[1] (although there is one chapter on potential flow) and nonlinear models in solid mechanics, such as plasticity.

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1. Max D. Gunzburger, *Finite element methods for viscous incompressible flows*, Academic Press, Boston, 1989. [Review **40**, Math. Comp. **57** (1991), 871–873.]

2. Claes Johnson, *Numerical solutions of partial differential equations by the finite element method*, Cambridge Univ. Press, 1987. [Review **1**, Math. Comp. **52** (1989), 247–249.]

3[65Cxx, 65Mxx, 65Nxx].—W. E. Schiesser, *The Numerical Method of Lines: Integration of Partial Differential Equations*, Academic Press, San Diego, 1991, xiii + 326 pp., $23\frac{1}{2}$ cm. Price $69.95.

The numerical method of lines for time-dependent PDEs consists of forming a spatially discrete system of ordinary differential (or possibly differential-algebraic) equations in time and then calling a suitable ODE or DAE integrator. The method is quite powerful and versatile, and is widely used in conjunction with the wide array of high-quality ODE and DAE solvers now available.

This book presents, mainly by example, one approach, or paradigm, for the numerical method of lines (NUMOL). It fills in the details of the method by establishing a complete Fortran program template in which the discrete representation of spatial operators is done by finite differences in a certain set of Fortran software (the DSS routines), and the coding for problem specification and I/O is to be inserted into a few specific spots, with heavy use of certain COMMON structures. Relegating the discretization to a set of black boxes and imposing a structure on the problem-specific coding makes it relatively straightforward to set up and solve a problem. But the price for this is a certain loss of flexibility in the areas of spatial discretization, boundary condition representation, and treatment of PDE systems.

Aside from certain limitations, and some minor errors and omissions, the book provides a very good introduction to the numerical method of lines. It assumes very little in the way of technical background of the reader. For example, the concepts of a PDE as a physical model, of Taylor series approximation, and of eigenvalues of a matrix, are all introduced from scratch as needed. Any scientist or engineer with a little Fortran background can read this book and quickly learn to solve some interesting problems. As an introduction to the subject, the book does not treat some of the more advanced aspects, such as mixed derivatives, irregular regions, or outflow boundary conditions, and it only briefly touches on the issues of nonuniform grid selection, differential-algebraic system problems, and the efficient treatment of large stiff systems. Fans of finite-element-type versions of the method of lines will not be accommodated by this book.

Chapter 1 does a good job of introducing the basic ideas of NUMOL, with the heat equation as the (much repeated) example. The various steps are explained in great detail, although some features of the procedure could use even more discussion.

A disturbing practice that appears in Chapter 1, and is continued later, is that of forming the discrete second derivative by two successive applications of a (central) first-derivative approximation ("stagewise"), rather than by direct differencing. This goes against the nearly unanimous opinion of the numerical PDE community. In the standard example, it makes the problem bandwidth five, instead of three, as in the standard central differencing, and has the potential for introducing spurious spatial oscillations (which can be demonstrated for the heat equation). Direct differencing for second derivatives is introduced in §3.4, but it is given less than equal emphasis. The relative simplicity and small size of the problems solved seems to allow stagewise differentiation to perform well here.

The first chapter ends by setting up the Fortran template for all subsequent examples. It is based on a largely fixed main program and interface routine FCN, problem-dependent routines INITAL, DERV, and PRINT; and a set of COMMON blocks with a fixed overall structure but problem-dependent details. Many readers will quibble with the programming style, but the template does do the job, and provides a start for the novice. One of the largest quibbles will be the nondynamic nature of the COMMON structure. Despite the appearance of problem size NEQN in the data file in all examples, this size cannot be varied at run time (except for a 1-D scalar PDE), because mesh sizes are built into the COMMON blocks. More serious is the fact that for a *system* of PDEs, the ordering of the dependent variables is by PDE variable first and then by mesh point, making the problem bandwidth far from minimal. In the stiff case (assuming fewer PDEs than mesh points), the solution with banded treatment of the Jacobian is far less costly if that ordering is transposed.

The issue of estimating and controlling the errors in a NUMOL solution is discussed, but somewhat too briefly. Early in Chapter 1 there is a good discussion of the important issue of setting error tolerances for the integrator. But it comes after an example where pure relative error control is specified on a problem that includes a vanishing solution component. Somehow this was not fatal, but it was partly to blame for high costs that are instead attributed to stiffness. Anyway, this mistake does not occur again after §1.6, as all subsequent examples use mixed tolerances. But the use of a scalar absolute tolerance throughout sets a dangerous example for one faced with a system having a wide range of magnitudes in the solution components. In Chapter 2, following the first nonlinear example, there is a short discussion on checking the validity of the NUMOL solution. It mentions use of physical intuition, but fails to mention an obvious numerical approach: refining the spatial and/or tightening the time integration tolerances. This usually gives a good idea of the error level, but of course is not guaranteed to.

The author's procedure for enforcing Dirichlet boundary conditions is strange at first sight, but as eventually adopted it is quite valid, though not thoroughly explained. Early on, this is done by resetting values of the dependent variable in the FCN routine, in violation of the usage instructions for most ODE integrators. However, with the introduction of the COMMON structure in §1.8, what is actually being done is to load boundary values into the appropriate components of the temporary arrays in COMMON/Y/, where they are used in the evaluation of all the other derivatives. The derivatives of those boundary variables are returned to the integrator as zero. Thus the ODE solver is integrating a dummy equation (with a constant solution) for each such boundary condition, while the true boundary value (which may or may not be constant) is absorbed into the remaining ODEs as needed. (Cases of nonconstant boundary values appear in §3.5 for ramp and pulse functions, and in Chapter 6 for the Burgers equation.) The correct boundary values get printed by the PRINT routine by virtue of the call to DERV just before the call to PRINT. In fact, this practice is not dangerous or erroneous, provided that it follows the given program structure. The alternative of having the ODE solver integrate the derivative of the boundary values can be impractical in the nonconstant case.

Chapter 2 gives several substantial examples that make the case for the ease of setup and application of NUMOL. These range from 1-D linear scalar PDEs to 2-D nonlinear systems, with hyperbolic and elliptic examples as well as parabolic. The wave equation example and the example with two coupled PDEs illustrate how two different forms (generic and problem-specific) of the COMMON blocks work together.

Chapter 3 is a primer on finite difference representations of 1-D derivatives, coupled with some of the DSS routines that implement them. Most of it assumes a uniform grid, though this is not clearly stated at the outset. For an introductory book, there is unexpected emphasis on higher-order differences. For example, the simplest case of direct second derivatives (three-point second-order, corresponding to DSS042) is not given, only the more involved five-point fourth-order case. Three-point differencing appears only in passing in §5.2. The final section of the chapter gives a good description of the obstacles involved with advection equations, and introduces noncentral difference schemes.

An interesting feature of all of the differentiation routines used is that they sacrifice bandwidth at the endpoints for the sake of preserving the order of accuracy. For example, for a parabolic problem with Neumann boundary conditions, DSS042 gives the standard 3-point second derivative approximation at each interior point, but also a 3-point difference expression at each boundary, making the system Jacobian pentadiagonal instead of tridiagonal. This tradeoff is probably a bad one in the stiff case, as the loss of efficiency is likely to offset the gain in accuracy at the boundary. In fact, one can easily have the best of both by keeping the tighter bandwidth but refining the mesh at the boundary if necessary, but doing this with the existing DSS routines would be awkward.

Chapter 4 does a good job of introducing the basic notions of explicit and implicit ODE methods, stability of ODE systems, and stability of numerical ODE methods. Moreover, it does this in a way that encourages the use of available ODE software. The important issue of stiffness in the NUMOL context is discussed, motivating the introduction of BDF methods. Following that are sections describing a number of available sophisticated ODE solvers. There are a few minor flaws in the presentation. In the sections on stability of the Euler methods, the equations connecting the problem eigenvalues λ to the characteristic growth factors β could have been obtained for *general* coefficients, by noting that the characteristic equation for β with the change of variable $\beta = 1 + \lambda\Delta t$ (explicit case) or $\beta = 1/(1 - \lambda\Delta t)$ (implicit case) is exactly the characteristic equation for λ. The section on BDF methods misstates the usual procedure for the initial guess; it is actually to extrapolate from existing data at order q, rather than use the base point. The section on the LSODE integrator incorrectly states that for a problem with banded coupling, programming a dense Jacobian is harder than a banded one; in fact both involve the same programming—of nonzero elements only.

The first two sections of Chapter 5 derive the von Neumann stability conditions for the advection equation (with centered and upwind differencing) and the heat equation with centered differencing. The results are correct, but the attempt to link them to actual NUMOL integrations is flawed. First, there is no note of the fact that, because the von Neumann analysis ignores boundary conditions, the actual eigenvalues of the ODE systems solved are somewhat different. There are two notes that attempt to make the connection in the case of the advection equation. But the first note incorrectly cites the real stability interval, whereas the better stability of RKF45 on the imaginary axis (vs explicit Euler) probably does account for its good performance. The second note gives undue credit to the variable stepsize algorithm for rescuing an otherwise unstable integration method. In all cases, the correct CFL condition is given, leading to conclusions on increasing stiffness with mesh refinement, but there is no discussion of convergence or Lax equivalence.

The bulk of Chapter 5 is devoted to the most complicated example problem in the book, a 1-D humidification column with three PDE variables plus a control variable. The example is noteworthy for its complexity and the fairly complete solution given. However, there is one improper feature, and a few omissions. The DERV routine includes lines that reset negative values of dependent variables. This is dangerous because it can make the integration unstable; the small negative values are harmless and if desired can be replaced by zero in the printing of output. The data lines displayed have four new entries ("1000 1 1 1 REL") that are not discussed. The computed spectrum displays damping modes with $\mathrm{Re}(\lambda) \sim -5000$, while the final time is $t = .5$. This raises questions of stiffness and efficiency that are not discussed. The solution was probably done with a nonstiff solver at a high price. The details of solving the problem with a stiff solver, including the use of the sparse structure of the Jacobian, would have made the example much more interesting and useful.

Chapter 6 begins with a heuristic classification of PDEs in terms of the appearance of dependencies, rather than rigorously in terms of discriminants of coefficient matrices. But for the novice this is probably more appropriate. The rest of the chapter gives various treatments of the Burgers equation that illustrate (again) handling different boundary conditions and setting up 2-D and 3-D problems. Finally, the adaptive grid solution of a Burgers equation provides an interesting and useful departure from all of the previous fixed-uniform-grid examples. But the program structure appears unable to permit the extension of the adaptive grid scheme to a system of PDEs.

In the appendices are (A) equations for the Laplacian in three coordinate systems, (B) a list of the DSS spatial differencing routines available, and (C) a list of over 200 applications from various disciplines, for which NUMOL solution programs are available.

Numerous typographical errors have crept into the book. I have a one-page list of corrections from the author, and have generated another (slightly longer) list in my own reading. Most of the errors are fairly innocuous, but a few are not. For example, in an exercise at the end of Chapter 1, the coding given for a modified Euler method has two errors (one in the time variable in the FCN call, and one in the dependent variable in the final loop). An error of a different sort was also made in my copy of the book: the binding was attached upside-down!

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4[34-00, 35-00].—DANIEL ZWILLINGER, *Handbook of Differential Equations*, 2nd ed., Academic Press, Boston, 1992, xx + 787 pp., $23\frac{1}{2}$ cm. Price $54.95.

The first edition of this book was published in 1989 and has been reviewed in [1]. That a second edition is appearing just three years after the first attests to the success of, and continued demand for, the book.

The principal changes made by the author are as follows. In Part I, dealing with basic concepts and transformations, new paragraphs have been added on chaos in dynamical systems, existence and uniqueness theorems in ODEs and PDEs, inverse problems, normal form of ODEs, and stability theorems for ODEs. Prüfer and modified Prüfer transformations, which originally appeared in Parts II and III, respectively, have been moved to Part I. Part II on exact methods has a new paragraph on exact first-order PDEs. The most extensive changes occur in Part IV, dealing with numerical methods, where one finds a reworked paragraph on available software, a long new paragraph on software classification, including excerpts from the GAMS manual, and new sections on finite difference methodology, grid generation, stability concepts in numerical ODEs, multigrid methods, parallel computer methods, and lattice gas dynamics (particle methods). In addition, many minor improvements have been made throughout the book: new examples, additional notes, and updated bibliographies. All in all, the text has expanded from the original 635 pages to 760 pages.

It should be clear from this brief review that the new edition of this reference work continues to be a useful aid to scientists and engineers and will be indispensable to anybody who needs to solve differential equations.

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1. W. F. Ames, Review **1**, Math. Comp. **54** (1990), 479–480.

5[65-06, 65Y05, 68-06].—JACK DONGARRA, PAUL MESSINA, DANNY C. SORENSEN & ROBERT G. VOIGT (Editors), *Parallel Processing for Scientific Computing*, SIAM, Philadelphia, PA, 1990, 454 pp., $25\frac{1}{2}$ cm. Price: Softcover $49.50.

This collection of 83 papers and short abstracts from the 1989 SIAM Conference on Parallel and Scientific Computing covers five areas: matrix computations, numerical methods, differential equations, massive parallelism, and performance and tools. Papers range from theoretical studies to performance evaluation to descriptions of software systems. Many of the major researchers in these fields are represented, and these papers give a good overview of research in this fast-changing area as of 1989. Many of the topics are still current, and