)$, $g \in L^\infty(0, T; L^2(\Gamma))$ (hence, $Dg \in L^\infty(0, T; H^{1/2}(\Omega))$), and $\varepsilon > 0$,

$$(21) \qquad \|y\|_{L^\infty(0,T;H^{1/2-\varepsilon}(\Omega))} \le C(\|v\|_{1/2} + \|g\|_{L^\infty(0,T;L^2(\Gamma))}).$$

Formula (19) and estimate (21) were obtained in [17] for the case where Γ is smooth.

Inequalities (21) and (20) give estimates on the regularity of the solution in terms of the data. We establish a priori bounds in Propositions 1 and 2 for the approximate solution, u, using the same data norms as in (21) and (20).

Finally we state results for the following backward in time parabolic problem that will be used in our analysis. If $f \in L^2(\mathcal{Q})$, then

$$w_t - Aw = f \quad \text{in } \Omega \times [0, T],$$
$$w = 0 \quad \text{on } \Gamma \times [0, T],$$
$$w(\cdot, T) = 0 \quad \text{on } \Omega$$

has a unique solution $w \in H^{2,1}(\mathcal{Q}) \cap H_0^1(\Omega)$. It is not difficult to prove that

$$(22) \qquad \|w\|_{2,1} \le C\|f\|_{0,0}$$

and

$$(23) \qquad \|w(\cdot, 0)\|_1 \le C\|f\|_{0,0}.$$

3. Approximation Properties

In this section we give a precise definition of our finite element space. We also list the required approximation properties of various projection operators we use and present several key technical results.

Let $0 < h < 1$ and V_h be the space of continuous piecewise linear functions relative to a quasiuniform triangulation T^h of Ω. That is, for some α (independent of h) each triangle $\tau \in T^h$ contains a disc of radius αh and is contained in a disc of radius h.

Define $P_h^0: L^2(\Omega) \to V_h^0$ to be the L^2 projection, and let $P_h^1: H^1(\Omega) \to V_h^0$ denote the elliptic projection:

$$a(P_h^1 v, \phi) = a(v, \phi) \quad \forall \phi \in V_h^0.$$

We now list, for later reference, some well-known (see Ciarlet [7, §3.1]) properties of V_h and V_h^0. For $z \in H^s(\Omega)$, $1 \le s \le 2$,

$$(24) \qquad \inf_{\phi \in V_h} (\|\phi - z\|_0 + h\|\phi - z\|_1) \le Ch^s \|z\|_s,$$

$$(25) \qquad |(I - Q_h)z|_0 \le Ch^{s-1/2}\|z\|_s.$$

We also use the inverse properties for $\chi \in V_h$:

(26) $$\|\chi\|_s \leq Ch^{-(s-l)}\|\chi\|_l, \qquad 0 \leq l \leq s \leq 1,$$

(27) $$|\chi|_{1/2} \leq Ch^{-1/2+r}|\chi|_r, \qquad 0 \leq r \leq \tfrac{1}{2}.$$

Remark. Inequality (25) follows from (24) by the use of (7) with $\varepsilon = h^{-1}$.

Remark. Using a duality argument together with (25), one obtains

(28) $$\left|(I - Q_h)\frac{\partial z}{\partial \nu_A}\right|_0 \leq Ch^{1/2}\|z\|_2, \qquad z \in H^2(\Omega).$$

Indeed, for $Aw = 0$ in Ω and $w = \partial z/\partial \nu_A$ on Γ, we have by (25) and elliptic estimates

$$\left|(I - Q_h)\frac{\partial z}{\partial \nu_A}\right|_0 = |(I - Q_h)w|_0 \leq Ch^{1/2}\|w\|_1 \leq Ch^{1/2}\left|\frac{\partial z}{\partial \nu_A}\right|_{1/2},$$

from which (28) follows by the trace theorem.

For V_h^0 it is well known that, for $z \in H^s(\Omega) \cap H_0^1(\Omega)$ with $1 \leq s \leq 2$,

(29) $$\inf_{\phi \in V_h^0}(\|\phi - z\|_0 + h\|\phi - z\|_1) \leq Ch^s\|z\|_s.$$

Remark. An immediate consequence of (29) and a standard duality argument is:

(30) $\|(I - P_h^1)z\|_l \leq Ch^{s-l}\|z\|_s, \qquad z \in H_0^1(\Omega) \cap H^s(\Omega), \ 0 \leq l \leq 1 \leq s \leq 2.$

From (30), the boundedness of P_h^0 in L^2, and the norm interpolation inequality (6) it follows that

(31) $$\|(I - P_h^0)z\|_0 \leq Ch^s\|z\|_s, \qquad 0 \leq s \leq 2, \ z \in \mathscr{D}(A).$$

By similar arguments it follows from (24) that

(32) $$\|(I - \mathscr{P}_h^0)z\|_0 \leq Ch^s\|z\|_s, \qquad 0 \leq s \leq 2, \ z \in H^s(\Omega).$$

Remark. We note that (25), (28), (30), (31), and (32) are valid for any space V_h satisfying (24), (26), (27), and (29).

Finally we list two approximation properties of the space V_k for a generic Hilbert space H:

(33) $\|(I - P_k)z\|_{L^1(0,T;H)} \leq Ck\|z_t\|_{L^1(0,T;H)}, \qquad z \in H^1(0,T;H),$

(34) $\|P_k z\|_{L^1(0,T;H)} \leq C\|z\|_{L^1(0,T;H)}, \qquad z \in L^1(0,T;H).$

We now prove three lemmas that involve the following splittings of an arbitrary $u \in V_h$:

$$u = u_h + u_p \quad \text{and} \quad u = u_H + u_P,$$

where $u_p = P_h^0 u$ and $u_P = P_h^1 u$. Note that

(35) $$(u_h, \chi) = 0, \qquad \chi \in V_h^0,$$

and

(36) $$a(u_H, \chi) = 0, \qquad \chi \in V_h^0.$$

The first two lemmas are slight generalizations of some results of [3].

Lemma 1. *Let* $u \in V_h$, *and suppose* $z \in H^2(\Omega)$ *satisfies* $a(z, \phi) = 0$ *for all* $\phi \in H_0^1(\Omega)$ *and* $z = u$ *on* Γ. *Then*

$$(37) \qquad\qquad \|z - u_H\|_1 \le Ch^{-1/2}|u|_0.$$

Proof. By (3.34) of [3], $u_H - z \in H_0^1(\Omega)$ satisfies

$$(38) \qquad\qquad \|u_H - z\|_1 \le Ca^{1/2}(z - u_H, z - u_H) \le C|u|_{1/2},$$

and (37) follows by the inverse property (27). □

The next result follows from Lemma 1 by the triangle inequality, elliptic estimates, and (27).

Lemma 2. *For any* $u \in V_h$,

$$(39) \qquad\qquad \|u_H\|_1 \le Ch^{-1/2}|u|_0.$$

Lemma 3. *Suppose* $u \in V_h$; *then*

$$(40) \qquad\qquad \|u_h\|_s \le Ch^{1/2-s}|u|_0, \qquad 0 \le s \le 1.$$

Proof. Since $C_0^\infty(\Omega)$ is dense in $H^p(\Omega)$, $p \le \frac{1}{2}$ (see [12]), there exists $\{v^n\} \subset C_0^\infty(\Omega)$ such that $\lim_{n\to\infty} \|v^n - u_H\|_{1/2} = 0$. Noting that $u_h = (I - P_h^0)u_H$, we then find

$$\|u_h\|_0 = \lim_{n\to\infty} \|(I - P_h^0)v^n\|_0 \le Ch^{1/2} \lim_{n\to\infty} \|v^n\|_{1/2} = Ch^{1/2}\|u_H\|_{1/2}.$$

As in the proof of Lemma 1, we have

$$\|u_H\|_{1/2} \le \|u_H - z\|_{1/2} + \|z\|_{1/2}.$$

By the elliptic estimate $\|z\|_{1/2} \le C|u|_0$ together with (37) we obtain (40) for $s = 0$. For the case $s = 1$ the result follows by the inverse property (26) and the case $s = 0$. Now the general result follows by norm interpolation. □

4. L^2 ESTIMATES

In this section we prove Theorem 1. We assume that $v \in H^{-1/2}(\Omega)$ and $g \in L^2(\Sigma)$, and hence $y \in L^2(0, T; H^{1/2}(\Omega))$ satisfies (13) and (20).

An integral part of the proof is a particular stability estimate for the numerical scheme. The stability analysis of the method is of some interest, independent of its use in proving Theorem 1, and we begin by stating the main stability bound.

Proposition 1 (L^2 *stability*). *There exists a constant* C *independent of* h, k, *and the data pair* (v, g) *such that if* (2) *holds then for* $0 \le n \le N$,

$$(41) \qquad \sum_{j=1}^n (\|u^j - u^{j-1}\|_0^2 + ka(u^j, u^j)) + \|u^n\|_0^2 \le Ch^{-1}(\|v\|_{-1/2}^2 + |g|_{0,0}^2).$$

Before proving this stability result, we state and prove an auxiliary lemma.

Lemma 4. *There exists a constant* C *such that for* $0 \le n \le N$,

$$(42) \qquad \sum_{j=1}^n (\|u_p^j - u_p^{j-1}\|_0^2 + ka(u_p^j, u_p^j)) + \|u_p^n\|_0^2 \le Ch^{-1}(\|v\|_{-1/2}^2 + |g|_{0,0}^2).$$

Proof. For any $\phi \in V_h^0$ we have

$$(u^j - u^{j-1}, \phi) + ka(u^j, \phi) = 0.$$

Substituting $u = u_h + u_p$ and using (35), we have

$$(u_p^j - u_p^{j-1}, \phi) + ka(u_p^j, \phi) = -ka(u_h^j, \phi).$$

Choosing $\phi = u_p^j$ and summing, we have

(43)
$$\sum_{j=1}^n \left[\frac{1}{2} \| u_p^j - u_p^{j-1} \|_0^2 + ka(u_p^j, u_p^j) \right] + \frac{1}{2} \| u_p^n \|_0^2$$

$$= \frac{1}{2} \| u_p^0 \|_0^2 - \sum_{j=1}^n ka(u_h^j, u_p^j).$$

Using the arithmetic-geometric mean inequality on the right side yields

(44)
$$\sum_{j=1}^n \left(\frac{1}{2} \| u_p^j - u_p^{j-1} \|_0^2 + \frac{1}{2} ka(u_p^j, u_p^j) \right) + \frac{1}{2} \| u_p^n \|_0^2$$

$$\leq \frac{1}{2} \| u_p^0 \|_0^2 + \frac{1}{2} \sum_{j=1}^n ka(u_h^j, u_h^j).$$

Since $u_p^0 = P_h^0 \mathscr{P}_h^0 v$, we have

$$\| u_p^0 \|_0 = \sup_{\psi \in L^2(\Omega)} \frac{(\mathscr{P}_h^0 v, P_h^0 \psi)}{\| \psi \|_0} \leq \sup_{\psi \in L^2(\Omega)} \frac{\| v \|_{-1/2} \| P_h^0 \psi \|_{1/2}}{\| \psi \|_0} \leq Ch^{-1/2} \| v \|_{-1/2},$$

where the last step follows from the inverse inequality (26).

By Lemma 3 and the fact that Q is bounded it follows that

$$\sum_{j=1}^n ka(u_h^j, u_h^j) = \sum_{j=1}^n \int_{I_J} a(u_h^j, u_h^j) \, dt \leq Ch^{-1} \sum_{j=1}^n \int_{I_J} |Qg|_0^2 \, dt$$

$$\leq Ch^{-1} \| g \|_{L^2(0, t_n; L^2(\Gamma))}^2. \quad \square$$

Proof of Proposition 1. By the triangle inequality and Lemma 4 it suffices to consider

$$\sum_{j=1}^n (\| u_h^j - u_h^{j-1} \|_0^2 + ka(u_h^j, u_h^j)) + \| u_h^n \|_0^2.$$

Using the inverse inequality (26) and assumption (2), we have

$$\sum_{j=1}^n (\| u_h^j - u_h^{j-1} \|_0^2 + ka(u_h^j, u_h^j)) + \| u_h^n \|_0^2 \leq C \sum_{j=0}^n \| u_h^j \|_0^2.$$

The proof of Proposition 1 follows by an application of Lemma 3:

$$\sum_{j=1}^n \| u_h^j \|_0^2 \leq C \left(h \sum_{j=1}^n |u^j|_0^2 + \| u^0 \|_0^2 \right)$$

$$\leq Ch^{-1} \left(\sum_{j=1}^n k |u^j|_0^2 + \| v \|_{-1/2}^2 \right)$$

$$\leq Ch^{-1} (\| g \|_{L^2(0, t_n; L^2(\Gamma))}^2 + \| v \|_{-1/2}^2). \quad \square$$

We are now in a position to prove the main result of this section.

Proof of Theorem 1. Letting $w \in H^{2,1}(\mathscr{Q}) \cap H_0^1(\Omega)$ with $w(\cdot, T) = 0$, we obtain

$$\int_0^T (u, w_t - Aw) \, dt = \sum_{n=1}^N \int_{I_n} [(u^n, w_t) - (u^n, Aw)] \, dt$$

$$= \sum_{n=1}^N \int_{I_n} \left[k^{-1}(u^n, w^n - w^{n-1}) - a(u^n, w) + \left\langle Qg, \frac{\partial w}{\partial \nu_A} \right\rangle \right] dt.$$

We apply summation by parts to the first term:

$$(45) \qquad \begin{aligned} &\sum_{n=1}^N \int_{I_n} k^{-1}(u^n, w^n - w^{n-1}) \, dt \\ &\qquad = (u^N, w^N) - (u^0, w^0) - \sum_{n=1}^N \int_{I_n} k^{-1}(u^n - u^{n-1}, w^{n-1}) \, dt. \end{aligned}$$

But $w^N = w(\cdot, T) = 0$, so we have

$$(46) \qquad \begin{aligned} &\int_0^T (u, w_t - Aw) \, dt \\ &\qquad = \sum_{n=1}^N \int_{I_n} \left(-k^{-1}(u^n - u^{n-1}, w^{n-1}) \right. \\ &\qquad\qquad \left. -a(u^n, w) + \left\langle Qg, \frac{\partial w}{\partial \nu_A} \right\rangle \right) dt - (u^0, w^0). \end{aligned}$$

Subtracting this from (13) and letting $w_t - Aw = e$, where $e = y - u$, we have

$$(47) \qquad \begin{aligned} \|e\|_{0,0}^2 &= \sum_{n=1}^N \int_{I_n} (k^{-1}(u^n - u^{n-1}, w^{n-1}) + a(u^n, w)) \, dt \\ &\quad + \sum_{n=1}^N \int_{I_n} \left\langle (I - Q)g, \frac{\partial w}{\partial \nu_A} \right\rangle dt - [(v, w(\cdot, 0)) - (\mathscr{P}_h^0 v, w(\cdot, 0))] \\ &= J_1 + J_2 - J_3. \end{aligned}$$

We estimate the individual terms in reverse order. Clearly,

$$|J_3| = |(v, (I - \mathscr{P}_h^0)w(\cdot, 0))| \le \|v\|_{-1/2} \|(I - \mathscr{P}_h^0)w(\cdot, 0)\|_{1/2}$$

$$\le Ch^{1/2} \|v\|_{-1/2} \|w(\cdot, 0)\|_1 \le Ch^{1/2} \|v\|_{-1/2} \|e\|_{0,0},$$

where the last step follows from the parabolic estimate (23).

We estimate J_2 by the approximation properties of P_k and Q_h, the interpolation inequality (8), and the parabolic estimate (22):

$$\begin{aligned} J_2 &= \int_0^T \left\langle (I - P_k)g, \frac{\partial w}{\partial \nu_A} \right\rangle dt + \int_0^T \left\langle P_k g, (I - Q_h)\frac{\partial w}{\partial \nu_A} \right\rangle dt \\ &= \int_0^T \left\langle (I - P_k)g, \frac{\partial}{\partial \nu_A}(I - P_k)w \right\rangle dt + \int_0^T \left\langle P_k g, (I - Q_h)\frac{\partial w}{\partial \nu_A} \right\rangle dt. \end{aligned}$$

Applying the Cauchy-Schwarz inequality to this, we obtain the estimate

$$|J_2| \le C|g|_{0,0} \left(\left| \frac{\partial}{\partial \nu_A}(I - P_k)w \right|_{0,0}^2 + \left| (I - Q_h)\frac{\partial w}{\partial \nu_A} \right|_{0,0}^2 \right)^{1/2}.$$

Inequality (8) yields

$$\left| \frac{\partial}{\partial \nu_A}(I - P_k)w \right|_{0,0}^2 \le \frac{C}{\varepsilon}\|(I - P_k)w\|_{2,0}^2 + \varepsilon^3\|(I - P_k)w\|_{0,0}^2$$

$$\le \frac{C}{\varepsilon}\|w\|_{2,0}^2 + \varepsilon^3 k^2\|w_t\|_{0,0}^2.$$

Choosing $\varepsilon = k^{-1/2}$ gives

$$\left| \frac{\partial}{\partial \nu_A}(I - P_k)w \right|_{0,0}^2 \le Ck^{1/2}\|w\|_{2,1}^2.$$

Since

$$\left| (I - Q_h)\frac{\partial w}{\partial \nu_A} \right|_{0,0} \le Ch^{1/2}\|w\|_{2,0},$$

we obtain, from assumption (2) and estimate (22),

$$|J_2| \le Ch^{1/2}\|w\|_{2,1}|g|_{0,0} \le Ch^{1/2}\|e\|_{0,0}|g|_{0,0}.$$

Finally, we turn to the estimation of J_1. We have

$$J_1 = \sum_{n=1}^{N}(u^n - u^{n-1}, w^{n-1}) + ka(u^n, P_k^n w)$$

$$= \sum_{n=1}^{N}(u^n - u^{n-1}, w^{n-1} - P_h^1 P_k^n w) + ka(u^n, (I - P_h^1)P_k^n w).$$

Application of the Cauchy-Schwarz inequality gives

(48) $$|J_1| \le T_1 \cdot T_2,$$

where

$$T_1 = \left(\sum_{n=1}^{N}(\|u^n - u^{n-1}\|_0^2 + ka(u^n, u^n)) \right)^{1/2}$$

and

$$T_2 = \left(\sum_{n=1}^{N}(\|w^{n-1} - P_h^1 P_k^n w\|_0^2 + ka((I - P_h^1)P_k^n w, (I - P_h^1)P_k^n w)) \right)^{1/2}.$$

In view of Proposition 1 it suffices to estimate T_2. First we note that by (30)

(49) $$\|w^{n-1} - P_h^1 P_k^n w\|_0 \le \|w^{n-1} - P_k^n w\|_0 + \|(I - P_h^1)P_k^n w\|_0$$

$$\le \|w^{n-1} - P_k^n w\|_0 + Ch^2\|P_k^n w\|_2,$$

and again by (30)

(50) $$a((I - P_h^1)P_k^n w, (I - P_h^1)P_k^n w) \le Ch^2\|P_k^n w\|_2^2.$$

It is straightforward to show that

(51) $$\|w^{n-1} - P_k^n w\|_0 \le k^{1/2} \|w_t\|_{L^2(I_n;L^2(\Omega))}$$

and

(52) $$\|P_k^n w\|_2 \le k^{-1/2} \|w\|_{L^2(I_n;H^2(\Omega))} .$$

Hence, by (2) and the previous estimates (49)–(52),

$$T_2 \le C \left(\sum_{n=1}^{N} (h^4 + kh^2) \|P_k^n w\|_2^2 + k \|w_t\|_{L^2(I_n;L^2(\Omega))}^2 \right)^{1/2}$$

$$\le Ch \left(\sum_{n=1}^{N} \|w\|_{L^2(I_n;H^2(\Omega))}^2 + \|w_t\|_{L^2(I_n;L^2(\Omega))}^2 \right)^{1/2}$$

$$\le Ch \|w\|_{2,1} .$$

By the parabolic estimate (22) we have $\|w\|_{2,1} \le C\|e\|_{0,0}$ and hence $T_2 \le Ch\|e\|_{0,0}$. Combining this with (48) and Proposition 1 completes the proof. $\square$

5. L^∞ ESTIMATES

In this section we prove Theorem 2 as well as an L^∞ stability estimate in time. We assume $v \in H^{1/2}(\Omega)$ and $g \in L^\infty(0,T;L^2(\Gamma))$, and hence $y \in L^\infty(0,T;H^{1/2-\varepsilon}(\Omega))$ satisfies (13) and (21). We will use the spectral properties of the discrete elliptic operator $A_h: V_h^0 \to V_h^0$ defined by

$$a(\chi,\phi) = (A_h\chi,\phi) \quad \forall \phi \in V_h^0 .$$

Let $\{\psi_i^h, \lambda_i^h\}_{i=1}^{N_h}$ be an orthonormal eigensystem for A_h, that is,

$$a(\psi_i^h,\phi) = \lambda_i^h(\psi_i^h,\phi)$$

for all $\phi \in V_h^0$ with $0 < \lambda_1^h \le \lambda_2^h \le \cdots \le \lambda_{N_h}^h$, where $N_h = \dim(V_h^0)$ and $(\psi_i^h,\psi_j^h) = \delta_{ij}$. In our analysis we will use the norm

$$|||\chi|||_s^2 = \sum_{i=1}^{N_h} (\chi,\psi_i^h)^2(\lambda_i^h)^s, \qquad 0 \le s \le 1,$$

where $\chi \in V_h^0$ is given by $\chi = \sum_{i=1}^{N_h}(\chi,\psi_i^h)\psi_i^h$. It is easy to see that

$$\|\chi\|_0^2 = \sum_{i=1}^{N_h}(\chi,\psi_i^h)^2 = |||\chi|||_0^2$$

and

$$a(\chi,\chi) = (A_h\chi,\chi) = |||\chi|||_1^2 .$$

By interpolation, Bank and Dupont [2] prove there exist constants C_0 and C_1 so that

$$C_0\|\chi\|_s \le |||\chi|||_s \le C_1\|\chi\|_s, \qquad 0 \le s \le 1, \ \chi \in V_h^0 .$$

Since A_h is symmetric positive definite, we may also define

$$A_h^q\chi = \sum_{i=1}^{N_h}(\chi,\psi_i^h)(\lambda_i^h)^q\psi_i^h, \qquad 0 \le q \le 1 .$$

It follows that

$$\|A_h^{s/2}\chi\|_0^2 = (A_h^s\chi\,,\,\chi) = \sum_{i=1}^{N_h}(\chi\,,\,\psi_i^h)^2(\lambda_i^h)^s = \|\|\chi\|\|_s^2\,.$$

The following lemma is critical to our proof of Theorem 2.

Lemma 5. *There exists a constant C such that for $0 \le q \le 1$*

(53) $$\|A_h^q(I + kA_h)^{-n}\| \le Ct_n^{-q}\,.$$

Remark. The inequality (53) is a discrete version of a key semigroup inequality used in [6, 17].

Proof of Lemma 5. We first assume $0 \le q < 1$. We have

$$\|A_h^q(I + kA_h)^{-n}\| = \sup_{\lambda \in \sigma(A_h)} f(\lambda)\,,$$

where $f(t) = t^q(1 + kt)^{-n}$, and $\sigma(A_h)$ denotes the spectrum of A_h. A short calculation shows that on $[0, \infty)$ the absolute maximum of f occurs at $\lambda^* = q/(k(n - q))$. It follows that

(54) $$f(\lambda) \le f(\lambda^*) = \left(\frac{q}{k(n - q)}\right)^q \left(1 + \frac{q}{n - q}\right)^{-n} = q^q(nk)^{-q}\left(\frac{n}{n - q}\right)^{q-n}\,.$$

Thus,

(55) $$\|A_h^q(I + kA_h)^{-n}\| \le q^q \left(\frac{1}{1 - q/n}\right)^{q-n} t_n^{-q} \le C_q t_n^{-q}\,.$$

The analysis above is valid for $q = 1$, except when $n = 1$ and f has no maximum. However, the estimate for this case is obvious. $\square$

The following lemma establishes the boundedness of the projection $\mathscr{P}_h^0$.

Lemma 6. *There exists a constant C such that for all $z \in H^s(\Omega)$*

(56) $$\|\mathscr{P}_h^0 z\|_s \le C\|z\|_s\,, \qquad 0 \le s \le 1\,.$$

Proof. Let $\mathscr{P}_h^1 \colon H^1(\Omega) \to V_h$ satisfy

$$(\mathscr{P}_h^1 z\,,\,\phi)_1 = (z\,,\,\phi)_1\,, \qquad \phi \in V_h\,,$$

where $(\cdot\,,\,\cdot)_1$ denotes the inner product on $H^1(\Omega)$.

It follows by a standard duality argument that

(57) $$\|(I - \mathscr{P}_h^1)z\|_0 \le Ch\|z\|_1\,, \qquad z \in H^1(\Omega)\,.$$

By definition, $\mathscr{P}_h^0$ is a bounded linear operator from $L^2(\Omega)$ into itself. Also we have

$$\begin{aligned}
\|\mathscr{P}_h^0 z\|_1 &\le \|(\mathscr{P}_h^0 - \mathscr{P}_h^1)z\|_1 + \|\mathscr{P}_h^1 z\|_1 \\
&\le Ch^{-1}(\|(I - \mathscr{P}_h^0)z\|_0 + \|(I - \mathscr{P}_h^1)z\|_0) + \|z\|_1 \\
&\le C\|z\|_1\,,
\end{aligned}$$

where we used the approximation properties of $\mathscr{P}_h^0$ and (57) in the last estimate. Thus, $\mathscr{P}_h^0$ is a bounded map from $H^1(\Omega)$ into itself. By interpolation (see Theorem 5.1 in Chapter 1 of [20]), $\mathscr{P}_h^0 \colon H^s(\Omega) \to H^s(\Omega)$ is bounded for $0 \le s \le 1$. $\square$

We now give the L^∞ in time stability estimate.

Proposition 2 (L^∞ stability). *Suppose $v \in H^{1/2}(\Omega)$ and $g \in L^\infty(0, T; L^2(\Gamma))$. Then for $0 \le n \le N$,*

$$
(58) \qquad \|u\|_{L^\infty(0, t_n; H^1(\Omega))} \le Ch^{-1/2} \left(\|v\|_{1/2} + \ln\left(\frac{1}{h}\right) \|g\|_{L^\infty(0, t_n; L^2(\Gamma))} \right).
$$

Proof. For $\phi \in V_h^0$ we have

$$
(u_p^n - u_p^{n-1}, \phi) + ka(u_p^n, \phi) = -ka(u_h^n, \phi),
$$

and since $-a(u_h^n, \phi) = a(u_p^n - u_P^n, \phi)$, we obtain

$$
(I + kA_h)u_p^n = u_p^{n-1} + kA_h(u_p^n - u_P^n).
$$

It now follows that

$$
u_p^n = (I + kA_h)^{-n}u_p^0 + k\sum_{j=1}^n (I + kA_h)^{-j}A_h(u_p^{n-j+1} - u_P^{n-j+1}),
$$

and hence

$$
(59) \qquad
\begin{aligned}
\||u_p^n\||_1 &\le \|(I + kA_h)^{-n}A_h^{1/4}\| \, \||u_p^0\||_{1/2} \\
&\quad + k\sum_{j=1}^n \|A_h(I + kA_h)^{-j}\| \, \||u_p^{n-j+1} - u_P^{n-j+1}\||_1.
\end{aligned}
$$

By Lemma 5 it follows that

$$
\|u_p^n\|_1 \le Ct_n^{-1/4}\||u_P^0\||_{1/2} + Ck\sum_{j=1}^n t_j^{-1}\||u_p^{n-j+1} - u_P^{n-j+1}\||_1.
$$

Now we estimate the terms $\||u_p^0\||_{1/2}$ and $\||u_p^j - u_P^j\||_1$. We have by norm equivalence

$$
\||u_p^0\||_{1/2} \le C\|u_p^0\|_{1/2} = C\|P_h^0 \mathscr{P}_h^0 v\|_{1/2} \le C(\|(I - P_h^0)\mathscr{P}_h^0 v\|_{1/2} + \|\mathscr{P}_h^0 v\|_{1/2}).
$$

By Lemma 6, the inverse property (26), and Lemma 3 we find

$$
\||u_p^0\||_{1/2} \le C(h^{-1/2}\|(I - P_h^0)\mathscr{P}_h^0 v\|_0 + \|v\|_{1/2}) \le C\|v\|_{1/2}.
$$

Also, since $u_p^m - u_P^m = u_H^m - u_h^m$, the triangle inequality and Lemmas 2 and 3 imply that

$$
(60) \qquad \||u_p^m - u_P^m\||_1 \le Ch^{-1/2}|u^m|_0 \le Ch^{-1/2}|g|_{L^\infty(I_m; L^2(\Gamma))}.
$$

Combining these estimates gives

$$
(61) \qquad \|u_p\|_{L^\infty(0, t_n; H^1(\Omega))} \le Ch^{-1/2}\left(\|v\|_{1/2} + \ln\left(\frac{1}{h}\right)\|g\|_{L^\infty(0, t_n; L^2(\Gamma))}\right),
$$

since $\sum_{j=1}^n kt_j^{-1} \le C\ln(\frac{1}{h})$. It is not difficult, using Lemma 3, to show that

$$
(62) \qquad \|u_h\|_{L^\infty(0, t_n; L_2(\Omega))} \le Ch^{1/2}\|g\|_{L^\infty(0, t_n; L^2(\Gamma))}
$$

and

$$
(63) \qquad \max_{0 \le t \le t_n} a(u_h(\cdot, t), u_h(\cdot, t)) \le Ch^{-1/2}\|g\|_{L^\infty(0, t_n; L^2(\Gamma))}.
$$

Finally, by the triangle inequality, we combine (61), (62), and (63) to give (58). $\square$

We complete this section by giving the proof of Theorem 2.

Proof of Theorem 2. By the triangle inequality,

$$\|(u - y)(\cdot, T)\|_0 \le \|(u - \mathscr{P}_h^0 y)(\cdot, T)\|_0 + \|(I - \mathscr{P}_h^0)y(\cdot, T)\|_0 .$$

Letting $\eta = (u - \mathscr{P}_h^0 y)(\cdot, T)$ and using the approximation properties of $\mathscr{P}_h^0$ along with the a priori estimate (21), we have

$$(64) \qquad \|(u - y)(\cdot, T)\|_0 \le \|\eta\|_0 + Ch^{1/2 - \varepsilon}(\|v\|_{1/2} + \|g\|_{L^\infty(0, T; L^2(\Gamma))}).$$

Thus, we must estimate η. Let $w \in H^{2,1}(\mathscr{Q}) \cap H_0^1(\Omega)$ satisfy $w_t - Aw = 0$ and $w(\cdot, T) = \eta$. Starting with (13), the discrete equations for u^n, and summation by parts, we obtain

$$((u - y)(\cdot, T), \eta) = \sum_{n=1}^{N} \int_{I_n} (k^{-1}(u^n - u^{n-1}, w^{n-1}) + a(u^n, w))\, dt$$

$$+ \sum_{n=1}^{N} \int_{I_n} \left\langle (I - Q)g, \frac{\partial w}{\partial \nu_A} \right\rangle dt - ((I - \mathscr{P}_h^0)v, w(\cdot, 0)).$$

Adding and subtracting $\mathscr{P}_h^0 y(\cdot, T)$ on the left side, we have, since

$$((I - \mathscr{P}_h^0)y(\cdot, T), \eta) = 0,$$

$$
\begin{aligned}
(65) \quad \|\eta\|_0^2 &= \sum_{n=1}^{N} \int_{I_n} (k^{-1}(u^n - u^{n-1}, w^{n-1}) + a(u^n, w))\, dt \\
&+ \int_0^T \left\langle (I - Q)g, \frac{\partial w}{\partial \nu_A} \right\rangle dt - ((I - \mathscr{P}_h^0)v, w(\cdot, 0)) \\
&= S_1 + S_2 + S_3 .
\end{aligned}
$$

We now estimate the terms S_j. For the last we have

$$(66) \qquad |S_3| \le \|(I - \mathscr{P}_h^0)v\|_0 \|w(\cdot, 0)\|_0 \le Ch^{1/2} \|v\|_{1/2} \|\eta\|_0 ,$$

where the estimate for $w(\cdot, 0)$ follows from (23). We now estimate S_2:

$$S_2 = \int_0^T \left\langle g, (I - P_k)\frac{\partial w}{\partial \nu_A} \right\rangle dt + \int_0^T \left\langle P_k g, (I - Q_h)\frac{\partial w}{\partial \nu_A} \right\rangle dt .$$

By the Cauchy-Schwarz and Hölder inequalities we have

$$
\begin{aligned}
(67) \quad |S_2| &\le \|g\|_{L^\infty(0, T; L^2(\Gamma))} \left(\left\| (I - P_k)\frac{\partial w}{\partial \nu_A} \right\|_{L^1(0, T; L^2(\Gamma))} \right. \\
&\left. + \left\| (I - Q_h)\frac{\partial w}{\partial \nu_A} \right\|_{L^1(0, T; L^2(\Gamma))} \right) .
\end{aligned}
$$

We estimate the terms on the right side of (67) separately. Using (8), we have

$$
\begin{aligned}
\left\| (I - P_k)\frac{\partial w}{\partial \nu_A} \right\|_{L^1(0, T; L^2(\Gamma))} &= \left\| \frac{\partial}{\partial \nu_A}(I - P_k)w \right\|_{L^1(0, T; L^2(\Gamma))} \\
&\le C\varepsilon^{-1/2} \|(I - P_k)w\|_{L^1(0, T; H^2(\Omega))} + \varepsilon^{3/2} \|(I - P_k)w\|_{L^1(0, T; L^2(\Omega))} .
\end{aligned}
$$

Choosing $\varepsilon = k^{-1/2}$, we obtain

$$\left\| (I - P_k)\frac{\partial w}{\partial \nu_A} \right\|_{L^1(0,T;L^2(\Gamma))}$$
$$\leq C k^{1/4}\|(I - P_k)w\|_{L^1(0,T;H^2(\Omega))} + k^{-3/4}\|(I - P_k)w\|_{L^1(0,T;L^2(\Omega))}.$$

Since $w_t = -Aw$, we have

$$\|(I - P_k)w\|_{L^1(0,T;L^2(\Omega))} \leq C k\|w_t\|_{L^1(0,T;L^2(\Omega))} \leq C k\|w\|_{L^1(0,T;H^2(\Omega))}$$

and

$$\|(I - P_k)w\|_{L^1(0,T;H^2(\Omega))} \leq C\|w\|_{L^1(0,T;H^2(\Omega))},$$

so that

$$(68) \qquad \left\| (I - P_k)\frac{\partial w}{\partial \nu_A} \right\|_{L^1(0,T;L^2(\Gamma))} \leq C k^{1/4}\|w\|_{L^1(0,T;H^2(\Omega))}.$$

To complete the estimation of S_2, we turn to the last term in (67). By (28) we obtain

$$(69) \qquad \left\| (I - Q_h)\frac{\partial w}{\partial \nu_A} \right\|_{L^1(0,T;L^2(\Gamma))} \leq C h^{1/2}\|w\|_{L^1(0,T;H^2(\Omega))}.$$

Combining estimates (68) and (69) in (67), we have

$$|S_2| \leq C(k^{1/4} + h^{1/2})\|g\|_{L^\infty(0,T;L^2(\Gamma))}\|w\|_{L^1(0,T;H^2(\Omega))}.$$

We now apply the estimate (11) with $s = 2$ and $l = \varepsilon$:

$$(70) \quad \|w\|_{L^1(0,T;H^2(\Omega))} \leq C \int_0^T (T-t)^{-(2-\varepsilon)/2}\|\eta\|_\varepsilon \, dt \leq C_T\|\eta\|_\varepsilon \leq C_T h^{-\varepsilon}\|\eta\|_0,$$

where we used the inverse inequality (26) in the last step on $\eta \in V_h$. Thus,

$$(71) \qquad |S_2| \leq C h^{1/2-\varepsilon}\|g\|_{L^\infty(0,T;L^2(\Gamma))}\|\eta\|_0.$$

Finally, we turn to estimating S_1. From (3) we have for any $\phi \in V_h^0 \otimes V_k$

$$S_1 = \sum_{n=1}^N ((u^n - u^{n-1}, w^{n-1} - \phi) + ka(u^n, P_k^n w - \phi))$$
$$= \sum_{n=1}^N ((u_p^n - u_p^{n-1}, P_h^0 w^{n-1} - \phi) + ka(u_P^n, P_h^1 P_k^n w - \phi))$$
$$+ \sum_{n=1}^N ((u_h^n - u_h^{n-1}, (I - P_h^0)w^{n-1}) + ka(u_H^n, (I - P_h^1)P_k^n w)).$$

The choice $\phi = P_h^0 w^{n-1}$ on I_n gives

$$S_1 = \sum_{n=1}^N ka(u_P^n, P_h^1 P_k^n w - P_h^0 w^{n-1})$$
$$+ \sum_{n=1}^N [(u_h^n - u_h^{n-1}, (I - P_h^0)w^{n-1}) + ka(u_H^n, (I - P_h^1)P_k^n w)]$$
$$= T_1 + T_2.$$

We estimate the terms T_1 and T_2. By the Cauchy-Schwarz and Hölder inequalities we have

$$(72) \qquad |T_1| \leq \|u\|_{L^\infty(0,T;H^1(\Omega))} \sum_{n=1}^{N} k \|P_h^1 P_k^n w - P_h^0 w^{n-1}\|_1 .$$

By the triangle inequality and inverse property (26),

$$\|P_h^1 P_k^n w - P_h^0 w\|_1 \leq \|(I - P_h^1)P_k^n w\|_1$$
$$+ Ch^{-1}(\|(I - P_h^0)P_k^n w\|_0 + \|P_h^0(P_k^n w - w^{n-1})\|_0),$$

and hence by (30), (31), and the boundedness of P_h^0,

$$(73) \qquad \|P_h^1 P_k^n w - P_h^0 w^{n-1}\|_1 \leq C(h\|P_k^n w\|_2 + h^{-1}\|P_k^n w - w^{n-1}\|_0).$$

It is not difficult to show

$$(74) \qquad \|P_k^n w\|_2 \leq k^{-1}\|w\|_{L^1(I_n;H^2(\Omega))}$$

and

$$(75) \qquad \|P_k^n w - w^{n-1}\|_0 \leq \|w_t\|_{L^1(I_n;L^2(\Omega))} .$$

Combining these estimates, we obtain

$$(76) \qquad \begin{aligned} &\sum_{n=1}^{N} k\|P_h^1 P_k^n w - P_h^0 w^{n-1}\|_1 \\ &\leq C \sum_{n=1}^{N} h\|w\|_{L^1(I_n;H^2(\Omega))} + kh^{-1}\|w_t\|_{L^1(I_n;L^2(\Omega))} \\ &\leq Ch\|w\|_{L^1(0,T;H^2(\Omega))}, \end{aligned}$$

where we used (2) and $w_t = -Aw$ in the last step. By (70), (72), (76), and Proposition 2 we complete the estimate of T_1:

$$|T_1| \leq Ch^{1/2-\varepsilon} \left(\|v\|_{1/2} + \ln\left(\frac{1}{h}\right) \|g\|_{L^\infty(0,T;L^2(\Gamma))} \right) \|\eta\|_0 .$$

For T_2 we have $|T_2| \leq F_1 \cdot F_2$, where

$$F_1 = \|u_h\|_{L^\infty(0,T;L^2(\Omega))} + h\|u_H\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch^{1/2}\|g\|_{L^\infty(0,T;L^2(\Omega))}$$

by Lemmas 2 and 3. Moreover (since $P_k^n w^{n-1} = w^{n-1}$),

$$\begin{aligned} F_2 &= \sum_{n=1}^{N} \|(I - P_h^0)P_k^n w^{n-1}\|_0 + kh^{-1}\|(I - P_h^1)P_k^n w\|_1 \\ &\leq \sum_{n=1}^{N} \|(I - P_h^0)P_k^n(w^{n-1} - w)\|_0 + \|(I - P_h^0)P_k^n w\|_0 + kh^{-1}\|(I - P_h^1)P_k^n w\|_1 \\ &\leq C \sum_{n=1}^{N} \|w^{n-1} - P_k^n w\|_0 + (h^2 + k)\|P_k^n w\|_2, \end{aligned}$$

where we used (30) and (31) in the last step. The last estimate together with (74), (75), and assumption (2) completes the estimation of F_2:

$$F_2 \leq C\|w\|_{L^1(0,T;H^2(\Omega))} .$$

Combining the estimates for F_1 and F_2, and using (70) gives

$$|T_2| \leq Ch^{1/2-\varepsilon}\|g\|_{L^\infty(0,T;L^2(\Gamma))}\|\eta\|_0.$$

This completes the estimation of S_1. Combining the estimates for S_1, S_2, S_3 in (65) gives the required bound for $\|\eta\|_0$. Substituting this in (64), we obtain the required estimate at $t_N = T$. Since the same argument applies for any t_n, the proof of Theorem 2 is complete. $\square$

6. SMOOTH SOLUTION ESTIMATES

In this section we show that an optimal-order convergence rate is obtained for our method when $y \in H^{2,1}(\mathcal{Q})$. Our analysis is quite similar to that contained in [14, §8.4; 15]. Note that there are no restrictions on k or h in this section. As usual, the error is split into two components:

$$e = y - u = y - \tilde{u} + \tilde{u} - u = \theta - \eta,$$

where $\tilde{u} \in V_h \otimes V_k$ is defined on I_n as follows:

$$\tilde{u} = \tilde{u}^n = \frac{1}{k}\int_{I_n} \hat{u}(\cdot, s)\, ds, \qquad n > 0,$$

and $\tilde{u}^0 = u^0$. The function $\hat{u}(\cdot, t) \in V_h$ is an elliptic projection:

$$a((\hat{u} - y)(\cdot, t), \phi) = 0, \qquad \phi \in V_h^0,$$

for $0 \leq t \leq T$ and $\hat{u} = Q_h g$ on Γ. The following lemma gives the approximation properties of $\hat{u}$.

Lemma 7. *There exists a constant C independent of h, $\hat{u}$, and y such that*

$$(77) \qquad \|(y - \hat{u})(\cdot, t)\|_0 \leq Ch^2\|y(\cdot, t)\|_2.$$

Proof. Define $\mathscr{P}_h \colon H^1(\Omega) \to V_h$ by the equation

$$a(\mathscr{P}_h w, \chi) + h^{-1}\langle \mathscr{P}_h w, \chi\rangle = a(w, \chi) + h^{-1}\langle w, \chi\rangle$$

for all $\chi \in V_h$. For $\xi \in H^1(\Omega)$ define the norm

$$\mathscr{N}(\xi) = (a(\xi, \xi) + h^{-1}|\xi|_0^2)^{1/2}.$$

It follows that for $w \in H^2(\Omega)$

$$\mathscr{N}((I - \mathscr{P}_h)w) = \inf_{\phi \in V_h} \mathscr{N}(\phi - w) \leq Ch\|w\|_2,$$

where the last estimate follows from (7) and (30) by choosing $\varepsilon = h^{-1/2}$ and $\phi = P_h^1 w$.

We note that $a(\mathscr{P}_h y, \phi) = a(y, \phi)$ for $\phi \in V_h^0$. Thus, $a(\hat{u} - \mathscr{P}_h y, \phi) = 0$ for all $\phi \in V_h^0$. So, $\hat{u} - \mathscr{P}_h y$ is discrete A-harmonic and therefore by Lemma 2

$$\begin{aligned}
a(\hat{u} - \mathscr{P}_h y, \hat{u} - \mathscr{P}_h y) &\leq Ch^{-1}|\hat{u} - \mathscr{P}_h y|_0^2 \\
&\leq Ch^{-1}(|(I - Q_h)y|_0^2 + |(I - \mathscr{P}_h)y|_0^2) \\
&\leq Ch^{-1}|(I - \mathscr{P}_h)y|_0^2.
\end{aligned}$$

This implies by the triangle inequality

$$\mathscr{N}(\hat{u} - y) \leq C\mathscr{N}((I - \mathscr{P}_h)y).$$

From this we obtain the optimal estimates

$$a(\hat{u} - y, \hat{u} - y) \leq Ch^2\|y\|_2^2$$

and

$$|\hat{u} - y|_0^2 \leq Ch^3\|y\|_2^2.$$

Let $w \in \mathscr{D}(A)$ satisfy $Aw = \hat{u} - y$ on Ω. Then for any $\phi \in V_h^0$

$$\|\hat{u} - y\|_0^2 = a(\hat{u} - y, w - \phi) + \left\langle (I - Q_h)y, (I - Q_h)\frac{\partial w}{\partial \nu_A} \right\rangle$$

$$\leq \mathscr{N}(\hat{u} - y)\left(a(w - \phi, w - \phi) + h\left| (I - Q_h)\frac{\partial w}{\partial \nu_A}\right|_0^2 \right)^{1/2}$$

$$\leq Ch\|w\|_2 \mathscr{N}(\hat{u} - y),$$

where we chose $\phi = P_h^1 w$ on the last step. Thus, by elliptic regularity,

$$\|\hat{u} - y\|_0 \leq Ch\mathscr{N}(\hat{u} - y).$$

Combining this with the previous estimates proves (77). □

Remark. The function $\hat{u}$ and the argument used in the proof of Lemma 7 were introduced in French and King [11]. Also, see the related work in Fix, Gunzburger, and Peterson [10].

By standard arguments we can now show, using the result of Lemma 7,

$$(78) \qquad \|\theta\|_{0,0} \leq C(k + h^2)\|y\|_{2,1},$$

$$(79) \qquad \sum_{n=1}^{N} k\|\theta^n\|_0^2 \leq C(k + h^2)\|y\|_{2,1},$$

and

$$(80) \qquad \|\theta\|_{L^\infty(0, t_n; L^2(\Omega))} \leq C(k + h^2)\|y\|_{L^\infty(0, T; H^2(\Omega))},$$

where $\theta^n = y(\cdot, t_n) - \tilde{u}^n$. In contrast to our analysis in §§4 and 5, the solution y satisfies the following weak form for $\phi \in V_h^0$:

$$(81) \qquad \int_{I_n} (k^{-1}(y^n - y^{n-1}, \phi) + a(y, \phi))\, dt = 0.$$

We are now in a position to state and prove the L^2 convergence theorem.

Theorem 3. *If* $y \in H^{2,1}(\mathscr{Q})$, *then there exists a constant* C *independent of* h, u, *and* y *such that*

$$(82) \qquad \|y - u\|_{0,0} \leq C(h^2 + k)\|y\|_{2,1}.$$

Proof. Define $z \in V_h^0 \otimes V_k$, so that $z^N = 0$ and

$$(83) \qquad k^{-1}(z^n - z^{n-1}, \phi) - a(z^{n-1}, \phi) = (\eta^n, \phi), \qquad \phi \in V_h^0.$$

Letting $\phi = \eta^n$ and integrating in time, we have

$$\|\eta\|_{0,0}^2 = \sum_{n=1}^{N} \int_{I_n} (k^{-1}(z^n - z^{n-1}, \eta^n) - a(z^{n-1}, \eta^n))\, dt$$

$$= \sum_{n=1}^{N} \int_{I_n} (k^{-1}(z^n - z^{n-1}, u^n) - a(z^{n-1}, u^n))\, dt$$

$$- \sum_{n=1}^{N} \int_{I_n} (k^{-1}(z^n - z^{n-1}, \tilde{u}^n) - a(z^{n-1}, \tilde{u}^n))\, dt$$

$$= -\sum_{n=1}^{N} \int_{I_n} (k^{-1}(z^{n-1}, u^n - u^{n-1}) + a(z^{n-1}, u^n))\, dt$$

$$+ \sum_{n=1}^{N} \int_{I_n} k^{-1}[(z^n, u^n) - (z^{n-1}, u^{n-1})]\, dt$$

$$- \sum_{n=1}^{N} \int_{I_n} (k^{-1}(z^n - z^{n-1}, \tilde{u}^n) - a(z^{n-1}, \tilde{u}^n))\, dt$$

$$= -(z^0, u^0) - \sum_{n=1}^{N} \int_{I_n} (k^{-1}(z^n - z^{n-1}, \tilde{u}^n) - a(z^{n-1}, \tilde{u}^n))\, dt,$$

where the last line follows from (3) and $z^N = 0$. Now from summation by parts we have

$$\|\eta\|_{0,0}^2 = -(z^0, u^0 - \tilde{u}^0) + \sum_{n=1}^{N} \int_{I_n} (k^{-1}(z^{n-1}, \tilde{u}^n - \tilde{u}^{n-1}) + a(z^{n-1}, \tilde{u}^n))\, dt$$

$$= -\sum_{n=1}^{N} \int_{I_n} k^{-1}(\theta^n - \theta^{n-1}, z^{n-1})\, dt.$$

In the last step we subtracted the equation

$$(84) \qquad 0 = \sum_{n=1}^{N} \int_{I_n} (k^{-1}(y^n - y^{n-1}, \phi) + a(y, \phi))\, dt$$

and used the fact that

$$(85) \qquad \int_{I_n} a(y - \tilde{u}^n, \phi)\, dt = 0$$

for $\phi = z^{n-1}$. Also, the first term is zero since $\tilde{u}^0 = u^0$. Summation by parts gives

$$(86) \qquad \|\eta\|_{0,0}^2 = \sum_{n=1}^{N} (z^n - z^{n-1}, \theta^{n-1}),$$

where we used the fact that $(\theta^0, z^0) = ((I - \mathscr{P}_h^0)v, z^0) = 0$.

By setting $\phi = z^{n-1} - z^n$ in (83) and using summation by parts, it follows that

$$\sum_{n=1}^{N} k^{-1}\|z^n - z^{n-1}\|_0^2 \leq \sum_{n=1}^{N} k\|\eta^n\|_0^2 = \|\eta\|_{0,0}^2.$$

By the Cauchy-Schwarz inequality applied to (86) we have

(87)
$$\|\eta\|_{0,0}^2 \leq \left(\sum_{n=1}^N k\|\theta^{n-1}\|_0^2\right)^{1/2} \left(\sum_{n=1}^N k^{-1}\|z^n - z^{n-1}\|_0^2\right)^{1/2}$$
$$\leq \sum_{n=1}^N k\|\theta^{n-1}\|_0^2 .$$

By (79) we have the required estimate for η, and the estimate (82) now follows from (78). $\square$

We now move on to the L^∞ estimate.

Theorem 4. *There exists a constant C independent of h, u, and y such that for any $0 \leq n \leq N$*

(88) $\|u^n - y^n\|_0 \leq C\left(1 + \ln\left(\dfrac{t_n}{k}\right)\right)(k + h^2)\|y\|_{L^\infty(0,T;H^2(\Omega))} .$

Proof. Let $z \in V_h^0 \otimes V_k$ satisfy $z^N = \eta^N$ and

(89) $k^{-1}(z^n - z^{n-1}, \phi) - a(z^{n-1}, \phi) = 0, \qquad \phi \in V_h^0 .$

Letting $\phi = u^n - \tilde{u}^n$ and integrating in time, we obtain

$$\sum_{n=1}^N \int_{I_n} [k^{-1}(z^n - z^{n-1}, u^n - \tilde{u}^n) - a(z^{n-1}, u^n - \tilde{u}^n)]\, dt = 0 .$$

Since $\tilde{u}^0 = u^0$, summing by parts gives

$$\|\eta^N\|_0^2 = -\sum_{n=1}^N \int_{I_n} [k^{-1}(u^n - \tilde{u}^n - (u^{n-1} - \tilde{u}^{n-1}), z^{n-1}) + a(z^{n-1}, u^n - \tilde{u}^n)]\, dt .$$

From (83) and (84) this becomes

$$\|\eta^N\|_0^2 = -\sum_{n=1}^N \int_{I_n} [k^{-1}(\theta^n - \theta^{n-1}, z^{n-1}) + a(\theta, z^{n-1})]\, dt .$$

Summing by parts and recalling that $(\theta^0, z^0) = ((I - \mathscr{P}_h^0)v, z^0) = 0$, we find

$$\|\eta^N\|_0^2 = (\theta^N, z^N) - \sum_{n=1}^N \int_{I_n} [k^{-1}(\theta^{n-1}, z^n - z^{n-1}) + a(\theta, z^{n-1})]\, dt .$$

From (85) we obtain, since $z^N = \eta^N$,

$$\|\eta^N\|_0^2 = (\theta^N, z^N) - \sum_{n=1}^N (z^n - z^{n-1}, \theta^{n-1})$$
$$\leq \|\theta^N\|_0\|\eta^N\|_0 + \max_{1 \leq n \leq N} \|\theta^n\|_0 \sum_{n=1}^N \|z^n - z^{n-1}\|_0 .$$

From [14, p. 166] we have

$$\sum_{n=1}^N \|z^n - z^{n-1}\|_0 \leq C\left(1 + \ln\left(\frac{T}{k}\right)\right)\|z^N\|_0 = C\left(1 + \ln\left(\frac{T}{k}\right)\right)\|\eta^N\|_0 ,$$

so

$$\|\eta^N\|_0 \le C \left(1 + \ln\left(\frac{T}{k}\right)\right) \max_{1 \le n \le N} \|\theta^n\|_0 .$$

From (80) and the fact that N can be replaced by any n the proof is complete. □

7. NUMERICAL RESULTS

In this section we discuss a practical implementation of our method on some problems with both smooth and nonsmooth boundary data. We let $A = -\Delta$. Our scheme, which consists of piecewise linear finite elements in space and the implicit backward Euler method in time, is standard except for the handling of the boundary conditions. We use a preconditioned conjugate gradient method to solve the linear systems that arise on each time step. An incomplete Cholesky decomposition provides the preconditioner.

The numerical results demonstrate clearly the advantages of a robust method. In a smooth example the optimal $O(h^2)$ convergence rate is achieved. In experiments with boundary data that has jumps and discontinuities in time and space, we found the rate of convergence over the range tested is more like $O(h^\gamma)$, where $\gamma > 1/2$. Previous methods that required the approximation to be zero on the boundary could achieve at most an $O(h^{1/2})$ convergence rate.

Our practical evaluation of the boundary conditions requires some discussion. To find u^{n+1} from u^n, we need the boundary function Qg restricted to the interval in time I_n, which is given by

$$\int_{I_n} \langle (I - Q)g , \chi \rangle \, dt = 0 , \qquad \chi \in V_h(\Gamma) .$$

Let $\{\phi_1 , \ldots , \phi_J\}$ be a basis for $V_h(\Gamma)$. We have, on I_n, $Qg = \sum_{j=1}^{J} c_j \phi_j$, and taking $\chi = \phi_i$ gives

$$\sum_{j=1}^{J} c_j k \langle \phi_j , \phi_i \rangle = \int_{I_n} \langle g , \phi_i \rangle \, dt .$$

To simplify the left side, we approximate the inner product by the trapezoid rule; the matrix then becomes diagonal. On the right side of the equation we used the trapezoid rule in the smooth case and the rectangle rule with a large number of subdivisions in the rough data case. In all experiments we took $\Omega = (0, 1) \times (0, 1)$, $T = 0.1$, used a uniform mesh to discretize Ω, and chose $k = Ch^2$.

The results for our experiment with smooth boundary data are displayed in Table 1 (see next page). We took as the known solution

$$y(x_1 , x_2) = e^{-\pi^2 t/2} \sin\left(\frac{\pi x_1}{2}\right) \sin\left(\frac{\pi x_2}{2}\right) .$$

The order of convergence was computed by the formula

$$\text{Rate} = \ln(E_2/E_1)/\ln(h_2/h_1) ,$$

where E_1 and E_2 are errors on successive meshes, and h_1 and h_2 are successive triangle diameters on these meshes. As predicted, we obtain an $O(h^2)$ rate of convergence.

TABLE 1. Smooth g

Mesh	N	Error	Rate
4×4	2	$0.456(-1)$	–
8×8	8	$0.112(-1)$	2.03
16×16	32	$0.280(-2)$	2.00
32×32	128	$0.701(-3)$	2.00

Our solution in the second experiment has a jump in time in its boundary data, $g = 0$ for $t < \tilde{t}$ and $g = 1$ for $t \geq \tilde{t}$, where $\tilde{t} = 0.07071$ and $v \equiv 0$. The true solution to this problem is obtained by separation of variables. We find $y(\cdot, t) = 1 + z(\cdot, t - \tilde{t})$, where

$$z(x_1, x_2, t) = -16 \sum_{n,m=1}^{\infty} \frac{1}{a_n b_m} e^{-(a_n^2 + b_m^2)t} \sin(a_n x_1)\sin(b_m x_2),$$

$a_n = (2n - 1)\pi$, $b_m = (2m - 1)\pi$, $z = 0$ on Γ, and $z = -1$ at $t = 0$ on Ω. We evaluated all boundary integrals using the trapezoid rule, splitting the integral on the interval I_j that contains $\tilde{t}$ into two pieces, one on each side of the jump in the boundary data. Table 2 has the results for this case. It is not surprising that the convergence is better than $h^{1/2}$, since the solution is smooth except near $\tilde{t}$.

TABLE 2. g discontinuous in time

Mesh	N	Error	Rate
4×4	2	0.728	–
8×8	8	$0.528(-1)$	3.79
16×16	32	$0.150(-1)$	1.82
32×32	128	$0.508(-2)$	1.56

In the final test the boundary data is given by

$$g(x_1, x_2, t) = \text{sgn}\left(\sin\left(\frac{4\pi t}{T} + \sqrt{3}\right)\sin(4\pi x_1 + \sqrt{2})\sin(6\pi x_2 + e)\right),$$

where sgn is the signum function. Here, g has discontinuities in both space and time. Since we do not know the true solution in this case, we compared our approximations to a finite element approximation obtained on a 64×64 mesh with 512 timesteps. Table 3 has the results, which again show the convergence rate is better than $h^{1/2}$.

TABLE 3. g rough

Mesh	N	Error	Rate
4×4	2	0.118(+1)	–
8×8	8	0.671	.814
16×16	32	0.274	1.29
32×32	128	0.132	1.05

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