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Supplement to

ADAPTIVE STREAMLINE DIFFUSION FINITE ELEMENT METHODS FOR STATIONARY CONVECTION-DIFFUSION PROBLEMS

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Proof of Lemma 3.3. We find that $\rho = u - \hat{u}$ solves the following problem:

$$\begin{aligned} \rho_x - \operatorname{div}(\varepsilon\nabla\rho) &= \phi \quad \text{in } \Omega, \\ \rho &= 0 \quad \text{on } \Gamma_- \cup \Gamma_0, \\ \frac{\partial\rho}{\partial n} &= 0 \quad \text{on } \Gamma_+, \end{aligned}$$

where $\phi = -\operatorname{div}((\hat{\varepsilon} - \varepsilon)\nabla\hat{u})$.

From Lemma 1.1 and the fact that $\|\rho\| \le C\|\rho_x\|$ we get that

$$\|\rho\| \le C\|\operatorname{div}((\hat{\varepsilon} - \varepsilon)\nabla\hat{u})\|. \tag{A}$$

Since ε is constant, we have that

$$\begin{aligned} \operatorname{div}((\hat{\varepsilon} - \varepsilon)\nabla\hat{u}) &= \nabla\hat{\varepsilon}\cdot\nabla\hat{u} + (\hat{\varepsilon} - \varepsilon)\Delta\hat{u} \\ &= \nabla\hat{\varepsilon}\cdot\nabla\hat{u} + \hat{\varepsilon}^{-1}(\hat{\varepsilon} - \varepsilon)(\hat{u}_x - f - \nabla\hat{\varepsilon}\cdot\nabla\hat{u}) \\ &= \hat{\varepsilon}^{-1}\varepsilon\nabla\hat{\varepsilon}\cdot\nabla\hat{u} + \hat{\varepsilon}^{-1}(\hat{\varepsilon} - \varepsilon)(\hat{u}_x - f). \end{aligned} \tag{B}$$

Further, with $\varphi = \hat{\varepsilon}^{-2}(\hat{\varepsilon} - \varepsilon)^2$ we have that

$$\begin{aligned} \|\hat{u}_x - f\|_\varphi^2 &= (\varphi(\hat{u}_x - f), \operatorname{div}(\hat{\varepsilon}\nabla\hat{u})) \\ &= \int_\Gamma \varphi(\hat{u}_x - f)\hat{\varepsilon}\frac{\partial\hat{u}}{\partial n}\,d\Gamma - (\nabla\varphi(\hat{u}_x - f), \hat{\varepsilon}\nabla\hat{u}) \\ &\quad - (\varphi\nabla\hat{u}_x, \hat{\varepsilon}\nabla\hat{u}) + (\varphi\nabla f, \hat{\varepsilon}\nabla\hat{u}) \\ &= I + II + III + IV. \end{aligned}$$

Here

$$III = -\frac{1}{2}\int_\Gamma \varphi\hat{\varepsilon}|\nabla\hat{u}|^2 n_1\,d\Gamma + \frac{1}{2}((\varphi\hat{\varepsilon})_x\nabla\hat{u}, \nabla\hat{u}) = V + VI,$$

and since f vanishes on Γ, $\partial\hat{u}/\partial n$ on Γ_+ and $\hat{u}$ on Γ_-, we get as in (1.8)

$$\begin{aligned} I + V &= \int_\Gamma \varphi\hat{\varepsilon}\left(\hat{u}_x\frac{\partial\hat{u}}{\partial n} - \frac{1}{2}|\nabla\hat{u}|^2 n_1\right)d\Gamma \\ &= -\frac{1}{2}\int_\Gamma \varphi\hat{\varepsilon}|\nabla\hat{u}|^2|n_1|\,d\Gamma \le 0. \end{aligned}$$

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Using obvious estimates for the terms II and IV, and the fact that $\varphi^{-1/2}\nabla\varphi = 2\varepsilon\hat{\varepsilon}^{-2}\nabla\hat{\varepsilon}$, we obtain

$$\begin{aligned}\|\hat{u}_x - f\|_\varphi^2 &\le \frac{1}{2}\|\hat{u}_x - f\|_\varphi^2 + 2\|\varepsilon\hat{\varepsilon}^{-1}\nabla\hat{\varepsilon}\cdot\nabla\hat{u}\|^2 \\ &+ \frac{1}{2}\|\varphi\hat{\varepsilon}^{1/2}\nabla f\|^2 + \frac{1}{2}\|\hat{\varepsilon}^{1/2}\nabla\hat{u}\|^2 + \frac{1}{2}((\varphi\hat{\varepsilon})_x\nabla\hat{u}, \nabla\hat{u}).\end{aligned}$$

From the identity (B) we may thus conclude, with σ as above, that

$$\begin{aligned}\|\mathrm{div}((\hat{\varepsilon} - \varepsilon)\nabla\hat{u})\|^2 &\le 2(\|\hat{\varepsilon}^{-1}\varepsilon\nabla\hat{\varepsilon}\cdot\nabla\hat{u}\|^2 + \|\hat{u}_x - f\|_\varphi^2) \\ &\le 2(\|\sigma\nabla\hat{u}\|_{\hat{\varepsilon}}^2 + \|\varphi\nabla f\|_{\hat{\varepsilon}}^2).\end{aligned}$$

Together with (A) this proves the desired error bound for ρ and completes the proof of Lemma 3.3. □

Proof of Lemma 3.4. We have that

$$(\theta_x, v + \delta v_x) + (\hat{\varepsilon}\nabla\theta, \nabla v) - (\mathrm{div}(\hat{\varepsilon}\nabla\theta), \delta v_x)_T = 0, \quad \forall v \in V.$$

Now let z be the solution of

$$\text{(C)}\qquad \begin{aligned} -z_x - \mathrm{div}(\hat{\varepsilon}\nabla z) &= -\mathrm{div}(\hat{\varepsilon}\nabla\theta) && \text{in } \Omega, \\ z &= 0 && \text{on } \Gamma_- \cup \Gamma_0, \\ \hat{\varepsilon}\frac{\partial z}{\partial n} + n_1 z &= \hat{\varepsilon}\frac{\partial\theta}{\partial n} && \text{on } \Gamma_+. \end{aligned}$$

We find that

$$\begin{aligned}\|\nabla\theta\|_{\hat{\varepsilon}}^2 &= (\theta, -z_x - \mathrm{div}(\hat{\varepsilon}\nabla z)) + \int_{\Gamma_+}\hat{\varepsilon}\theta\frac{\partial\theta}{\partial n}d\Gamma = (\theta_x, z) + (\hat{\varepsilon}\nabla\theta, \nabla z) \\ &= (\theta_x, z - \tilde{z} - \delta\tilde{z}_x) + (\hat{\varepsilon}\nabla\theta, \nabla(z - \tilde{z})) + (\mathrm{div}(\hat{\varepsilon}\nabla\theta), \delta\tilde{z}_x)_T \\ &= (f - U_x + \nabla\hat{\varepsilon}\cdot\nabla U, z - \tilde{z} - \delta\tilde{z}_x) + \sum_\tau\int_\tau \hat{\varepsilon}[\frac{\partial U}{\partial n_\tau}](z - \tilde{z})d\Gamma \\ &= I + II.\end{aligned}$$

With $r = |f - U_x + \nabla\hat{\varepsilon}\cdot\nabla U|$, we have that

$$|I| \le (r, |z - \tilde{z}| + \delta|\tilde{z}_x|) \le C\|\min_*(1, h\hat{\varepsilon}^{-1/2})r\|\,(\|z\| + \|\nabla z\|_{\hat{\varepsilon}}).$$

Similarly

$$|II| \le C\|\min_*(1, h\hat{\varepsilon}^{-1/2})\hat{\varepsilon}D_h^2 U\|\,(\|z\| + \|\nabla z\|_\varepsilon).$$

Multiplying the equation in (C) by z and integrating over Ω we obtain, using the given boundary conditions,

$$\frac{1}{2}\int_{\Gamma_+} z^2 n_1 + \|\nabla z\|_{\hat{\varepsilon}}^2 = (\hat{\varepsilon}\nabla z, \nabla\theta) \le \|\nabla z\|_{\hat{\varepsilon}}\,\|\nabla\theta\|_{\hat{\varepsilon}},$$

from which follows at once that $\|\nabla z\|_{\hat{\varepsilon}} \le \|\nabla\theta\|_{\hat{\varepsilon}}$. If we repeat the arguments in the first part of the proof of Lemma 1.2 we easily derive the same control for z, or $\|z\| \le C\|\nabla\theta\|_{\hat{\varepsilon}}$. Together our estimates now show that

$$\|\nabla\theta\|_{\hat{\varepsilon}} \le C(\|\min_*(1, h\hat{\varepsilon}^{-1/2})r\| + \|\min_*(1, h\hat{\varepsilon}^{-1/2})\hat{\varepsilon}D_h^2 U\|).$$

Clearly, by estimating the terms I and II together, we can derive the more precise estimate (3.9). This completes the proof of Lemma 3.4. □