# PARABOLIC APPROXIMATIONS
# OF THE CONVECTION-DIFFUSION EQUATION

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ABSTRACT. We propose an approximation of the convection-diffusion operator which consists in the product of two parabolic operators. This approximation is much easier to solve than the full convection-diffusion equation, which is elliptic in space. We prove that this approximation is of order three in the viscosity and that the classical parabolic approximation is of order one in the viscosity. Numerical examples are given to demonstrate the effectiveness of our new approximation.

## 1. INTRODUCTION

The purpose of this paper is to develop an approximation of the steady convection-diffusion equation (1.1) which consists in paraxial (or parabolic) equations (i.e., equations which are evolution equations in the direction $x$),

$$\mathscr{L}(u) = a(x, y)\frac{\partial u}{\partial x} - \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f, \qquad 0 \le x \le L, \ y \in \mathbf{R},$$

(1.1.a)
$$a(x, y) \ge a_0 > 0,$$

$$u(0, y) = U_0(y) \quad \text{at } x = 0$$

with an open boundary condition at $x = L$ of the form

(1.1.b)
$$\alpha\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial y^2} = 0$$

and the requirement that $u$ be bounded at infinity (in $y$).

Equation (1.1) models, for instance, the concentration of a pollutant or a colorant in natural environmental flows (see, for instance, [2]). When the viscosity $\nu$ is small, a classical parabolic approximation to (1.1) is made by neglecting the diffusion in the direction of the flow with respect to the transport term (see, for instance, [1, 9–11]). We have ($p$ stands for parabolic)

(1.2)
$$\mathscr{L}_p(u_p) = a(x, y)\frac{\partial u_p}{\partial x} - \nu\frac{\partial^2 u_p}{\partial y^2} = f, \qquad 0 \le x \le L, \ y \in \mathbf{R},$$

$$u_p(0, y) = U_0(y) \quad \text{at } x = 0.$$

The main numerical advantage of (1.2) is that it can be solved faster and de-
mands less computer memory than (1.1). Indeed, problem (1.1) is elliptic in
space, while problem (1.2) is elliptic only in the direction $y$.

Let us describe formally the error $u_p - u$. We make an asymptotic expansion
of the form

$$u = u_0 + \nu u_1 + \nu^2 u_2 + \cdots \quad \text{and} \quad u_p = u_{p0} + \nu u_{p1} + \nu^2 u_{p2} + \cdots,$$

which we introduce respectively in (1.1) and (1.2). We set equal to zero the
terms of each order in $\nu$. At order zero we have

$$a\frac{\partial u_0}{\partial x} = f \quad \text{and} \quad a\frac{\partial u_{p0}}{\partial x} = f,$$

and hence $u_{p0}$ is equal to $u_0$:

$$u_0(x, y) = u_{p0}(x, y) = U_0(y) + \int_0^x \frac{f(s, y)}{a(s, y)}\, ds.$$

At order one the terms are different, since we have

$$a\frac{\partial u_1}{\partial x} = \frac{d^2 U_0}{dy^2}(y) + \int_0^x \frac{\partial^2}{\partial y^2}\left(\frac{f}{a}\right)(s, y)\, ds + \frac{\partial}{\partial x}\left(\frac{f}{a}\right)(x, y)$$

and

$$a\frac{\partial u_{p1}}{\partial x} = \frac{d^2 U_0}{dy^2}(y) + \int_0^x \frac{\partial^2}{\partial y^2}\left(\frac{f}{a}\right)(s, y)\, ds.$$

Thus, in the general case the error seems to be of order 1 in the viscosity $\nu$.

The goal of this paper is to introduce an approximation of (1.1) of order 3 in
$\nu$ which is almost as easy to solve as (1.2). It generalizes the case $f = 0$ and the
case where the velocity $a$ depends only on $y$, which has been considered in [7,
8]. In §2 we derive the form of the approximation in the case of constant $a$. We
are able to approximately factor the operator $a\partial_x - \nu\Delta$ as a product of operators
$-\nu(\partial_x - \Lambda_1^+)(\partial_x - \Lambda_1^-)$, where $\Lambda_1^+$ (resp. $\Lambda_1^-$) is a positive (resp. negative)
differential operator in the $y$-direction. The exact form of this approximation
is

$$(1.3) \qquad -\nu\left(\frac{\partial}{\partial x} - \frac{a}{\nu} + \frac{\nu}{a}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial}{\partial x} - \frac{\nu}{a}\frac{\partial^2}{\partial y^2}\right)u_p = f$$

and is of order 3 with respect to $\nu$. When $a$ is not constant, this form yields
an error of order 2. To increase the order of the error, we replace (1.3) by

$$(1.4) \qquad -\nu\left(\frac{\partial}{\partial x} - \frac{a}{\nu} + \frac{\nu}{\alpha}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial}{\partial x} - \frac{\nu}{\alpha}\frac{\partial^2}{\partial y^2}\right)u_p = f,$$

where $\alpha$ is to be determined. From the mathematical point of view, factoring
partial differential operators is a well-known technique in the theory of pseudo-
differential operators. However, we use this idea in a somewhat unusual fashion
here, since we are interested in global estimates and we consider singular per-
turbation. Equation (1.4) can be numerically solved by a double sweep over the
computational domain. Solving (1.4) costs only twice as much as solving (1.2)
and is much less costly than solving (1.1). The paper is organized as follows:
in §2, (1.4) is introduced and $\alpha$ is "optimized" with the aid of an asymptotic
development. In §3, we consider the well-posedness of (1.4) and we prove that

the $L^2$-norm of the difference between the solutions of (1.1) and (1.4) is of order 3 in $\nu$ and that the difference between the solutions of (1.1) and (1.2) is of order 1 as mentioned. In §4, we present numerical results.

## 2. Design of the approximate operator

### 2.1. Form of the approximate operator.

We denote by $k$ the dual variable of $y$ for the Fourier transform in $y$ and by $\lambda$ the dual variable of $x$ for the Laplace transform in $x$.

The case when $a$ is a positive constant can be treated by performing a Fourier transform in $y$ and a Laplace transform in $x$. The convection-diffusion operator is transformed to

$$(2.1) \qquad \mathscr{P}(\lambda, k) = a\lambda - \nu\lambda^2 + \nu k^2.$$

The zeros of this polynomial are

$$(2.2) \qquad \begin{aligned} \lambda^+(k) &= \frac{a}{2\nu}\left(1 + \sqrt{1 + \frac{4k^2\nu^2}{a^2}}\right) \geq 0, \\ \lambda^-(k) &= \frac{a}{2\nu}\left(1 - \sqrt{1 + \frac{4k^2\nu^2}{a^2}}\right) \leq 0. \end{aligned}$$

Let $\Lambda^+$ and $\Lambda^-$ be the pseudodifferential operators in $y$ of respective symbols $\lambda^+$ and $\lambda^-$. We have the exact factorization

$$(2.3) \qquad \mathscr{L} = -\nu\mathscr{L}^+\mathscr{L}^-,$$

where $\mathscr{L}^+ = \partial_x - \Lambda^+$ and $\mathscr{L}^- = \partial_x - \Lambda^-$. Problem (1.1) can now easily be solved with the following boundary conditions:

$$u(0, y) = U_0(y) \quad \text{and} \quad \mathscr{L}^-(u)(L, y) = 0.$$

This boundary condition is an exact open boundary condition, see [3]. We introduce $w(x, y) = \mathscr{L}^-(u)$, so that solving the elliptic problem (1.1) is equivalent to solving two successive parabolic problems in the $x$ direction (one in the direction of negative $x$ and the other in the direction of positive $x$):

$$(2.4) \qquad \begin{aligned} \mathscr{L}^+(w) &= -\frac{f}{\nu} \text{ and } w(L, y) = 0, \quad \text{and then} \\ \mathscr{L}^-(u) &= w \text{ and } u(0, y) = U_0(y). \end{aligned}$$

As the operators $\Lambda^+$ and $\Lambda^-$ are pseudodifferential, they are difficult to use in a numerical computation except if spectral methods are used. Even worse, they are very difficult to generalize to the case of nonconstant coefficients. To use only differential operators, $\lambda^+$ and $\lambda^-$ are approximated for small $\nu$ by

$$\lambda^- \cong \lambda_1^-(k) = -\frac{\nu k^2}{a} \quad \text{and} \quad \lambda^+ \cong \lambda_1^+(k) = \frac{a}{\nu} + \frac{\nu k^2}{a}.$$

Equality (2.3) becomes

$$\mathscr{L} = -\nu\mathscr{L}_1^+\mathscr{L}_1^- - \nu^3\mathscr{R},$$

where $\mathscr{L}_1^\pm$ has for symbol $(\partial_x - \lambda_1^\pm(k))$ and $\mathscr{R}$ is an error term with symbol $\frac{k^4}{a^2}$. When $a$ depends on $x$ and $y$, we let

$$(2.5) \quad \mathscr{L}_1^+ = \frac{\partial}{\partial x} - \frac{a(x, y)}{\nu} + \frac{\nu}{a(x, y)}\frac{\partial^2}{\partial y^2} \quad \text{and} \quad \mathscr{L}_1^- = \frac{\partial}{\partial x} - \frac{\nu}{a(x, y)}\frac{\partial^2}{\partial y^2}.$$

Then we have

$$\mathscr{L} = -\nu\mathscr{L}_1^+\mathscr{L}_1^- - \nu^3\mathscr{R} - \nu^2\mathscr{R}_1,$$

where

$$(2.6) \quad \mathscr{R} = \frac{1}{a(x,y)}\frac{\partial^2}{\partial y^2}\left(\frac{1}{a(x,y)}\frac{\partial^2}{\partial y^2}\right) \quad \text{and} \quad \mathscr{R}_1 = \partial_x\left(\frac{1}{a(x,y)}\right)\frac{\partial^2}{\partial y^2}.$$

Thus, if we replace the operator $\mathscr{L}$ by its approximate factorization $-\nu\mathscr{L}_1^+\mathscr{L}_1^-$ and consider the following approximate problem:

$$-\nu\mathscr{L}_1^+\mathscr{L}_1^-(u_p) = f,$$

$$(2.7)$$

$$u_p(0,y) = U_0(y) \quad \text{and} \quad \mathscr{L}_1^-(u_p) = 0 \quad \text{at } x = L,$$

we may expect the difference between $u$ and $u_p$ to be $O(\nu^2)$. When $a$ depends only on $y$, the error term $\mathscr{R}_1$ is zero and the difference between the solutions of (2.7) and (1.1) is a priori of order $\nu^3$. In the next subsection, we design an approximate problem which yields an error still of $O(\nu^3)$ when $a$ depends on $x$, as in the constant $a$ case.

## 2.2. Construction of a more precise approximate operator.

We introduce a new function $\alpha$ to be determined, and we define a new approximation

$$(2.8) \qquad -\nu\left(\frac{\partial}{\partial x} - \frac{a}{\nu} + \frac{\nu}{\alpha}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial}{\partial x} - \frac{\nu}{\alpha}\frac{\partial^2}{\partial y^2}\right)u_p = f,$$

$$u_p(0,y) = U_0(y) \quad \text{at } x = 0,$$

$$(2.9)$$

$$\alpha\frac{\partial u_p}{\partial x} - \nu\frac{\partial^2 u_p}{\partial y^2} = 0 \qquad \text{at } x = L$$

and $u_p$ bounded at infinity (in $y$). We consider now problem (1.1) with the following boundary condition:

$$(2.9.\text{bis}) \qquad \alpha\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{at } x = L.$$

We shall look for an $\alpha(x,y,\nu)$ so that the error $e = u - u_p$ is in $\nu^3$ and the problem (2.8)–(2.9) is well posed. Let us first remark that we have

$$a\frac{\partial u_p}{\partial x} - \nu\left(\frac{\partial^2 u_p}{\partial x^2} + \frac{\partial^2 u_p}{\partial y^2}\right)$$

$$(2.10)$$

$$+ \nu^2\left[\partial_x\left(\frac{1}{\alpha}\right) + \frac{\alpha - a}{\nu\alpha}\right]\frac{\partial^2 u_p}{\partial y^2} + \frac{\nu^3}{\alpha}\frac{\partial^2}{\partial y^2}\left(\frac{1}{\alpha}\frac{\partial^2 u_p}{\partial y^2}\right) = f.$$

To begin with, we write a formal asymptotic expansion of the solutions of (1.1) and (2.8) in the form

$$u = u_0 + \nu u_1 + \nu^2 u_2 + \cdots \quad \text{and} \quad u_p = u_{p0} + \nu u_{p1} + \nu^2 u_{p2} + \cdots.$$

We wish to choose $\alpha$ such that

$$(2.11) \qquad g(x,y,\nu) = \frac{1}{\nu}\left[\partial_x\left(\frac{1}{\alpha}\right) + \frac{1}{\nu} - \frac{a}{\nu}\frac{1}{\alpha}\right]$$

is uniformly bounded, independently of $\nu$. Possible choices are

$$(2.12) \qquad \alpha = a + \nu\frac{a_x}{a}$$

or

$$(2.13) \qquad \alpha = a \frac{1}{1 - \nu \frac{a_x}{a^2}} .$$

Moreover, we require $\alpha > 0$. This is automatically satisfied for small $\nu$ for either choice (2.12), (2.13). From a numerical point of view, it may be interesting to switch between these two expressions according to the sign of $a_x$: this guarantees that $\alpha$ will be positive for all $\nu$ at the expense of a loss of regularity.

The results of the identification in the asymptotic expansions are as follows:

$$a \frac{\partial u_{p0}}{\partial x} = f, \qquad\qquad\qquad a \frac{\partial u_0}{\partial x} = f;$$

$$a \frac{\partial u_{p1}}{\partial x} = \Delta u_{p0}, \qquad\qquad\qquad a \frac{\partial u_1}{\partial x} = \Delta u_0;$$

$$a \frac{\partial u_{p2}}{\partial x} = \Delta u_{p1}, \qquad\qquad\qquad a \frac{\partial u_2}{\partial x} = \Delta u_1;$$

$$a \frac{\partial u_{p3}}{\partial x} = \Delta u_{p2} \qquad\qquad\qquad a \frac{\partial u_3}{\partial x} = \Delta u_2.$$
$$+ g_0 \frac{\partial^2 u_{p0}}{\partial y^2} + \frac{1}{\alpha_0} \frac{\partial^2}{\partial y^2} \left( \frac{1}{\alpha_0} \frac{\partial^2 u_{p0}}{\partial y^2} \right) ,$$

Here, $g_0(x, y) = g(x, y, 0)$ and $\alpha_0(x, y) = \alpha(x, y, 0) = a(x, y)$.

*Remark.* We now relate the value of $\alpha$ given by our procedure to previous results. For the time-dependent convection-diffusion equation with constant coefficients,

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} - \nu \Delta u = 0,$$

Halpern [3] has given the following infinite family of open boundary condition operators: $(\frac{\partial}{\partial t} + a \frac{\partial}{\partial x})^n$. The case $n = 1$ corresponds to the Euler approximation of (1.1). If $a$ depends on $x$ and $y$, $n = 2$, and we reduce our problem to the stationary case, the open boundary operator becomes

$$(2.14) \qquad a_x \frac{\partial}{\partial x} + a \frac{\partial^2}{\partial x^2} .$$

Multiplying (2.14) by $-\frac{\nu}{a^2}$ and subtracting $\mathscr{L}$, we get

$$\left( a + \nu \frac{a_x}{a} \right) \frac{\partial}{\partial x} - \nu \frac{\partial^2}{\partial y^2} = 0,$$

which corresponds to the choice (2.12) of $\alpha$. For a justification, see [6].

## 3. ERROR ESTIMATES

It is first necessary to consider the well-posedness of the different problems.

### 3.1. Well-posedness of the approximate problem.

We make the following strong assumptions:

$(3.1)$  $a$ belongs to $W^{\infty, \infty}((0, L) \times \mathbf{R})$ and there exists $a_0 > 0$ such that $a \geq a_0 > 0$; $f$ belongs to $H^\infty((0, L) \times \mathbf{R})$ and $U_0$ to $C_c^\infty(\mathbf{R})$.

For $\nu$ small enough, there exists $\alpha(x, y, \nu)$ such that:

(3.2)
  $\alpha$ belongs to $W^{\infty,\infty}((0, L) \times \mathbf{R})$,

  there exists $\alpha_0 > 0$ independent of $\nu$ such that $\alpha \geq \alpha_0 > 0$,

  for each integer $k$, $\alpha$ is uniformly bounded in $W^{k,\infty}((0, L) \times \mathbf{R})$,

  $g$ defined by (2.11) is uniformly bounded in $L^{\infty}$.

Problem (2.8)–(2.9) can be decomposed in two parabolic problems in the $x$-direction,

$$-\nu \left( \frac{\partial}{\partial x} - \frac{a}{\nu} + \frac{\nu}{\alpha} \frac{\partial^2}{\partial y^2} \right) w = f, \qquad w(L, y) = 0,$$

and

$$\left( \frac{\partial}{\partial x} - \frac{\nu}{\alpha} \frac{\partial^2}{\partial y^2} \right) u_p = w, \qquad u_p(0, y) = U_0(y).$$

For $\nu$ small enough, both problems are well posed since they are uniformly parabolic for sufficiently small $\nu$ (see, for instance, [4, 5]), and we have for each problem a unique solution in $H^{\infty}((0, L) \times \mathbf{R})$. As for problem (1.1)–(2.9.bis), it is easy to see that for classical solutions the maximum principle holds and it is thus natural to suppose that there is a unique solution to (1.1)-(2.9.bis) in $H^{\infty}((0, L) \times \mathbf{R})$.

### 3.1. Majorations.

We set $e = u - u_p$. By subtracting (2.10) from (1.1) we can see that the error $e$ satisfies

(3.3)
$$a \frac{\partial e}{\partial x} - \nu \left( \frac{\partial^2 e}{\partial x^2} + \frac{\partial^2 e}{\partial y^2} \right) = \nu^2 \left[ \partial_x \left( \frac{1}{\alpha} \right) + \frac{\alpha - a}{\nu \alpha} \right] \frac{\partial^2 u_p}{\partial y^2}$$
$$+ \frac{\nu^3}{\alpha} \frac{\partial^2}{\partial y^2} \left( \frac{1}{\alpha} \frac{\partial^2 u_p}{\partial y^2} \right),$$

$$e(0, y) = 0, \qquad \alpha \frac{\partial e}{\partial x} - \nu \frac{\partial^2 e}{\partial y^2} = 0 \quad \text{at } x = L.$$

Under assumption (2.11) we obtain

$$\nu^2 \left[ \partial_x \left( \frac{1}{\alpha} \right) + \frac{\alpha - a}{\nu \alpha} \right] = \nu^3 g(x, y, \nu).$$

The main result of this section is:

**Theorem 3.1.** *Assume* (3.1) *and* (3.2) *and denote by* $u$ *the solution of* (1.1)–(2.9.bis) *and by* $u_p$ *the solution of* (2.8)–(2.9). *Then there exists* $\nu_0 > 0$ *such that for* $0 < \nu < \nu_0$ *there exists* $M_1 > 0$ *independent of* $\nu$ *such that* ($e = u - u_p$)

$$\left( \int_0^L \int_{\mathbf{R}} \left( \frac{\partial e}{\partial x} \right)^2 \right)^{1/2} \leq \nu^3 M_1.$$

The order in $\nu$ of the error estimate is the same as when $a$ is constant or depends only on $y$ (see [7, 8]).

For the proof of the theorem we shall need five intermediate lemmas.

**Lemma 3.2.** *Let $e$ belong to $H^{\infty}((0, L) \times \mathbf{R})$ and satisfy $\mathscr{L}(e) = z(x, y, \nu) \in L^2((0, L) \times \mathbf{R})$, $e(0, y) = 0$, and $\alpha \frac{\partial e}{\partial x} - \nu \frac{\partial^2 e}{\partial y^2} = 0$ at $x = L$. Then there exists $\nu_0 > 0$ such that $e$ for any $0 < \nu < \nu_0$ satisfies the following estimate:*

$$
(3.4) \quad \int_0^L \int_{\mathbf{R}} \frac{a(x, y)}{2} \left( \frac{\partial e}{\partial x} \right)^2
$$
$$
+ \frac{\nu^2}{2a} \left( \left( \frac{\partial^2 e}{\partial x^2} \right)^2 + \left( \frac{\partial^2 e}{\partial y^2} \right)^2 + 2 \left( \frac{\partial^2 e}{\partial x \partial y} \right)^2 \right) \leq \int_0^L \int_{\mathbf{R}} \frac{z^2}{a}.
$$

*Proof.* We square $\mathscr{L}(e) = z$, divide it by $a$, and then integrate by parts over the vertical strip. We obtain

$$
\int_0^L \int_{\mathbf{R}} \frac{1}{a} \left( a \frac{\partial e}{\partial x} - \nu \frac{\partial^2 e}{\partial y^2} \right)^2 + \frac{\nu^2}{a} \left( \frac{\partial^2 e}{\partial x^2} \right)^2 - 2\nu \int_0^L \int_{\mathbf{R}} \frac{\partial^2 e}{\partial x^2} \left( \frac{\partial e}{\partial x} - \frac{\nu}{a} \frac{\partial^2 e}{\partial y^2} \right)
$$
$$
= \int_0^L \int_{\mathbf{R}} a \left( \frac{\partial e}{\partial x} \right)^2 + \frac{\nu^2}{a} \left( \frac{\partial^2 e}{\partial y^2} \right)^2 + \frac{\nu^2}{a} \left( \frac{\partial^2 e}{\partial x^2} \right)^2
$$
$$
+ \nu \int_{\mathbf{R}} \left( \frac{\partial e}{\partial y} \right)^2 (L, y) dy - 2\nu \int_0^L \int_{\mathbf{R}} \frac{\partial^2 e}{\partial x^2} \left( \frac{\partial e}{\partial x} - \frac{\nu}{a} \frac{\partial^2 e}{\partial y^2} \right).
$$

We now rewrite the last integral:

$$
-2\nu \int_0^L \int_{\mathbf{R}} \frac{\partial^2 e}{\partial x^2} \left( \frac{\partial e}{\partial x} - \frac{\nu}{a} \frac{\partial^2 e}{\partial y^2} \right)
$$
$$
= 2\nu \int_0^L \int_{\mathbf{R}} \frac{\partial e}{\partial x} \left( \frac{\partial^2 e}{\partial x^2} - \frac{\partial}{\partial x} \left( \frac{\nu}{a} \frac{\partial^2 e}{\partial y^2} \right) \right) + 2\nu \int_{x=0} \left( \frac{\partial e}{\partial x} \right)^2 dy
$$
$$
- 2\nu \int_{x=L} \left( \frac{\partial e}{\partial x} \right)^2 dy + 2\nu \int_{x=L} \left( \frac{\partial e}{\partial x} \right)^2 \frac{\alpha}{a} dy
$$
$$
= \nu \int_{x=L} \left( \frac{\partial e}{\partial x} \right)^2 dy + \nu \int_{x=0} \left( \frac{\partial e}{\partial x} \right)^2 dy - 2\nu^2 \int_0^L \int_{\mathbf{R}} \frac{\partial e}{\partial x} \partial_x \left( \frac{1}{a} \right) \frac{\partial^2 e}{\partial y^2}
$$
$$
- 2\nu^2 \int_0^L \int_{\mathbf{R}} \frac{\partial e}{\partial x} \frac{1}{a} \frac{\partial^3 e}{\partial x \partial y^2} - 2\nu \int_{x=L} \left( \frac{\partial e}{\partial x} \right)^2 \left( 1 - \frac{\alpha}{a} \right) dy
$$
$$
= \nu \int_{x=0} \left( \frac{\partial e}{\partial x} \right)^2 - 2\nu^2 \int_0^L \int_{\mathbf{R}} \frac{\partial e}{\partial x} \partial_x \left( \frac{1}{a} \right) \frac{\partial^2 e}{\partial y^2} + 2\nu^2 \int_0^L \int_{\mathbf{R}} \left( \frac{\partial^2 e}{\partial x \partial y} \right)^2 \frac{1}{a}
$$
$$
+ 2\nu^2 \int_0^L \int_{\mathbf{R}} \frac{\partial^2 e}{\partial x \partial y} \partial_y \left( \frac{1}{a} \right) \frac{\partial e}{\partial x}
$$
$$
+ \nu \int_{x=L} \left( \frac{\partial e}{\partial x} \right)^2 - 2\nu^2 \int_{x=L} \left( \frac{\partial e}{\partial x} \right)^2 \left( \frac{a - \alpha}{\nu a} \right).
$$

Finally, we get the following equality:

$$
\int_0^L \int_{\mathbf{R}} \frac{1}{a} \left( a \frac{\partial e}{\partial x} - \nu \Delta e \right)^2
$$

$$
= \int_0^L \int_{\mathbf{R}} a \left( \frac{\partial e}{\partial x} \right)^2 + \frac{\nu^2}{a} \left( \frac{\partial^2 e}{\partial y^2} \right)^2 + \frac{\nu^2}{a} \left( \frac{\partial^2 e}{\partial x^2} \right)^2 + 2 \frac{\nu^2}{a} \left( \frac{\partial^2 e}{\partial x \, \partial y} \right)^2
$$

$$
+ \nu \int_{x=L} \left( \frac{\partial e}{\partial x} \right)^2 - 2\nu^2 \int_0^L \int_{\mathbf{R}} \frac{\partial e}{\partial x} \partial_x \left( \frac{1}{a} \right) \frac{\partial^2 e}{\partial y^2}
$$

$$
+ 2\nu^2 \int_0^L \int_{\mathbf{R}} \frac{\partial^2 e}{\partial x \, \partial y} \partial_y \left( \frac{1}{a} \right) \frac{\partial e}{\partial x} - 2\nu^2 \int_{x=L} \left( \frac{\partial e}{\partial x} \right)^2 \frac{a - \alpha}{\nu a}
$$

$$
+ \nu \int_{x=L} \left( \frac{\partial e}{\partial y} \right)^2 + \nu \int_{x=0} \left( \frac{\partial e}{\partial x} \right)^2 .
$$

With the hypothesis on $a$ and $\alpha$, the ratio $\frac{a-\alpha}{\nu a}$ is uniformly bounded in $\nu$ and the last integral of the second line can be controlled by the last integral of the first line for small $\nu$. Moreover, by using the inequality $\xi \eta \leq \frac{1}{2}(\xi^2 + \eta^2)$, we can control the other integrals of the second line, and there exists $\nu_0 > 0$ such that for $\nu_0 > \nu > 0$,

$$
\int_0^L \int_{\mathbf{R}} \frac{1}{a} \left( a \frac{\partial e}{\partial x} - \nu \Delta e \right)^2
$$

$$
\geq \frac{1}{2} \left( \int_0^L \int_{\mathbf{R}} a \left( \frac{\partial e}{\partial x} \right)^2 + \frac{\nu^2}{a} \left( \frac{\partial^2 e}{\partial y^2} \right)^2 + \frac{\nu^2}{a} \left( \frac{\partial^2 e}{\partial x^2} \right)^2 + 2 \frac{\nu^2}{a} \left( \frac{\partial^2 e}{\partial x \, \partial y} \right)^2 \right),
$$

and we get (3.4). $\quad \square$

To end the proof of Theorem 3.1, we need to estimate the right-hand side of equation (3.3). Let us introduce $w$ defined by

$$
-\nu \left( \frac{\partial}{\partial x} - \frac{a}{\nu} + \frac{\nu}{\alpha} \frac{\partial^2}{\partial y^2} \right) w = f .
$$

We first have to estimate the auxiliary unknown $w$ and its derivative with respect to $y$. The equation for $w$ corresponds to a parabolic problem in the direction of negative $x$. We shall use

**Lemma 3.3.** *Let $\beta(x, y, \nu)$ and $\gamma(x, y, \nu)$ belong to $W^{\infty, \infty}([0, L] \times \mathbf{R})$ and have derivates of any order uniformly bounded with respect to $\nu$. Let $q(x, y, \nu)$ be in $L^\infty((0, L), L^2(\mathbf{R}))$ for any $\nu$ in $(0, \nu_0)$. Assume that there exists a constant $C$ independent of $\nu$ such that*

$$
\left( \int_{\mathbf{R}} q^2(x, y, \nu) \, dy \right)^{1/2} \leq C \quad \text{for any } \nu \text{ in } (0, \nu_0) \text{ and } x \text{ in } (0, L).
$$

*For $\nu$ in $(0, \nu_0)$, let $p$ satisfy*

$$
-\nu \left( \frac{\partial}{\partial x} - \frac{a}{\nu} + \frac{\nu}{\alpha} \frac{\partial^2}{\partial y^2} + \nu \beta \frac{\partial}{\partial y} + \nu \gamma \right) p = q , \qquad 0 \leq x \leq L , \ y \in \mathbf{R} ,
$$

$$
p(L, y) = 0 .
$$

*Then, under assumptions* (3.1) *and* (3.2), *for* $\nu$ *small enough, we have the following estimate*:

$$\left(\int_{\mathbf{R}} p^2(x, y)\, dy\right)^{1/2} \le \frac{2C}{a_0} \quad \textit{for any } x \textit{ in } (0, L).$$

*Proof.* Observe first that this result seems natural, since as $\nu$ tends to zero we have formally $ap = q$. It is useful to make the change of variables $x \to L - x$. The equation becomes, with the same notations,

$$\nu \left(\frac{\partial}{\partial x} + \frac{a}{\nu} - \frac{\nu}{\alpha} \frac{\partial^2}{\partial y^2} - \nu \beta \frac{\partial}{\partial y} - \nu \gamma\right) p = q, \qquad 0 \le x \le L, \ y \in \mathbf{R},$$

$$p(0, y) = 0.$$

Multiply the equation by $p$ and integrate by parts over $\mathbf{R}$,

$$\frac{d}{dx} \frac{\nu}{2} \int_{\mathbf{R}} p^2 + \int_{\mathbf{R}} \left(a + \frac{\nu^2}{2} \frac{\partial \beta}{\partial y} - \frac{\nu^2}{2} \frac{\partial}{\partial y}\left(\frac{1}{\alpha}\right) - \nu^2 \gamma\right) p^2$$
$$+ \nu^2 \int_{\mathbf{R}} \frac{1}{\alpha} \left(\frac{\partial p}{\partial y}\right)^2 = \int_{\mathbf{R}} qp,$$

so that we have for $\nu$ small enough

$$\frac{d}{dx} \frac{\nu}{2} \int_{\mathbf{R}} p^2 + \frac{a_0}{2} \int_{\mathbf{R}} p^2 \le \left(\int_{\mathbf{R}} q^2\right)^{1/2} \left(\int_{\mathbf{R}} p^2\right)^{1/2}.$$

Let

$$h(x) = \left(\int_{\mathbf{R}} p^2(x, y)\, dy\right)^{1/2}.$$

We then have

$$\nu \frac{d\sqrt{h}}{dx} + \frac{a_0}{2} \sqrt{h} \le C.$$

By Gronwall's lemma we obtain

$$\sqrt{h} \le \frac{2C}{a_0},$$

which enables us to conclude the proof. $\square$

With Lemma 3.3, it is easy to establish Lemma 3.4 below. Recall that $w$ is defined by

$$-\nu \left(\frac{\partial}{\partial x} - \frac{a}{\nu} + \frac{\nu}{\alpha} \frac{\partial^2}{\partial y^2}\right) w = f, \qquad w(L, y) = 0.$$

**Lemma 3.4.** *Assume* (3.1) *and* (3.2); *then for* $\nu$ *small enough, $w$ satisfies the following estimate*:

$$\left(\int_{\mathbf{R}} \left(\frac{\partial^i w}{\partial y^i}\right)^2 (x, y)\, dy\right)^{1/2} \le C_i \sup_{0 \le x \le L} \|f(x, \cdot)\|_{H^i(\mathbf{R})}$$

$$\textit{for any } x \textit{ in } (0, L) \textit{ and } i = 0, 1, \ldots, 4,$$

*with* $C_i$ *independent of* $\nu$.

*Proof.* For $i = 0$, this reduces to Lemma 3.3. Suppose the assertion is true for some $i \geq 0$. Differentiating $i + 1$ times with respect to $y$ the equation satisfied by $w$, we get

$$-\nu \left( \frac{\partial}{\partial x} - \frac{a}{\nu} + \frac{\nu}{\alpha} \frac{\partial^2}{\partial y^2} + \nu(i+1) \frac{\partial}{\partial y} \left( \frac{1}{\alpha} \right) \frac{\partial}{\partial y} + \nu \frac{i(i+1)}{2} \frac{\partial^2}{\partial y^2} \left( \frac{1}{\alpha} \right) \right) \frac{\partial^{i+1} w}{\partial y^{i+1}}$$

$$= \frac{\partial^{i+1} f}{\partial y^{i+1}} + \sum_{j=0}^{i} \beta_j(x, y, \nu) \frac{\partial^j w}{\partial y^j},$$

$$\frac{\partial^{i+1} w}{\partial y^{i+1}}(L, y) = 0,$$

where the functions $\beta_j$ are uniformly bounded. To complete the proof, it suffices to apply Lemma 3.3 with

$$\beta = (i+1) \frac{\partial}{\partial y} \left( \frac{1}{\alpha} \right), \qquad \gamma = \frac{1}{2} i(i+1) \frac{\partial^2}{\partial y^2} \left( \frac{1}{\alpha} \right),$$

and

$$q = \frac{\partial^{i+1} f}{\partial y^{i+1}} + \sum_{j=0}^{i} \beta_j(x, y, \nu) \frac{\partial^j w}{\partial y^j}. \qquad \square$$

We now consider the parabolic equation in the direction of positive $x$. We will prove two lemmas.

**Lemma 3.5.** *Let $p_0(y)$ belong to $L^2(\mathbf{R})$, and let $\beta(x, y, \nu)$ and $\gamma(x, y, \nu)$ belong to $W^{\infty, \infty}([0, L] \times \mathbf{R})$ and have derivates of any order uniformly bounded with respect to $\nu$. Let $q(x, y, \nu)$ be in $L^{\infty}((0, L), L^2(\mathbf{R}))$ for any $\nu$ in $(0, \nu_0)$. Assume that there exists a constant $C$ independent of $\nu$ such that*

$$\left( \int_{\mathbf{R}} q^2(x, y, \nu) \, dy \right)^{1/2} \leq C \quad \text{for any } \nu \text{ in } (0, \nu_0) \text{ and } x \text{ in } (0, L).$$

*For $\nu$ in $(0, \nu_0)$, let $p$ satisfy*

$$\left( \frac{\partial}{\partial x} - \frac{\nu}{\alpha} \frac{\partial^2}{\partial y^2} + \nu \beta \frac{\partial}{\partial y} + \nu \gamma \right) p = q, \qquad 0 \leq x \leq L, y \in \mathbf{R},$$

$$p(0, y) = p_0(y).$$

*Then, under assumptions (3.1) and (3.2), for $\nu$ small enough, there exists $K$ independent of $\nu$ such that we have the following estimate:*

$$\left( \int_{\mathbf{R}} p^2(x, y) \, dy \right)^{1/2} \leq \left( \int_{\mathbf{R}} p_0^2(y) \, dy \right)^{1/2} e^{\nu_0 K L} + \frac{C}{\nu_0 K} (e^{\nu_0 K L} - 1).$$

*Proof.* As in Lemma 3.3, we multiply by $p$, integrate by parts, and get

$$\frac{d\sqrt{h}}{dx} - \nu K \sqrt{h} \leq C,$$

where

$$h(x) = \int_{\mathbf{R}} p^2(x, y) \, dy \quad \text{and} \quad K = \sup_{x, y} \left\{ \left| \frac{1}{2} \partial_{yy} \left( \frac{1}{\alpha} \right) \right| + \left| \frac{1}{2} \partial_y(\beta) \right| + |\gamma| \right\}.$$

Thanks to Gronwall's lemma we obtain

$$\left( \int_{\mathbf{R}} p^2(x, y)\, dy \right)^{1/2} \leq \left( \int_{\mathbf{R}} p_0^2(y)\, dy \right)^{1/2} e^{\nu K x} + \frac{C}{\nu K}(e^{\nu K x} - 1).$$

Since $0 \leq x \leq L$ and the functions $e^x$ and $(e^x - 1)/x$ are increasing, the proof of the estimate of Lemma 3.5 is complete. $\square$

We can now estimate the derivatives with respect to $\nu$ of the solution to problem (2.8)–(2.9).

**Lemma 3.6.** *Assume* (3.1) *and* (3.2); *then for* $\nu$ *small enough, there exist constants* $C_i$ *independent of* $\nu$ *such that*

$$\left( \int_{\mathbf{R}} \frac{\partial^i u_p^2}{\partial y^i}(x, y)\, dy \right)^{1/2} \leq C_i \quad \text{for any } x \text{ in } (0, L) \text{ and } i = 0, 1, \ldots, 4,$$

*with* $C_i$ *independent of* $\nu$.

*Proof.* With the help of Lemmas 3.4 and 3.5, the proof is very similar to the one of Lemma 3.4, and is omitted here. $\square$

To finish the proof of Theorem 3.1, we apply Lemma 3.2 with $z$ equal to the right-hand side of (3.3), and then Lemma 3.6 to problem (2.8)–(2.9).

3.2. **Lower bounds.** In this subsection, we shall prove that the difference between the solution of the convection-diffusion equation (1.1) and the solution of (1.2) is of order one in the viscosity, for $\nu$ small enough, when $\partial_x(f/a)$ is different from zero. We rewrite both equations:

$$\mathscr{L}(u) = a(x, y)\frac{\partial u}{\partial x} - \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = f,$$

$$0 \leq x \leq L, \quad y \in \mathbf{R}, \quad a(x, y) \geq a_0 > 0,$$

$$u(0, y) = U_0(y) \quad \text{and} \quad \alpha \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial y^2} = 0$$

and

$$a(x, y)\frac{\partial v}{\partial x} - \nu \frac{\partial^2 v}{\partial y^2} = f, \qquad 0 \leq x \leq L, \quad y \in \mathbf{R},$$

$$v(0, y) = U_0(y).$$

For this purpose, we set up, as in §1, an asymptotic expansion of the form

$$u = u_0 + \nu u_1 + \cdots \quad \text{and} \quad v = v_0 + \nu v_1 + \cdots.$$

At order zero, we get

$$u_0(x, y) = v_0(x, y) = U_0(y) + \int_0^x \frac{f(s, y)}{a(s, y)}\, ds,$$

and at order one,

$$a\frac{\partial u_1}{\partial x} = \frac{d^2 U_0}{dy^2}(y) + \int_0^x \frac{\partial^2}{\partial y^2}\left( \frac{f}{a} \right)(s, y)\, ds + \frac{\partial}{\partial x}\left( \frac{f}{a} \right)(x, y),$$

$$u_1(0, y) = 0$$

and

$$a\frac{\partial v_1}{\partial x} = \frac{d^2 U_0}{dy^2}(y) + \int_0^x \frac{\partial^2}{\partial y^2}\left(\frac{f}{a}\right)(s,y)\,ds, \qquad v_1(0,y) = 0.$$

We now write the difference $u - v$ in the form

$$u - v = \{u - u_0 - \nu u_1\} - \{v - v_0 - \nu v_1\} + \nu(u_1 - v_1)$$

(we used the equality of $u_0$ and $v_0$). We can estimate the terms inside the curly brackets with the help of Lemmas 3.2 and 3.5. Indeed, $v - v_0 - \nu v_1 = v_e$ satisfies

$$a(x,y)\frac{\partial v_e}{\partial x} - \nu\frac{\partial^2 v_e}{\partial y^2} = \nu^2\frac{\partial^2 v_1}{\partial y^2},$$

and $u - u_0 - \nu u_1 = u_e$ satisfies

$$a(x,y)\frac{\partial u_e}{\partial x} - \nu\left(\frac{\partial^2 u_e}{\partial x^2} + \frac{\partial^2 u_e}{\partial y^2}\right) = \nu^2 \Delta u_1.$$

By Lemma 3.2 for $u_e$, and by Lemma 3.5 for $v_e$, we know that for $\nu$ small enough there exist $M_u$ and $M_v$ (independent of $\nu$) such that

$$\left\|\frac{\partial u_e}{\partial x}\right\| \le \nu^2 M_u \quad \text{and} \quad \left\|\frac{\partial v_e}{\partial x}\right\| \le \nu^2 M_v,$$

where $\|\cdot\|$ denotes the $L^2$ norm in $L^2((0,L) \times \mathbf{R})$. Finally,

$$\left\|\frac{\partial(u-v)}{\partial x}\right\| \ge -\nu^2 M_u - \nu^2 M_v + \nu\left\|\frac{1}{a}\frac{\partial}{\partial x}\left(\frac{f}{a}\right)\right\|.$$

Thus, for $\nu$ small enough, since $\partial_x(f/a)$ is different from zero, we have

$$\left\|\frac{\partial(u-v)}{\partial x}\right\| \ge \frac{\nu}{2}\left\|\frac{1}{a}\frac{\partial}{\partial x}\left(\frac{f}{a}\right)\right\|.$$

## 4. NUMERICAL EXPERIMENTS

In this section we consider a convection-diffusion problem in $\Omega = (0,1) \times (0,1)$ (see Figure 1). We denote:

$$\Gamma_0 = \{0\} \times (0,1), \quad \Gamma_1 = \{1\} \times (0,1), \quad \Gamma_2 = (0,1) \times \{0,1\}.$$
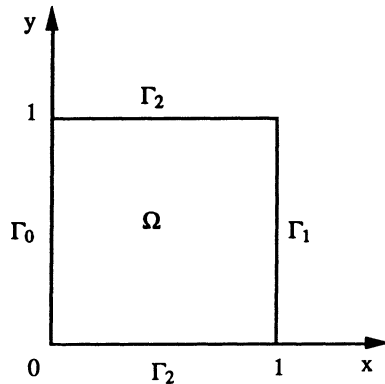


FIGURE 1. Computational domain

The convection-diffusion problem is:

$$(4.1) \quad \begin{cases} au_x - \nu\Delta u = f & \text{in } \Omega, \\ u = u_0 & \text{on } \Gamma_0, \\ au_x - \nu u_{yy} = 0 & \text{on } \Gamma_1, \\ u_y = 0 & \text{on } \Gamma_2. \end{cases}$$

Starting from a given solution, $\tilde{u}$, of (4.1), we deduce $f$ and $u_0$. We compare $\tilde{u}$ with the numerical solutions of the parabolized approximations to (4.1). These are:

1. "*Single-sweep*":

$$(4.2) \quad \begin{cases} u_x - \dfrac{\nu}{a}u_{yy} = \dfrac{f}{a} & \text{in } \Omega, \\ u = u_0 & \text{on } \Gamma_0, \\ u_y = 0 & \text{on } \Gamma_2. \end{cases}$$

2. "*Double-sweep*":

$$\begin{cases} -v_x + \dfrac{a}{\nu}v - \dfrac{\nu}{a}v_{yy} = \dfrac{f}{\nu} & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_1, \\ v_y = 0 & \text{on } \Gamma_2 \end{cases}$$

followed by

$$(4.3) \quad \begin{cases} u_x - \dfrac{\nu}{a}u_{yy} = v & \text{in } \Omega, \\ u = u_0 & \text{on } \Gamma_0, \\ u_y = 0 & \text{on } \Gamma_2. \end{cases}$$

3. "*Optimized double-sweep*":

$$\begin{cases} -v_x + \dfrac{a}{\nu}v - \dfrac{\nu}{\alpha}v_{yy} = \dfrac{f}{\nu} & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma_1, \\ v_y = 0 & \text{on } \Gamma_2 \end{cases}$$

(4.4)    followed by

$$\begin{cases} u_x - \dfrac{v}{\alpha}u_{yy} = v & \text{in } \Omega, \\ u = u_0 & \text{on } \Gamma_0, \\ u_y = 0 & \text{on } \Gamma_2. \end{cases}$$

Here, $\alpha$ is a function which satisfies (2.11). When replacing (4.1) by (4.2) or (4.3) or (4.4), we introduce a theoretical error which is $O(\nu)$, $O(\nu^2)$, $O(\nu^3)$, respectively. We are going to verify these estimates in numerical experiments. We choose $a$ and $\tilde{u}$ such that

(i) $a$ depends on $x$ only;
(ii) $\tilde{u}(x, y) = \tilde{u}_1(x)\cos \pi y$.

With these assumptions, we can write

$$f(x, y) = f_1(x)\cos \pi y,$$

and we have, in (4.2), (4.3), and (4.4),

$$u(x, y) = u_1(x) \cos \pi y \quad \text{and} \quad v(x, y) = v_1(x) \cos \pi y.$$

Furthermore, the function $\alpha$, in "optimized double-sweep", depends on $x$ only. Now, we may replace (4.2), (4.3), (4.4) respectively by the following ordinary differential equations:

1. *"Single-sweep"*:

(4.5)
$$\begin{cases} u_1' + \dfrac{\nu \pi^2}{a} u_1 = \dfrac{f_1}{a} & \text{in } (0, 1), \\ u_1(0) = u_0. \end{cases}$$

2. *"Double-sweep"*:

$$\begin{cases} -v_1' + \left( \dfrac{a}{\nu} + \dfrac{\nu \pi^2}{a} \right) v_1 = \dfrac{f_1}{\nu} & \text{in } (0, 1), \\ v_1(1) = 0 \end{cases}$$

(4.6)    followed by

$$\begin{cases} u_1' + \dfrac{\nu \pi^2}{a} u_1 = v_1 & \text{in } (0, 1), \\ u_1(0) = u_0. \end{cases}$$

3. *"Optimized double-sweep"*:

(4.7)
$$\begin{cases} -v_1' + \left( \dfrac{a}{\nu} + \dfrac{\nu \pi^2}{\alpha} \right) v_1 = \dfrac{f_1}{\nu} & \text{in } (0, 1), \\ v_1(1) = 0 \end{cases}$$

followed by

(4.8)
$$\begin{cases} u_1' + \dfrac{\nu \pi^2}{\alpha} u_1 = v_1 & \text{in } (0, 1), \\ u_1(0) = u_0. \end{cases}$$

Numerical solutions are computed by using a Crank-Nicolson scheme. It is well known that this scheme is of second order and is unconditionally stable. In numerical experiments we choose the mesh width $h = 10^{-4}$, so that the discretization error is negligible.

For the function $\alpha$ which appears in (4.8), we propose two choices:

1. *First choice*:

(4.9)
$$\alpha = a + \nu \frac{a'}{a} \quad \text{if } a' \geq 0,$$

$$\alpha = a \frac{1}{1 - \nu \frac{a'}{a^2}} \quad \text{if } a' \leq 0.$$

2. *Second choice*: $\alpha$ is such that

(4.10)
$$\begin{cases} \left( \dfrac{1}{\alpha} \right)' - a \left( \dfrac{1}{\alpha} \right) + \dfrac{1}{\nu} = 0 & \text{in } (0, 1), \\ \left( \dfrac{1}{\alpha} \right)(1) = \left( \dfrac{1}{a} \right)(1). \end{cases}$$
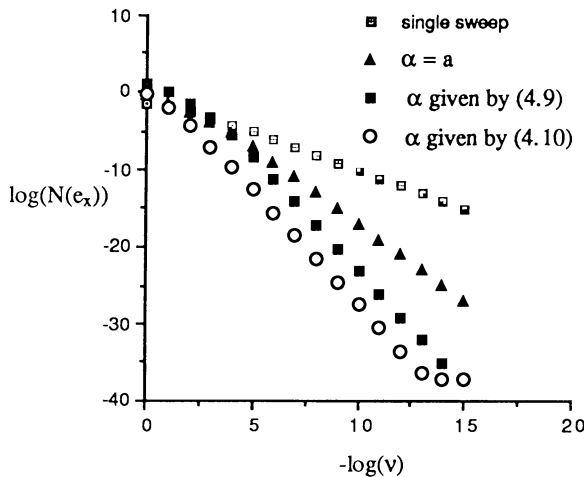
FIGURE 2. Log of the error as a function of the log of the viscosity for the first example

In this case, we compute $(\frac{1}{\alpha})$ by using a Crank-Nicolson scheme. (In fact, we chose $g = 0$ in (2.11).)

If we denote by $e$ the difference between the exact solution $\tilde{u}$ of (4.1) and the solution of (4.2), (4.3), or (4.4), we define by analogy to Theorem 3.1:

$$N(e_x) = \int_\Omega \frac{a}{2}|e_x|^2.$$

We have performed many numerical experiments. In every case, we obtained results which are very similar to the results given here. We plot $\log(N(e_x))$ as a function of $-\log\nu$.

1. *First example.* Here we have chosen

$$\tilde{u}(x) = (x - 1)^2$$

and

$$a(x) = \tfrac{1}{2}(3 - \cos 50(x - 1)).$$

We can see in Figure 2 that the error $N(e_x)$ is $O(\nu)$ for the "single-sweep", $O(\nu^2)$ for the "double-sweep", and $O(\nu^3)$ for the "optimized double-sweep". The choice (4.10) of $\alpha$ is better than (4.9).

2. *Second example.* Here we have

$$\tilde{u}(x) = (x - 1)^2 + \nu\pi^2(x - 1) - 1$$

and

$$a(x) = 1 + x(1 - x).$$

When $0 < -\log\nu < 8$, we can see in Figure 3 (next page) results which are similar to the previous ones. For larger values of $-\log\nu$ (or smaller values of $\nu$) we can observe that the choice (4.10) of $\alpha$ is clearly better than (4.9). We can explain this by the following remark: for the choice (4.10) of $\alpha$, the conditions along $\Gamma_1$ in (4.1) and (4.7) are similar because $\alpha(1) = a(1)$. This is not the case for (4.9), and we are not in the conditions of Theorem 3.1.
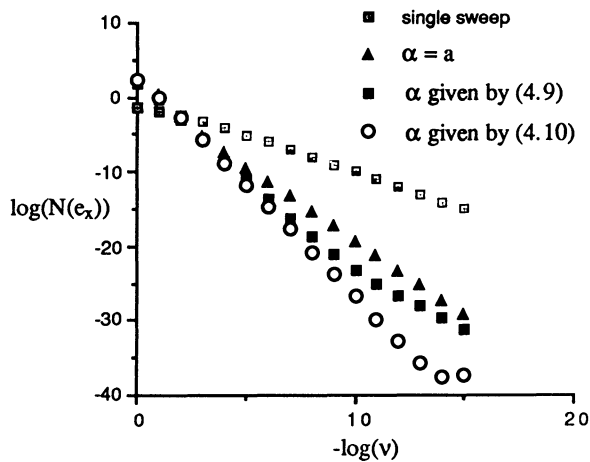
FIGURE 3. Log of the error as a function of the log of the viscosity for the second example

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