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ON THE MINIMAL ELEMENTS FOR THE SEQUENCE OF ALL POWERS IN THE LEMOINE-KÁTAI ALGORITHM

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ABSTRACT. It is proved, with the help of a computer, that for $m = 20$ the first m minimal elements for the sequence of all powers in an integer-representing algorithm are given by $y_i = i$, $i = 1, 2, 3$, $y_{i+1} = (y_i^2 + 6y_i + 1)/4$, $i = 3, \dots, m-1$. This extends an earlier result of the author (for $m = 10$).

1. INTRODUCTION

Let $1 = a_1 < a_2 < \cdots$ be an infinite strictly increasing sequence of positive integers. Let n be a positive integer. We write

$$(1) \qquad n = a_{(1)} + a_{(2)} + \cdots + a_{(s)},$$

where $a_{(1)}$ is the greatest element of the sequence $\leq n$, $a_{(2)}$ is the greatest element $\leq n - a_{(1)}$, and, generally, $a_{(i)}$ is the greatest element $\leq n - a_{(1)} - a_{(2)} - \cdots - a_{(i-1)}$. This algorithm for additive representation of positive integers was introduced in 1969 by Kátai [2, 3, 4]. Lemoine had earlier considered the special cases $a_i = i^k$, $k \geq 2$ [5, 6], and $a_i = i(i+1)/2$ [7]. (See [10, 11, 12, and 13] for further information and note also [1].) The above algorithm is, in turn, a special case of a more general algorithm introduced by Nathanson [9] in 1975.

The following basic definitions and results are taken from [8 and 10]. We denote here the set of positive integers by **N**.

Let $1 = a_1 < a_2 < \cdots$ be an infinite strictly increasing sequence of positive integers with the first element equal to 1. We call it an *A-sequence* and denote by A the sequence itself or sometimes the set consisting of the elements of the sequence. We denote the number s of terms in (1) by $h(n)$. If the set $\{n \in \mathbf{N} \mid h(n) = m\}$ is nonempty for some $m \in \mathbf{N}$, we say that y_m *exists* and define y_m to be the *smallest* element of this set.

Theorem 1 (Lord). *Let y_k be given $(k \in \mathbf{N})$. Then y_{k+1} exists if and only if there exists a number $n \in \mathbf{N}$ such that $a_{n+1} - a_n - 1 \geq y_k$. Furthermore, if y_{k+1} exists, then $y_{k+1} = y_k + a_m$, where m is the smallest number in the set $\{n \in \mathbf{N} \mid a_{n+1} - a_n - 1 \geq y_k\}$.*

Proof. See [8; 10, p. 9]. □

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If y_m exists for every $m \in \mathbf{N}$, we say that the *Y-sequence exists* and we denote the sequence $1 = y_1 < y_2 < \cdots$ by Y. The elements y_m are also called *minimal elements.*

Corollary 1. *The Y-sequence exists if and only if the set* $\{a_{n+1} - a_n \mid n \in \mathbf{N}\}$ *is not bounded.*

If the A-sequence is well behaved, then, using Theorem 1, it may be possible to determine all the elements of the Y-sequence (see [10] for many examples). In particular, we have the following result (see [10, p. 20]):

Theorem 2 (Lemoine). *Let* $a_i = i^2$, $i = 1, 2, \ldots$. *The Y-sequence is given by*

$$(2) \qquad y_1 = 1, \quad y_2 = 2, \quad y_3 = 3, \quad y_{i+1} = \frac{y_i^2 + 6y_i + 1}{4}, \qquad i \geq 3.$$

Consider now the *A-sequence of all powers*, that is, the sequence formed from all integers s^k, where $s, k \in \mathbf{N}$ and $k \geq 2$. This sequence is not very well behaved, starting as A : 1, 4, 8, 9, 16, 25, 27, 32, 36, 49, 64, 81, 100, 121, 125, 128, 144, 169, 196, ... Using Corollary 1, we proved in [10, p. 49] that the Y-sequence exists for the A-sequence of all powers. Moreover, we established (with the help of a pocket calculator and somewhat to our surprise) the following result:

Theorem 3. *Let* $m = 10$. *The first* m *elements of the Y-sequence for the A-sequence of all powers are given by*

$$y_1 = 1, \quad y_2 = 2, \quad y_3 = 3, \quad y_{i+1} = \frac{y_i^2 + 6y_i + 1}{4}, \qquad i = 3, \ldots, m-1.$$

Using the computer, we have now extended Theorem 3 to

Theorem 4. *Theorem* 3 *is true with* $m = 20$.

Our purpose in this paper is to show how Theorem 4 can be established. In §2 we explain the method, and in §3 we illustrate the method by reestablishing Theorem 3. (No numerical details were given in [10].) Finally, in §4, we indicate what kind of calculations are used in the extension of Theorem 3 to Theorem 4. We remark at this point that y_{20} is a number with 26681 digits.

2. The method

Let $A : 1 = a_1 < a_2 < \cdots$ be the A-sequence of all powers and let Y: $1 = y_1 < y_2 < \cdots$ be its Y-sequence. Let Y^*: $1 = y_1^* < y_2^* < \cdots$ be the Y-sequence for the A-sequence of squares (see Theorem 2). We use Theorem 1 in the following

Definition 1. Let $k_1, k_2, \ldots$ be the sequence of positive integers defined by

$$(3) \qquad y_{i+1}^* = y_i^* + k_i^2, \qquad i = 1, 2, \ldots .$$

Theorem 5. *We have*

$$(4) \qquad k_1 = k_2 = 1, \quad k_3 = 2, \quad k_{i+1} = \frac{k_i^2}{2} + k_i, \qquad i \geq 3.$$

Proof. This follows immediately from (3) and (2). □

Definition 2. We let $C(n) = \{i \in \mathbf{N} \mid k_n^2 < a_i < (k_n + 1)^2\}$.

The following result forms the basis of our method:

Theorem 6. *Suppose that we have, for some* $n \in \mathbf{N}$, $n > 3$, $y_n = y_n^*$. *Then* $y_{n+1} = y_{n+1}^*$ *if and only if* $C(n) = \varnothing$.

Remark. The "if" part of Theorem 6 is from [10, p. 52] (note the slightly different notation). The "only if" part was recently established by Ernst S. Selmer [15] and is published here with his kind permission. The following proof of the "only if" part is actually a somewhat shortened version by the present author of Selmer's original proof.

Proof of Theorem 6. Let $B(n) = \{i \in \mathbf{N} \mid k_n^2 < a_i \leq y_{n+1}^*\}$. We have (see [10, p. 51])

$$y_{n+1} = y_{n+1}^* \quad \text{if and only if} \quad B(n) = \varnothing. \tag{5}$$

It follows easily from (2) and (3) that

$$y_{n+1}^* = (k_n + 1)^2 - 2 \quad \text{for } n \geq 3. \tag{6}$$

Therefore, $B(n) \subset C(n)$ and the "if" part follows from (5).

To prove the "only if" part, we note that using (5) and (6), we only have to prove that, if $n > 3$, then

$$(k_n + 1)^2 - 1 \notin A. \tag{7}$$

But if $(k_n + 1)^2 - 1 \in A$, then $(k_n + 1)^2 - 1 = s^p$, where $s \in \mathbf{N}$ and p is an odd prime. Now we use the fact (see [14, p. 197 and p. 206]) that if the Diophantine equation

$$x^2 = y^p + 1$$

has a solution in natural numbers x and y, then $p = 3$, $x = 3$, and $y = 2$. It follows that $k_n = 2$ and $n = 3$, which contradicts our assumption $n > 3$. Therefore, (7) holds, and the proof is completed. □

3. Proof of Theorem 3

It is easy to see, by means of Theorem 1, that

$$y_i = y_i^* = i \quad \text{for } i = 1, 2, 3,$$
$$y_4 = y_4^* = 7, \quad y_5 = y_5^* = 23, \quad y_6 = y_6^* = 167.$$

We may therefore start using Theorem 6 with $n = 6$. Using Theorem 6, we try to show that between certain consecutive squares there are no elements from the sequence A, that is, no higher powers. Obviously, it is enough to show that there are no powers with exponent p for p *an odd prime.* This we do by finding an integer $x \in \mathbf{N}$ such that

$$x^p < k_n^2 \quad \text{and} \quad (x+1)^p > (k_n + 1)^2. \tag{8}$$

With n fixed, we use the following notation:

$$a = k_n^2 - x^p, \qquad b = (x+1)^p - (k_n + 1)^2. \tag{9}$$

For example, with $n = 6$, we have, from (4), $k_6 = 84$, and if $p = 3$, then $x = 19$ satisfies (8), so that $a = 84^2 - 19^3 = 197$ and $b = 20^3 - 85^2 = 775$.

To prove Theorem 3, we show that $C(n) = \varnothing$ for $n = 6, 7, 8,$ and 9. This will be seen from Tables 1, 2, 3, and 4, respectively. There are three things to note in these tables:

1° . If the *same* x corresponds to two different primes p_1 and p_2, $p_1 < p_2$, then the same x also corresponds to any prime p, $p_1 < p < p_2$. Therefore, any such prime may be suppressed from the table. For example, in Table 2, $p = 19$ is suppressed.

2° . If, for some prime p, we have $x = 1$, then we can clearly stop.

3° . The only interesting thing about the numbers a and b from (9) in this connection is that they are *positive*. Therefore (except for Table 1), only an approximation is given. (However, their exact values were calculated by the computer to check that they are indeed positive.)

TABLE 1. Proving that $C(6) = \varnothing$

p	x	a	b
3	19	197	775
5	5	3931	551
7	3	4869	9159
11	2	5008	169922
13	1	7055	967

TABLE 2. Proving that $C(7) = \varnothing$

p	x	a	b
3	235	68669	90487
5	26	$1.16517 \cdot 10^6$	$1.29514 \cdot 10^6$
7	10	$3.04654 \cdot 10^6$	$6.43340 \cdot 10^6$
11	4	$8.85224 \cdot 10^6$	$3.57744 \cdot 10^7$
13	3	$1.14522 \cdot 10^7$	$5.40551 \cdot 10^7$
17	2	$1.29155 \cdot 10^7$	$1.16086 \cdot 10^8$
23	2	$4.65794 \cdot 10^6$	$9.41301 \cdot 10^{10}$
29	1	$1.30465 \cdot 10^7$	$5.23817 \cdot 10^8$

TABLE 3. Proving that $C(8) = \varnothing$

p	x	a	b
3	34925	$2.49546 \cdot 10^8$	$3.39677 \cdot 10^9$
5	531	$3.84650 \cdot 10^{11}$	$1.43472 \cdot 10^{10}$
7	88	$1.73266 \cdot 10^{12}$	$1.63111 \cdot 10^{12}$
11	17	$8.32832 \cdot 10^{12}$	$2.16682 \cdot 10^{13}$
13	11	$8.07750 \cdot 10^{12}$	$6.43930 \cdot 10^{13}$
17	6	$2.56736 \cdot 10^{13}$	$1.90030 \cdot 10^{14}$
19	5	$2.35267 \cdot 10^{13}$	$5.66760 \cdot 10^{14}$
23	3	$4.25061 \cdot 10^{13}$	$2.77685 \cdot 10^{13}$
29	2	$4.25997 \cdot 10^{13}$	$2.60301 \cdot 10^{13}$
43	2	$3.38041 \cdot 10^{13}$	$3.28257 \cdot 10^{20}$
47	1	$4.26002 \cdot 10^{13}$	$9.81373 \cdot 10^{13}$

TABLE 4. Proving that $C(9) = \varnothing$

p	x	a	b
3	768401051	$1.06232 \cdot 10^{18}$	$7.08959 \cdot 10^{17}$
5	214463	$2.05557 \cdot 10^{21}$	$8.52194 \cdot 10^{21}$
7	6428	$2.43108 \cdot 10^{23}$	$2.50925 \cdot 10^{23}$
11	265	$1.10521 \cdot 10^{24}$	$1.80400 \cdot 10^{25}$
13	112	$1.73455 \cdot 10^{25}$	$3.61063 \cdot 10^{25}$
17	36	$1.67183 \cdot 10^{26}$	$2.79309 \cdot 10^{24}$
19	25	$8.98970 \cdot 10^{25}$	$3.12772 \cdot 10^{26}$
23	14	$2.24109 \cdot 10^{26}$	$6.68579 \cdot 10^{26}$
29	8	$2.98952 \cdot 10^{26}$	$4.25643 \cdot 10^{27}$
31	7	$2.95919 \cdot 10^{26}$	$9.44983 \cdot 10^{27}$
37	5	$3.80935 \cdot 10^{26}$	$6.14329 \cdot 10^{28}$
41	4	$4.48859 \cdot 10^{26}$	$4.50210 \cdot 10^{28}$
43	4	$3.76324 \cdot 10^{26}$	$1.13641 \cdot 10^{30}$
47	3	$4.53668 \cdot 10^{26}$	$1.93533 \cdot 10^{28}$
53	3	$4.34312 \cdot 10^{26}$	$8.11292 \cdot 10^{31}$
59	2	$4.53695 \cdot 10^{26}$	$1.36767 \cdot 10^{28}$
83	2	$4.44023 \cdot 10^{26}$	$3.99084 \cdot 10^{39}$
89	1	$4.53695 \cdot 10^{26}$	$1.65275 \cdot 10^{26}$

4. ON THE PROOF OF THEOREM 4

The proof of Theorem 4, that is, the proof that $C(n) = \varnothing$ for $n = 10, \ldots, 19$, is too long to be published in its entirety. To give some idea of the nature of calculations, we show, in Table 5, the beginning and the end of the case $C(19) = \varnothing$. To save space, only approximations for the numbers x are given in the first part of Table 5. Those who look at Table 5 might like to know that the last prime p there, 88643, is the 8585th prime number.

TABLE 5. Beginning and end of the proof that $C(19) = \varnothing$

p	x	a	b
3	$3.46421 \cdot 10^{8893}$	$3.42990 \cdot 10^{17787}$	$1.70322 \cdot 10^{17786}$
5	$1.32973 \cdot 10^{5336}$	$6.91975 \cdot 10^{21344}$	$8.71246 \cdot 10^{21344}$
7	$3.28831 \cdot 10^{3811}$	$8.66010 \cdot 10^{22869}$	$1.89784 \cdot 10^{22868}$
11	$3.24191 \cdot 10^{2425}$	$8.14941 \cdot 10^{24255}$	$5.95661 \cdot 10^{24255}$
13	$2.26617 \cdot 10^{2052}$	$1.57522 \cdot 10^{24629}$	$8.09640 \cdot 10^{24628}$
...	...	...	...
...	...	...	...
...	...	...	...
44351	3	$4.15732 \cdot 10^{26680}$	$9.17646 \cdot 10^{26701}$
55903	3	$4.15732 \cdot 10^{26680}$	$9.11371 \cdot 10^{33656}$
55921	2	$4.15732 \cdot 10^{26680}$	$8.36502 \cdot 10^{26680}$
88609	2	$4.15731 \cdot 10^{26680}$	$1.72687 \cdot 10^{42277}$
88643	1	$4.15732 \cdot 10^{26680}$	$1.59145 \cdot 10^{26684}$

To eliminate errors, we used different languages (MACSYMA, LISP, Reduce, and Mathematica, all capable of handling integers exactly) as well as different computers (Sun 4/390 and VAX 8650 of the Centre for Scientific Computing and DECstation 3100 of the Physics Computation Unit). Only Mathematica, however, was used in the last two steps ($C(18) = \varnothing$ and $C(19) = \varnothing$).

Remark. Theorem 4 leaves open the question whether $y_n = y_n^*$ for all n or whether there exists an integer n such that $y_n \neq y_n^*$. We consider the latter case to be more likely.

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