

A LEAST SQUARES PETROV-GALERKIN FINITE ELEMENT METHOD FOR THE STATIONARY NAVIER-STOKES EQUATIONS

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ABSTRACT. In this paper, a Galerkin/least squares-type finite element method is proposed and analyzed for the stationary Navier-Stokes equations. The method is consistent and stable for any combination of discrete velocity and pressure spaces (without requiring a Babuška-Brezzi stability condition). The existence, uniqueness and convergence (at optimal rate) of the discrete solution is proved in the case of sufficient viscosity (or small data).

1. INTRODUCTION

For mixed finite element methods solving the stationary Navier-Stokes equations, it is an important convergence stability condition that the Babuška-Brezzi inequality holds for the combination of finite element subspaces (see [1, 13]). Recently, in an attempt to circumvent this constraint, the so-called CBB [6] or stabilized finite element methods [2-5] have been developed, motivated by SD (or SUPG) methods [7, 8]. In addition to works [3-6] on the Stokes problems, the paper [9] proposed and analyzed a stabilized SD method for time-dependent N-S equations, and a stabilized, piecewise discontinuous vorticity-stream function formulation of mesh-dependent type for stationary N-S equations has been discussed in the paper [10] based on so-called homology families of generalized variational principles.

The present paper considers the stationary N-S equations in primitive variables. In this direction, L. Tobiska and G. Lube [12] proposed a penalty finite element method of streamline diffusion type. It is a stabilized method in which the finite element spaces of velocity and pressure are not required to satisfy the discrete B-B condition. But it is not consistent with the exact solution, owing to the addition of the penalty term $\alpha(\nabla p, \nabla q)$, and the optimal estimates of convergence rate cannot be achieved. In this paper, another stabilized finite element method is studied, which is different from the method in [11, 12]. It is an application of the Galerkin/least squares method [14] and its alternative [16] to nonlinear equations. Least squares forms of residuals are added to the Galerkin method for enhancing its stability without degrading accuracy.

For the following presentation we introduce the following notation: $X = V \times Q$, $V = H_0^1(\Omega)^n$, $Q = L_0^2(\Omega) = \{q \in L^2(\Omega) \mid \int_{\Omega} q \, dx = 0\}$, $(\circ, \circ)_G$ the

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inner product in $L^2(G)$ and $L^2(G)^n$, respectively, $G \subseteq \Omega$. Let $\|w\|_{k,p,G}$ and $|w|_{k,p,G}$ be the usual norm and seminorm on the Sobolev space $W^{k,p}(G)$, respectively. For vector-valued functions $u = (u_1, \dots, u_n) \in W^{k,p}(G)^n$ and $v = (v_1, \dots, v_n) \in L^\infty(G)^n$ we use the following norms and seminorms, respectively: $\|u\|_{k,p,G}^p = \sum_{i=1}^n \|u_i\|_{k,p,G}^p$, $|u|_{k,p,G}^p = \sum_{i=1}^n |u_i|_{k,p,G}^p$, $\|v\|_{0,\infty,G} = \max_i \|v_i\|_{0,\infty,G}$. In the case of $G = \Omega$ and $p = 2$ we omit the indices G and p .

Throughout the paper, C indicates a positive constant, possibly different at different occurrences, which is independent of the mesh parameter h , but may depend on Ω , on the Reynolds number and other parameters introduced in this paper. Notations not especially explained are used with their usual meanings.

An outline of the paper follows. In §2 we present the new finite element variational formulation for the N-S equations. The existence and uniqueness of the finite element solution is studied in §3. Its error analysis is performed in §4, and concluding remarks are made in §5.

2. FINITE ELEMENT FORMULATION

Let Ω be a convex domain with boundary Γ in R^n ($n = 2, 3$). We consider the following stationary Navier-Stokes equations with boundary conditions:

$$(2.1) \quad \begin{aligned} -\mu\Delta u + (u\nabla)u + \nabla p &= f & \text{in } \Omega, \\ \operatorname{div} u &= 0 & \text{in } \Omega, \\ u &= 0 & \text{on } \Gamma, \end{aligned}$$

where $u = (u_1, \dots, u_n)$ is the velocity vector, p the pressure, $f = (f_1, \dots, f_n)$ the body force, and μ the constant inverse Reynolds number. Problem (2.1) is equivalent to the following variational problem:

Find $(u, p) \in X$ such that for all $(v, q) \in X$

$$(2.2) \quad \mu a(u, v) + b(u; u, v) - (p, \operatorname{div} v) + (q, \operatorname{div} u) = (f, v),$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \nabla v \, dx, \\ \tilde{b}(u; v, w) &= \sum_{i,j=1}^n \int_{\Omega} u_i \partial v_j / \partial x_i w_j \, dx \quad \forall u, v, w \in V, \\ b(u; v, w) &= \frac{1}{2} \{ \tilde{b}(u; v, w) - \tilde{b}(u; w, v) \} \quad \forall u, v, w \in V. \end{aligned}$$

We define

$$N = \operatorname{Sup}_{u,v,w \in V} \frac{b(u; v, w)}{|u|_1 |v|_1 |w|_1}, \quad \|f\|_* = \operatorname{Sup}_{v \in V} \frac{(f, v)}{|v|_1}.$$

Theorem 2.1 [1]. *If $f \in H^{-1}(\Omega)^n$, then the problem (2.2) has a solution which, in addition, is unique provided that $\mu^{-2} N \|f\|_* < 1$.*

Let $\{\mathcal{F}_h\}$ be a family of triangulations of Ω into affinely equivalent finite elements K with $\Omega = \bigcup_{K \in \mathcal{F}_h} K$, which is assumed to be regular in the usual sense, and let $h_K = \operatorname{diam} K$. We also assume that $h/h_K < C$, $\forall K \in \mathcal{F}_h$, $h = \max_K h_K$, so that we can use inverse inequalities.

We introduce the following finite element spaces of velocity and pressure:

$$\begin{aligned} V_h(\Omega) &= \{v \in H_0^1(\Omega)^n : v|_K \in P_l(x), \forall K \in \mathcal{F}_h\}, \\ Q_h(\Omega) &= \{q \in Q \cap H^1(\Omega) : q|_K \in P_k(x), \forall K \in \mathcal{F}_h\}. \end{aligned}$$

Here, $P_l(x)$, $P_k(x)$ denote piecewise polynomials of degree l and k , respectively. We let $X_h = V_h \times Q_h$.

Paper [12] proposed the following penalty finite element method of streamline diffusion type with the penalty term $\alpha(\nabla p, \nabla q)$ for (2.2):

Find $\hat{u}_h = (u_h, p_h) \in X_h$ such that for all $\hat{v} \in X_h$

$$\begin{aligned} (2.3) \quad & \mu a(u_h, v) + b(u_h; u_h, v) - (p_h, \operatorname{div} v) + (q, \operatorname{div} u_h) \\ & + \sum_K \delta_K (-\mu \Delta u_h + (u_h \nabla) u_h + \nabla p_h, (u_h \nabla) v + \nabla q)_K \\ & + \alpha(\nabla p_h, \nabla q) \\ & = (f, v) + \sum_K \delta_K (f, (u_h \nabla) v + \nabla q)_K. \end{aligned}$$

In order to establish existence, uniqueness, and convergence of the solution of (2.3), the parameter δ_K is required to satisfy the condition $0 < \delta_K < C_1 \mu^{-1} h^2$, where C_1 is a certain constant.

In this paper, we present the following Petrov-Galerkin finite element formulation for problem (2.1): Find $\hat{u}_h = (u_h, p_h) \in X_h$ such that for all $\hat{v} \in X_h$

$$\begin{aligned} (2.4) \quad & \mu a(u_h, v) + b(u_h; u_h, v) - (p_h, \operatorname{div} v) + (q, \operatorname{div} u_h) \\ & + \sum_K \delta_K (-\mu \Delta u_h + (u_h \nabla) u_h + \nabla p_h, -\mu \Delta v + (u_h \nabla) v + \nabla q)_K \\ & = (f, v) + \sum_K \delta_K (f, -\mu \Delta v + (u_h \nabla) v + \nabla q)_K, \end{aligned}$$

where $\delta_K = \alpha h_K^2$, and $\alpha > 0$ is arbitrary.

For $u \in V$, $u_h \in V_h$, $\hat{v} = (v, q)$, $\hat{w} = (w, r) \in V \times (Q \cap H^1(\Omega))$, we define

$$\begin{aligned} B_\delta(u, u_h; \hat{v}, \hat{w}) &= \mu a(v, w) + b(u; v, w) - (q, \operatorname{div} w) + (r, \operatorname{div} v) \\ & + \sum_K \delta_K (-\mu \Delta v + u \nabla v + \nabla q, -\mu \Delta w + u_h \nabla w + \nabla r)_K, \\ L_\delta(u_h; \hat{w}) &= (f, w) + \sum_K \delta_K (f, -\mu \Delta w + u_h \nabla w + \nabla r)_K, \end{aligned}$$

where δ is the piecewise constant function defined by $\delta|_K = \delta_K$. Then (2.4) can be rewritten in the following form: Find $\hat{u}_h = (u_h, p_h) \in X_h$ such that

$$(2.5) \quad B_\delta(u_h, u_h; \hat{u}_h, \hat{w}) = L_\delta(u_h; \hat{w}) \quad \forall \hat{w} \in X_h.$$

Remark 1. Assume f belongs to $L^2(\Omega)^n$ and the solution (u, p) of (2.1) belongs to $(V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega))$; i.e., there holds

$$-\mu \Delta u + u \nabla u + \nabla p = f \quad \text{in } L^2(\Omega)^n.$$

Then $\hat{u} = (u, p)$ satisfies

$$(2.6) \quad B_\delta(u, u_h; \hat{u}, \hat{w}) = L_\delta(u_h; \hat{w}) \quad \forall u_h \in V_h, \hat{w} \in X_h,$$

i.e., (2.5) is consistent with the exact solution of problem (2.1) or (2.2).

3. EXISTENCE AND UNIQUENESS OF THE FINITE ELEMENT SOLUTION

For the discrete problem (2.5) we will establish existence and uniqueness of an approximate solution without requiring the B-B condition.

Theorem 3.1. *If $f \in L^2(\Omega)^n$, then (2.5) has at least one solution $\hat{u}_h = (u_h, p_h) \in X_h$.*

Proof. We use Brouwer’s fixed point theorem to prove our theorem. The proof proceeds in two steps.

(I) For a given $\hat{v}_h \in X_h$, the following linearized problem of (2.5) has a unique solution:

Find $\hat{u}_h = (u_h, p_h) \in X_h$ such that

$$(3.1) \quad B_\delta(v_h, v_h; \hat{u}_h, \hat{w}) = L_\delta(v_h; \hat{w}) \quad \forall \hat{w} \in X_h.$$

In fact, we have

$$B_\delta(v_h, v_h; \hat{u}_h, \hat{u}_h) = \mu|u_h|_1^2 + \|\delta^{1/2}(-\mu\Delta u_h + v_h \nabla u_h + \nabla p_h)\|_{0,h}^2,$$

where $\|\circ\|_{0,h} = (\sum_K \|\circ\|_{0,K}^2)^{1/2}$. By virtue of the coercivity of the bilinear form $B_\delta(v_h, v_h; \hat{u}, \hat{w})$ there exists a unique solution of (3.1), and since

$$\begin{aligned} L_\delta(v_h; \hat{u}_h) &= (f, u_h) + \sum_K \delta_K(f, -\mu\Delta u_h + v_h \nabla u_h + \nabla p_h)_K \\ &\leq (\mu^{-1}\|f\|_*^2 + \|\delta^{1/2}f\|_0^2)^{1/2} \\ &\quad \times (\mu|u_h|_1^2 + \|\delta^{1/2}(-\mu\Delta u_h + v_h \nabla u_h + \nabla p_h)\|_{0,h}^2)^{1/2}, \end{aligned}$$

we get

$$\begin{aligned} &(\mu|u_h|_1^2 + \|\delta^{1/2}(-\mu\Delta u_h + v_h \nabla u_h + \nabla p_h)\|_{0,h}^2)^{1/2} \\ &\leq (\mu^{-1}\|f\|_*^2 + \|\delta^{1/2}f\|_0^2)^{1/2} \end{aligned}$$

or

$$(3.2) \quad \begin{aligned} &(|u_h|_1^2 + \mu^{-1}\|\delta^{1/2}(-\mu\Delta u_h + v_h \nabla u_h + \nabla p_h)\|_{0,h}^2)^{1/2} \\ &\leq \frac{1}{\mu}(\|f\|_*^2 + \mu\|\delta^{1/2}f\|_0^2)^{1/2}. \end{aligned}$$

Therefore, for arbitrarily given $\hat{v}_h \in X_h$, the solution of (3.1) determines a map $F: \hat{v}_h \rightarrow \hat{u}_h = F(\hat{v}_h)$.

For convenience, let $1 = (\|f\|_*^2 + \mu\|\delta^{1/2}f\|_0^2)^{1/2}$, $R = 1/\mu$.

(II) For the set $B_R = \{\hat{v}_h \in X_h: |v_h|_1 \leq R\}$, F is a continuous map from B_R to B_R . In fact, by virtue of the estimate (3.2), it is easy to show that $F: B_R \rightarrow B_R$. Thus, we only need to prove F is continuous.

Let for arbitrarily given $\hat{v}_h^i \in B_R$ the approximations $\hat{u}_h^i = F(\hat{v}_h^i)$ ($i = 1, 2$) be defined by (3.1); then we have

$$(3.3) \quad B_\delta(v_h^i, v_h^i; \hat{u}_h^i, \hat{w}) = L_\delta(v_h^i; \hat{w}) \quad \forall \hat{w} \in X_h$$

and

$$(3.4) \quad (|u_h^i|_1^2 + \mu^{-1}\|\delta^{1/2}(-\mu\Delta u_h^i + v_h^i \cdot \nabla u_h^i + \nabla p_h^i)\|_{0,h}^2)^{1/2} \leq R \quad (i = 1, 2).$$

By (3.3), we have

$$(3.5) \quad \begin{aligned} & B_\delta(v_h^1, v_h^1, \hat{u}_h^1, \hat{w}) - B_\delta(v_h^2, v_h^2; \hat{u}_h^2, \hat{w}) \\ &= \sum_K \delta_K(f, (v_h^1 - v_h^2)\nabla w)_K \quad \forall \hat{w} \in X_h. \end{aligned}$$

Now we choose $\hat{w} = \hat{u}_h^1 - \hat{u}_h^2$, i.e., $\hat{w} = (w, r) = (u_h^1 - u_h^2, p_h^1 - p_h^2)$. We have

$$(3.6) \quad B_\delta(v_h^1, v_h^1; \hat{w}, \hat{w}) = \mu|w|_1^2 + \|\delta^{1/2}(-\mu\Delta w + v_h^1\nabla w + \nabla r)\|_{0,h}^2$$

and

$$(3.7) \quad \begin{aligned} & B_\delta(v_h^1, v_h^1; \hat{w}, \hat{w}) = B_\delta(v_h^1, v_h^1; \hat{u}_h^1, \hat{w}) - B_\delta(v_h^1, v_h^1; \hat{u}_h^2, \hat{w}) \\ &= B_\delta(v_h^2, v_h^2; \hat{u}_h^2, \hat{w}) - B_\delta(v_h^1, v_h^1; \hat{u}_h^2, \hat{w}) \\ &\quad + \sum_K \delta_K(f, (v_h^1 - v_h^2)\nabla w)_K \quad (\text{by 3.5}) \\ &= b(v_h^2 - v_h^1; u_h^2, w) \\ &\quad + \sum_K \delta_K((v_h^2 - v_h^1)\nabla u_h^2, -\mu\Delta w + v_h^1\nabla w + \nabla r)_K \\ &\quad + \sum_K \delta_K(-\mu\Delta u_h^2 + v_h^2\nabla u_h^2 + \nabla p_h^2, (v_h^2 - v_h^1)\nabla w)_K \\ &\quad + \sum_K \delta_K(f, (v_h^1 - v_h^2)\nabla w)_K \\ &=: S_1 + S_2 + S_3 + S_4. \end{aligned}$$

For S_4 , it is easy to get

$$(3.8) \quad |S_4| \leq \|v_h^1 - v_h^2\|_{0,\infty} |w|_1 \|\delta f\|_0.$$

By means of Sobolev's embedding theorem and an inverse inequality, we can prove that

$$(3.9) \quad \|v\|_{0,\infty} \leq C_0 h^{-\chi} |v|_1 \quad \forall v \in V_h$$

with $\chi > 0$ arbitrary in the case of $n = 2$ and $\chi = 1/2$ in the case of $n = 3$. Thus, (3.8) yields

$$(3.10) \quad |S_4| \leq C_0 h^{-\chi} \|\delta f\|_0 |v_h^1 - v_h^2|_1 |w|_1 \leq C_0 \mu^{1/2} \delta_M^{1/2} h^{-\chi} R |v_h^1 - v_h^2|_1 |w|_1,$$

where $\delta_M = \max_{x \in \Omega} \delta = \alpha h^2$. Similarly, by (3.4), (3.9) and the Cauchy-Schwarz inequality, we get

$$(3.11) \quad |S_1| \leq NR |v_h^2 - v_h^1|_1 |w|_1,$$

$$(3.12) \quad |S_2| \leq C_0 R h^{-\chi} \delta_M^{1/2} |v_h^2 - v_h^1|_1 \|\delta^{1/2}(-\mu\Delta w + v_h^1\nabla w + \nabla r)\|_{0,h},$$

$$(3.13) \quad |S_3| \leq C_0 \mu^{1/2} \delta_M^{1/2} h^{-\chi} R |v_h^2 - v_h^1|_1 |w|_1.$$

Combining (3.10), (3.11), and (3.13), we have

$$|S_1| + |S_3| + |S_4| \leq (NR + 2C_0 \mu^{1/2} \delta_M^{1/2} h^{-\chi} R) |v_h^1 - v_h^2|_1 |w|_1,$$

i.e.,

$$(3.14) \quad |S_1| + |S_3| + |S_4| \leq M(R) |v_h^1 - v_h^2|_1 |w|_1.$$

By (3.12), (3.14) and Cauchy's inequality, we obtain

$$(3.15) \quad \sum_{i=1}^4 |S_i| \leq L(R) |v_h^1 - v_h^2|_1 (\mu |w|_1^2 + \|\delta^{1/2}(-\mu\Delta w + v_h^1 \nabla w + \nabla r)\|_{0,h}^2)^{1/2}.$$

Then, by (3.6), (3.7), (3.15), we finally get

$$(3.16) \quad (\mu |w|_1^2 + \|\delta^{1/2}(-\mu\Delta w + v_h^1 \nabla w + \nabla r)\|_{0,h}^2)^{1/2} \leq L(R) |v_h^1 - v_h^2|_1,$$

where $M(R)$, $L(R)$ are constants independent of \hat{v}_h^i and \hat{u}_h^i . Noting that $\hat{w} = \hat{u}_h^1 - \hat{u}_h^2$ and (3.16), we conclude that F is a continuous map from B_R to B_R . By Brouwer's fixed point theory, this implies that F has at least one fixed point $\hat{u}_h = F(\hat{u}_h)$, i.e., the problem (2.5) has at least one solution $\hat{u}_h = (u_h, p_h) \in X_h$. \square

Theorem 3.2. *Assume that $f \in L^2(\Omega)^n$, $\mu^{-2}N\|f\|_* < 1$. Then there is a constant $h_0 > 0$ such that for all $h \leq h_0$ the problem (2.5) is uniquely solvable, and the solution $\hat{u}_h = (u_h, p_h)$ satisfies the estimate*

$$(3.17) \quad (\mu |u_h|_1^2 + \|\delta^{1/2}(-\mu\Delta u_h + u_h \nabla u_h + \nabla p_h)\|_{0,h}^2)^{1/2} \leq (\mu^{-1}\|f\|_*^2 + \|\delta^{1/2}f\|_0^2)^{1/2}.$$

Proof. From Theorem 3.1 we know that the problem (2.5) has at least one solution $\hat{u}_h = (u_h, p_h) \in X_h$ such that

$$(3.18) \quad B_\delta(u_h, u_h; \hat{u}_h, \hat{w}) = L_\delta(u_h; \hat{w}) \quad \forall \hat{w} \in X_h.$$

Setting $\hat{w} = \hat{u}_h$ in (3.18), we have

$$(3.19) \quad B_\delta(u_h, u_h; \hat{u}_h, \hat{u}_h) = L_\delta(u_h; \hat{u}_h),$$

$$(3.20) \quad B_\delta(u_h, u_h; \hat{u}_h, \hat{u}_h) = \mu |u_h|_1^2 + \|\delta^{1/2}(-\mu\Delta u_h + u_h \nabla u_h + \nabla p_h)\|_{0,h}^2,$$

$$(3.21) \quad L_\delta(u_h; \hat{u}_h) \leq (\mu^{-1}\|f\|_*^2 + \|\delta^{1/2}f\|_0^2)^{1/2} \times (\mu |u_h|_1^2 + \|\delta^{1/2}(-\mu\Delta u_h + u_h \nabla u_h + \nabla p_h)\|_{0,h}^2)^{1/2}.$$

By (3.19), (3.20), and (3.21), we have

$$(3.22) \quad (\mu |u_h|_1^2 + \|\delta^{1/2}(-\mu\Delta u_h + u_h \nabla u_h + \nabla p_h)\|_{0,h}^2)^{1/2} \leq (\mu^{-1}\|f\|_*^2 + \|\delta^{1/2}f\|_0^2)^{1/2}.$$

If we let $R = \mu^{-1}(\|f\|_*^2 + \mu\|\delta^{1/2}f\|_0^2)^{1/2}$, then (3.22) can be rewritten as follows:

$$(3.23) \quad (|u_h|_1^2 + \mu^{-1}\|\delta^{1/2}(-\mu\Delta u_h + u_h \nabla u_h + \nabla p_h)\|_{0,h}^2)^{1/2} \leq R.$$

In order to prove $\hat{u}_h \in X_h$ to be the unique solution of (2.5), we suppose \hat{u}_h^i ($i = 1, 2$) are two solutions of (2.5); by the above results we easily conclude that

$$(3.24) \quad (|u_h^i|_1^2 + \mu^{-1}\|\delta^{1/2}(-\mu\Delta u_h^i + u_h^i \nabla u_h^i + \nabla p_h^i)\|_{0,h}^2)^{1/2} \leq R \quad (i = 1, 2)$$

and

$$(3.25) \quad \begin{aligned} & B_\delta(u_h^1, u_h^1; \hat{u}_h^1, \hat{w}) - B_\delta(u_h^2, u_h^2; \hat{u}_h^2, \hat{w}) \\ &= \sum_K \delta_K(f, (u_h^1 - u_h^2) \nabla w)_K \quad \forall \hat{w} \in X_h. \end{aligned}$$

Now let $\hat{w} = \hat{u}_h^1 - \hat{u}_h^2$, i.e., $\hat{w} = (w, r) = (u_h^1 - u_h^2, p_h^1 - p_h^2)$. We have

$$(3.26) \quad B_\delta(u_h^1, u_h^1; \hat{w}, \hat{w}) = \mu|w|_1^2 + \|\delta^{1/2}(-\mu\Delta w + u_h^1\nabla w + \nabla r)\|_{0,h}^2,$$

and

$$(3.27) \quad B_\delta(u_h^1, u_h^1; \hat{w}, \hat{w}) = B_\delta(u_h^1, u_h^1; \hat{u}_h^1, \hat{w}) - B_\delta(u_h^1, u_h^1; \hat{u}_h^2, \hat{w}).$$

By using the same arguments as in the proof of Theorem 3.1.(II), based on (3.24)–(3.27), we can get an estimate similar to (3.16):

$$(3.28) \quad \begin{aligned} \mu|w|_1^2 + \frac{1}{2}\|\delta^{1/2}(-\mu\Delta w + u_h^1\nabla w + \nabla r)\|_{0,h}^2 \\ \leq (NR + 2C_0\mu^{1/2}\delta_M^{1/2}h^{-x}R + \frac{1}{2}C_0^2\delta_M h^{-2x}R^2)|w|_1^2. \end{aligned}$$

Since

$$R = \mu^{-1}(\|f\|_*^2 + \mu\|\delta^{1/2}f\|_0^2)^{1/2} \leq \mu^{-1}\|f\|_* + \mu^{-1/2}\delta_M^{1/2}\|f\|_0,$$

(3.28) becomes

$$(3.29) \quad \begin{aligned} \mu|w|_1^2(1 - \mu^{-2}N\|f\|_* - \mu^{-3/2}\delta_M^{1/2}N\|f\|_0 \\ - 2c_0\mu^{-1/2}\delta_M^{1/2}h^{-x}R - \frac{1}{2}C_0^2\mu^{-1}\delta_M h^{-2x}R^2) \\ + \frac{1}{2}\|\delta^{1/2}(-\mu\Delta w + u_h^1\nabla w + \nabla r)\|_{0,h}^2 \leq 0. \end{aligned}$$

As $\mu^{-2}N\|f\|_* < 1$, there exists a constant $\omega_1 \in (0, 1)$ such that $\mu^{-2}N\|f\|_* \leq \omega_1 < 1$. Since $\delta_M = \alpha h^2$, $\delta_M^{1/2}h^{-x} \rightarrow 0$ as $h \rightarrow 0$, there is a constant $h_0 > 0$ such that for all $h \leq h_0$

$$(3.30) \quad \mu^{-3/2}\delta_M^{1/2}N\|f\|_0 + 2C_0\mu^{-1/2}\delta_M^{1/2}h^{-x}R + \frac{1}{2}C_0^2\mu^{-1}\delta_M h^{-2x}R^2 \leq \frac{1}{2}(1 - \omega_1).$$

By (3.29) and (3.30), we obtain

$$(3.31) \quad \frac{1}{2}(1 - \omega_1)[\mu|w|_1^2 + \|\delta^{1/2}(-\mu\Delta w + u_h^1\nabla w + \nabla r)\|_{0,h}^2] \leq 0.$$

This means that $|w|_1^2 = \|\mu^{1/2}(-\mu\Delta w + u_h^1\nabla w + \nabla r)\|_{0,h}^2 = 0$, that is, $\hat{u}_h^1 = \hat{u}_h^2$. \square

4. CONVERGENCE OF THE METHOD

This section is devoted to establishing convergence results on the Galerkin/least squares finite element approximation for any choice of discrete velocity and pressure spaces.

Theorem 4.1. *Assume that $f \in L^2(\Omega)^n$, and let $\{\hat{u}_h\}$ be a sequence of solutions of (2.5) as h tends to zero. Then there is a subsequence $\{\hat{u}_h\}$ which converges strongly to a solution \hat{u} of (2.2) in the sense of*

$$(4.1) \quad \lim_{h \rightarrow 0} (\|u - u_h\|_1 + \alpha^{1/2}h\|\nabla p - \nabla p_h\|_0) = 0.$$

Proof. From Theorem 3.2 we see that $\{\hat{u}_h\}$ is uniformly bounded with respect to h , i.e., there exists a constant C independent of h and \hat{u}_h such that

$$(4.2) \quad \left(\mu|u_h|_1^2 + \sum_K \delta_K \|-\mu\Delta u_h + u_h\nabla u_h + \nabla p_h\|_{0,2,K}^2 \right)^{1/2} \leq C.$$

Using $h/h_K \leq C$, $\delta_K = \alpha h_K^2$, (4.2) and an inverse inequality, we obtain

$$(4.3) \quad |u_h|_1 + \alpha^{1/2} h \|\nabla p_h\|_0 \leq C.$$

By (4.3) we have

$$(4.4) \quad \|p_h\|_0 \leq C(\alpha),$$

where $C(\alpha)$ denotes a constant dependent on α and independent of h and \hat{u}_h . Therefore, we get

$$(4.5) \quad |u_h|_1 + \|p_h\|_0 \leq C(\alpha).$$

Consequently, by (4.5) there is a weakly converging subsequence in $V \times Q$, which for simplicity we denote again by $\{\hat{u}_h\}$. We will show that the weak limit \hat{u} is a solution of (2.2). For this, let $I_h = (I_h^1, I_h^2): (V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega)) \rightarrow V_h \times Q_h$ be the usual Lagrangian interpolation operator [15]. Setting $\hat{w} = I_h \hat{v} = (I_h^1 \hat{v}, I_h^2 \hat{q})$ in (2.5), we conclude that, as h tends to zero,

$$(4.6) \quad \mu a(u, v) + b(u; u, v) - (p, \operatorname{div} v) + (q, \operatorname{div} u) = (f, v) + \lim_{h \rightarrow 0} F_h^1,$$

where

$$F_h^1 = \sum_K \delta_K (f + \mu \Delta u_h - u_h \nabla u_h - \nabla p_h, -\mu \Delta I_h^1 v + u_h \nabla I_h^1 v + \nabla I_h^2 q)_K.$$

By (4.2) and Cauchy's inequality and an inverse inequality, we have

$$(4.7) \quad \begin{aligned} |F_h^1| &\leq \sum_K \delta_K \|f\|_{0,K} \|-\mu \Delta I_h^1 v + u_h \nabla I_h^1 v + \nabla I_h^2 q\|_{0,K} \\ &\quad + \left(\sum_K \delta_K \|-\mu \Delta u_h + u_h \nabla u_h + \nabla p_h\|_{0,K}^2 \right)^{1/2} \\ &\quad \times \left(\sum_K \delta_K \|-\mu \Delta I_h^1 v + u_h \nabla I_h^1 v + \nabla I_h^2 q\|_{0,K}^2 \right)^{1/2} \\ &\leq (\delta_M \|f\|_0 + C \delta_M^{1/2}) (\|-\mu \Delta I_h^1 v\|_{0,h} + \|u_h \nabla I_h^1 v + \nabla I_h^2 q\|_0) \\ &\leq (\delta_M \|f\|_0 + C \delta_M^{1/2}) (\|-\mu \Delta (v - I_h^1 v)\|_{0,h} + \|-\mu \Delta v\|_0 \\ &\quad + C_0 h^{-\chi} |I_h^1 v|_1 + |q - I_h^2 q|_1 + |q|_1) \\ &\leq (C \delta_M^{1/2} + \delta_M \|f\|_0) (C \|v\|_2 + C h^{-\chi} \|v\|_1 + C \|q\|_1) \\ &\leq C(f, \hat{v}) \delta_M^{1/2} h^{-\chi} \leq C(f, \hat{v}) h^{1-\chi}, \end{aligned}$$

where $C(f, \hat{v})$ denotes a constant dependent on f and \hat{v} , but independent of h . This gives $\lim_{h \rightarrow 0} F_h^1 = 0$, since $1 - \chi > 0$; i.e., for $\hat{v} = (v, q) \in (V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega))$, (4.6) becomes

$$(4.8) \quad \mu a(u, v) + b(u; u, v) - (p, \operatorname{div} v) + (q, \operatorname{div} u) = (f, v).$$

Since $(V \cap H^2(\Omega)^n) \times (Q \cap H^1(\Omega))$ is dense in $V \times Q$, we obtain that

$$(4.9) \quad \begin{aligned} \mu a(u, v) + b(u; u, v) - (p, \operatorname{div} v) + (q, \operatorname{div} u) \\ = (f, v) \quad \forall \hat{v} = (v, q) \in V \times Q; \end{aligned}$$

i.e., (u, p) is a weak solution.

Now we prove that $\lim_{h \rightarrow 0} (|u - u_h|_1 + \alpha^{1/2} h |p - p_h|_1) = 0$. In fact, for $f \in L^2(\Omega)^n$, the solution $\hat{u} = (u, p)$ of (2.2) belongs to $(H_0^1(\Omega) \cap H^2(\Omega))^n \times (H^1(\Omega)/R)$ and there holds $\mu a(u, u) = (f, u)$. Therefore,

$$\begin{aligned}
 & \mu |u_h - I_h^1 u|_1^2 + \|\delta^{1/2} \nabla(p_h - I_h^2 p)\|_{0,h}^2 \\
 &= \mu a(u_h - I_h^1 u, u_h - I_h^1 u) + \sum_K \delta_K (\nabla(p_h - I_h^2 p), \nabla(p_h - I_h^2 p))_K \\
 (4.10) \quad &= \mu a(u_h, u_h) + \sum_K \delta_K (\nabla p_h, \nabla p_h) - \mu a(2u_h - I_h^1 u, I_h^1 u) \\
 &\quad - \sum_K \delta_K (\nabla(2p_h - I_h^2 p), \nabla I_h^2 p)_K.
 \end{aligned}$$

Recalling that $\hat{u}_h = (u_h, p_h)$ satisfies (2.5), we have

$$(4.11) \quad \mu a(u_h, u_h) = (f, u_h) + F_h^2,$$

where

$$F_h^2 = \sum_K \delta_K (f + \mu \Delta u_h - u_h \nabla u_h - \nabla p_h, -\mu \Delta u_h + u_h \nabla u_h + \nabla p_h)_K$$

and

$$\begin{aligned}
 (4.12) \quad F_h^2 &= - \sum_K \delta_K (\nabla p_h, \nabla p_h) + \sum_K \delta_K (f + \mu \Delta u_h - u_h \nabla u_h, \nabla p_h)_K \\
 &\quad + \sum_K \delta_K (f + \mu \Delta u_h - u_h \nabla u_h - \nabla p_h, -\mu \Delta u_h + u_h \nabla u_h)_K.
 \end{aligned}$$

By (4.10), (4.11), and (4.12), we get

$$\begin{aligned}
 & \mu |u_h - I_h^1 u|_1^2 + \|\delta^{1/2} \nabla(p_h - I_h^2 p)\|_{0,h}^2 \\
 &= (f, u_h) - \mu a(2u_h - I_h^1 u, I_h^1 u) - \sum_K \delta_K (\nabla(2p_h - I_h^2 p), \nabla I_h^2 p)_K \\
 (4.13) \quad &+ \sum_K \delta_K (f + \mu \Delta u_h - u_h \nabla u_h, \nabla p_h)_K \\
 &+ \sum_K \delta_K (f + \mu \Delta u_h - u_h \nabla u_h - \nabla p_h, -\mu \Delta u_h + u_h \nabla u_h)_K \\
 &=: (f, u_h) - \mu a(2u_h - I_h^1 u, I_h^1 u) + F_h^3.
 \end{aligned}$$

By using the same techniques used in deriving the estimates of F_h^1 , we obtain

$$(4.14) \quad |F_h^3| \leq C \delta_M^{1/2} h^{-x}.$$

Thus, we conclude from (4.13) and (4.14) that as $h \rightarrow 0$

$$(4.15) \quad \lim_{h \rightarrow 0} (\mu |u_h - I_h^1 u|_1^2 + \|\delta^{1/2} \nabla(p_h - I_h^2 p)\|_{0,h}^2) = (f, u) - \mu a(u, u) = 0.$$

By virtue of the definition of δ and the assumption $h/h_K \leq C$, (4.15) yields

$$\lim_{h \rightarrow 0} (|u_h - I_h^1 u|_1 + \alpha^{1/2} h |p_h - I_h^2 p|_1) = 0.$$

Finally, $\lim_{h \rightarrow 0} (|u - u_h|_1 + \alpha^{1/2} h |p - p_h|_1) = 0$. \square

Remark 2. In the proof of (4.14), we used (4.2) and $\|u_h\|_{0,\infty} \leq C_0 h^{-\alpha} |u_h|_1$ and $(\sum_K \|\Delta u_h\|_{0,K}^2)^{1/2} \leq C h^{-1} |u_h|_1$.

Theorem 4.2. Assume that $\mu^{-2} N \|f\|_* < 1$ and that the exact solution $\hat{u} = (u, p)$ of (2.2) belongs to the space $(W_0^{1,\infty}(\Omega) \cap W^{l+1}(\Omega))^n \times W^{k+1}(\Omega)$, $l, k \in \mathbb{N}$. Then there is a positive constant h^* such that the following error estimate holds for the solution $\hat{u}_h = (u_h, p_h)$ of (2.5) for $h \leq h^*$:

$$\begin{aligned} & (\mu |u - u_h|_1^2 + \|\delta^{1/2} [-\mu \Delta(u - u_h) + u_h \nabla(u - u_h) + \nabla(p - p_h)]\|_{0,h}^2)^{1/2} \\ & \leq C(h^l + h^{k+1}), \end{aligned}$$

where the constant C depends on the seminorms $|u|_{1,\infty}$, $|u|_{l+1}$, $|p|_{k+1}$ of the exact solution of (2.2).

Proof. According to Theorem 2.1 and Theorem 3.2, both problems (2.2) and (2.5) have uniquely determined solutions. Let $\hat{w}_h = I_h \hat{u} - \hat{u}_h$, i.e., $\hat{w}_h = (w_h, r_h) = (I_h^1 u - u_h, I_h^2 p - p_h)$. It is easy to see that

$$(4.16) \quad S_1 =: B_\delta(u_h, u_h; \hat{w}_h, \hat{w}_h) = \mu |w_h|_1^2 + \|\delta^{1/2} (-\mu \Delta w_h + u_h \nabla w_h + \nabla r_h)\|_{0,h}^2,$$

and from (2.5), (2.6) we derive that

$$(4.17) \quad \begin{aligned} S_1 &= B_\delta(u_h, u_h; I_h \hat{u}, \hat{w}_h) - B_\delta(u_h, u_h; \hat{u}_h, \hat{w}_h) \\ &= B_\delta(u_h, u_h; I_h \hat{u}, \hat{w}_h) - B_\delta(u, u_h; \hat{u}, \hat{w}_h); \end{aligned}$$

i.e.,

$$S_1 = S_2 + S_3 + S_4,$$

where

$$\begin{aligned} S_2 &= \mu a(I_h^1 u - u, w_h) - (I_h^2 p - p, \operatorname{div} w_h), \\ S_3 &= \sum_K \delta_K (-\mu \Delta(I_h^1 u - u) + u_h \nabla I_h^1 u - u \nabla u + \nabla(I_h^2 p - p), -\mu \Delta w_h + u_h \nabla w_h + \nabla r_h)_K, \\ S_4 &= b(u_h; I_h^1 u, w_h) - b(u; u, w_h) + (r_h, \operatorname{div}(I_h^1 u - u)). \end{aligned}$$

By using well-known interpolation error estimates [15], it is easy to get

$$(4.18) \quad |S_2| \leq C(h^l + h^{k+1}) |w_h|_1,$$

$$(4.19) \quad \begin{aligned} |S_3| &\leq \frac{1}{4} \sum_K \delta_K \|-\mu \Delta w_h + u_h \nabla w_h + \nabla r_h\|_{0,K}^2 + C \delta_M (h^{2l-2} + h^{2k}) \\ &\quad + C \delta_M \|u_h \nabla(I_h^1 u - u) + (u_h - u) \nabla u\|_0^2 \\ &\leq \frac{1}{4} \|\delta^{1/2} (-\mu \Delta w_h + u_h \nabla w_h + \nabla r_h)\|_{0,h}^2 \\ &\quad + C \delta_M |w_h|_1^2 + C \delta_M (h^{2l-2} + h^{2k} + h^{2l-2\alpha} + h^{2l+2}). \end{aligned}$$

It should be mentioned that the constants C in (4.18) and (4.19) depend on the seminorms of the exact solution.

For S_4 , by using Green's formula, we have

$$(r_h, \operatorname{div}(I_h^1 u - u)) = (u_h \nabla w_h, I_h^1 u - u) - (u_h \nabla w_h + \nabla r_h, I_h^1 u - u),$$

and then

$$S_4 = b(u_h; I_h^1 u - u, w_h) + b(I_h^1 u - u; u, w_h) - b(w_h; u, w_h) \\ + (u_h \nabla w_h, I_h^1 u - u) - (u_h \nabla w_h + \nabla r_h, I_h^1 u - u).$$

Recalling that \hat{u}_h is bounded and that for the exact solution \hat{u} we have $|u|_1 \leq \mu^{-1} \|f\|_*$, and using an inverse estimate, we get

$$(4.20) \quad |S_4| \leq Ch^l |w_h|_1 + \mu^{-1} N \|f\|_* |w_h|_1^2 \\ + \left| \sum_K (-\mu \Delta w_h + u_h \nabla w_h + \nabla r_h, I_h^1 u - u)_K \right| \\ + \left| \sum_K (-\mu \Delta w_h, I_h^1 u - u)_K \right| \\ \leq Ch^l |w_h|_1 + \mu^{-1} N \|f\|_* |w_h|_1^2 + C \delta_{\min}^{-1} h^{2l+2} \\ + \frac{1}{4} \|\delta^{1/2} (-\mu \Delta w_h + u_h \nabla w_h + \nabla r_h)\|_{0,h}^2,$$

where $\delta_{\min} = \min_{x \in \Omega} \delta = \inf_K \delta_K$. Combining (4.16) with (4.17)–(4.20), we have

$$(4.21) \quad \mu |w|_1^2 (1 - \mu^{-2} N \|f\|_* - C \delta_M) + \frac{1}{2} \|\delta^{1/2} (-\mu \Delta w_h + u_h \nabla w_h + \nabla r_h)\|_{0,h}^2 \\ \leq C(h^l + h^{k+1}) |w_h|_1 + C \delta_M (h^{2l-2} + h^{2k} + h^{2l-2\alpha}) + C \delta_{\min}^{-1} h^{2l+2}.$$

Taking into account $\delta_M = \alpha h^2$ and $\mu^{-2} N \|f\|_* < 1$, we may conclude that there exists a sufficiently small $h^* > 0$ such that

$$(\mu |w_h|_1^2 + \|\delta^{1/2} (-\mu \Delta w_h + u_h \nabla w_h + \nabla r_h)\|_{0,h}^2)^{1/2} \\ \leq C(h^l + h^{k+1} + \delta_{\min}^{-1/2} h^{l+1})$$

for all $h \leq h^*$. Since $\delta_K = \alpha h_K^2$, $h/h_K \leq C$, we then obtain

$$(4.22) \quad (\mu |w_h|_1^2 + \|\delta^{1/2} (-\mu \Delta w_h + u_h \nabla w_h + \nabla r_h)\|_{0,h}^2)^{1/2} \\ \leq C(h^l + h^{k+1}).$$

Noting that $\hat{w}_h = I_h \hat{u} - \hat{u}_h$, and using the triangle inequality, we finally get

$$(\mu |u - u_h|_1^2 + \|\delta^{1/2} [-\mu \Delta (u - u_h) + u_h \nabla (u - u_h) + \nabla (p - p_h)]\|_{0,h}^2)^{1/2} \\ \leq C(h^l + h^{k+1}). \quad \square$$

Remark 3. By using Nitsche's duality technique, we can also get L^2 -error estimates for velocity and pressure.

Remark 4. If the finite element pressure subspace Q belongs only to $L_0^2(\Omega)$, we need to add the boundary integral term $\sum_K \beta h_K \oint_{\partial K} [q][r] ds$ to $B_\delta(u, u_h; \hat{v}, \hat{w})$ (where $\beta > 0$, $[q] = q_+ - q_-$) in order to obtain corresponding convergence results.

5. CONCLUSION

A finite element method of Galerkin/least squares-type for approximating the stationary N-S equations in primitive variables is presented with the following characteristics:

(i) The method exhibits stable and convergent approximation with optimal rate for any choice of the discrete velocity and pressure spaces, in contrast with the Galerkin mixed methods, in which the discrete B-B condition is required. For the 3-dimensional analysis, this point has important significance because of the implementational simplicity of lower and equal-order interpolations.

(ii) The method is variationally consistent, and the parameter $\alpha > 0$ can be arbitrarily chosen, yielding practical convenience and improved convergence error estimates compared to the associated penalty-type method [12].

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