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CHEBYSHEV EXPANSIONS FOR MODIFIED STRUVE AND RELATED FUNCTIONS

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ABSTRACT. We consider the approximation of the modified Struve functions $\mathbf{L}_0$ and $\mathbf{L}_1$, and the related functions $I_0 - \mathbf{L}_0$ and $I_1 - \mathbf{L}_1$, where I_0, I_1 are modified Bessel functions. Chebyshev expansions are derived to an accuracy of 20D for these functions. By using generalized bilinear and biquadratic maps we optimize the number of coefficients for 20D accuracy.

1. INTRODUCTION

The modified Struve functions $\mathbf{L}_n(x)$ satisfy the equation

$$x^2\mathbf{L}_n'' + x\mathbf{L}_n' - (x^2 + n^2)\mathbf{L}_n = \frac{4(x/2)^{n+1}}{\sqrt{\pi}\Gamma(n + \frac{1}{2})} \tag{1.1}$$

and are clearly closely related to the modified Bessel functions of the first kind, $I_n(x)$. The functions $\mathbf{L}_0$ and $\mathbf{L}_1$ appear in the fluid dynamics of water waves; see, for example, Hirata [5] or Shaw [10]. The functions $I_0 - \mathbf{L}_0$ and $I_1 - \mathbf{L}_1$ appear in surface wave problems, Wehausen and Laitone [11], and in unsteady aerodynamics, Ahmadi and Widnall [2]. Tables of values for $\mathbf{L}_0$ and $\mathbf{L}_1$ (small x) and $I_0 - \mathbf{L}_0$, $I_1 - \mathbf{L}_1$ (large x) appear in Chapter 12 of Abramowitz and Stegun [1]. Luke [6] gives coefficients for Chebyshev expansions for $\mathbf{L}_0$ $\mathbf{L}_1$ in the range $0 \leq |x| \leq 8$. The computation of $I_0 - \mathbf{L}_0$, $I_1 - \mathbf{L}_1$ by separate computation of the I and $\mathbf{L}$ functions leads to severe cancellation problems. Desmarais [4] developed expansions for these functions, but his results are incomplete.

In this paper, we derive Chebyshev expansions for the computation of $\mathbf{L}_0$ and $\mathbf{L}_1$ for all x, and $I_0 - \mathbf{L}_0$ and $I_1 - \mathbf{L}_1$ for $x \geq 0$. The coefficients are derived to an accuracy of 20D, with the number of coefficients minimized by generalized mappings. Test procedures show these values give the required accuracy.

2. RESULTS FOR $\mathbf{L}_0$, $I_0 - \mathbf{L}_0$

In this section, we describe in detail the procedures used for the functions of order 0. The methods are then applied to the order-1 functions, so only the relevant results are given in the next section.

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$\mathbf{L}_0(x)$ has the power series

$$\mathbf{L}_0(x) = \frac{2x}{\pi}\left(1 + \frac{x^2}{9} + \frac{x^4}{225} + \cdots\right), \tag{2.1}$$

which is convergent for all x. Thus, $\mathbf{L}_0$ is an odd function, so we need only restrict attention to approximating it for $x \geq 0$, then use $\mathbf{L}_0(-|x|) = -\mathbf{L}_0(|x|)$.

Asymptotically, we have

$$\mathbf{L}_0(x) \sim I_0(x) - \frac{1}{\pi}\sum_{k=0}^{\infty}\left(\frac{(-1)^k\Gamma\left(k+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}-k\right)}\right)\left(\frac{2}{x}\right)^{2k+1}. \tag{2.2}$$

The power series for $\mathbf{I}_0$ gives

$$I_0 - \mathbf{L}_0 = 1 - \frac{2x}{\pi} + \frac{x^2}{4} + \cdots, \tag{2.3}$$

which is decreasing for $x \geq 0$, the physically significant range.

Equation (2.1) suggests that we approximate $\mathbf{L}_0$ as

$$\mathbf{L}_0(x) = xg(x), \qquad 0 \leq x \leq a, \tag{2.4}$$

where we can expand $g(x)$ in terms of Chebyshev polynomials, and use the fact that g will be even. Schonfelder and Razaz [8] showed, however, that such expansions can give rise to serious error amplification if the function g varies greatly in size, as we have with $\mathbf{L}_0$. They recommend extracting an explicit exponential term which will absorb most of the function variation, leaving a more stable function to be expanded.

This idea, together with (2.2) and (2.3), suggests the following set of approximations:

$$\mathbf{L}_0 = \frac{2x}{\pi}e^x g_1(x), \qquad 0 \leq x \leq P, \tag{2.5}$$

$$I_0 - \mathbf{L}_0 = g_2(x), \qquad 0 \leq x \leq P, \tag{2.6}$$

$$I_0 - \mathbf{L}_0 = \frac{2}{\pi x}g_3(x), \qquad x > P, \tag{2.7}$$

where values of $\mathbf{L}_0$ for $x > P$ can be derived from (2.7) and one of the readily available approximations to I_0. The functions g_1, g_2, and g_3 are to be expanded as Chebyshev series. Thus, the intervals $[0, P]$ and (P, ∞) need to be transformed into $[-1, 1]$. For g_1 and g_2 the simple standard transform is $t = 2x/P - 1$, while the nature of (2.2) gives for g_3 the standard form $t = 2P^2/x^2 - 1$.

Scraton [9] and Schonfelder [7] have both shown the advantages of more general bilinear and biquadratic maps, so we consider the forms

$$t = \frac{ax - b}{x + b}, \qquad a = 1 + 2b/P, \ b > 0 \tag{2.8}$$

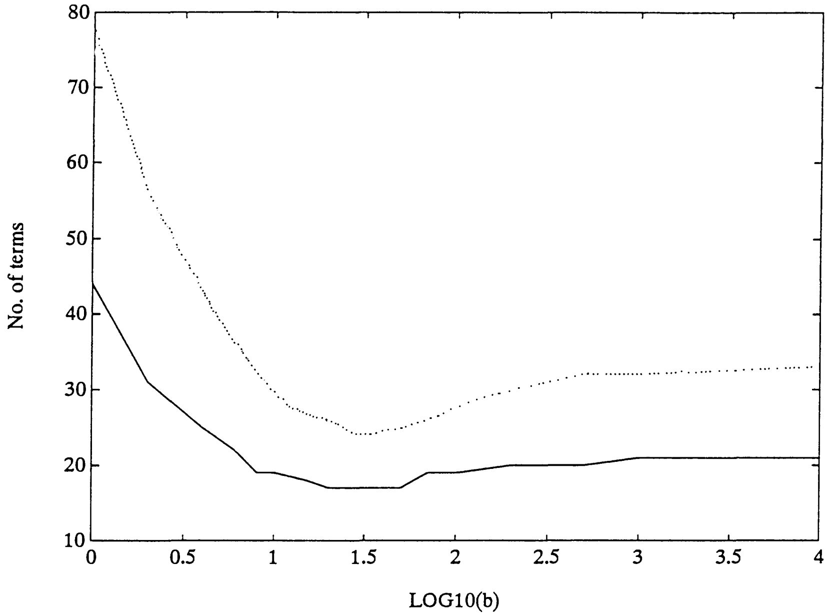

FIGURE 1. Plots for map (2.8)

for $[0, P]$, while for (P, ∞) we consider

$$t = \frac{x^2 - c}{d - x^2}, \qquad d = 2P^2 - c, \ c > P^2. \tag{2.9}$$

To see the effect of such maps, consider $P = 4$ and $P = 16$, and the expansions for g_2 and g_3. The coefficients of the Chebyshev expansion tend to zero fairly quickly, so we can count how many terms are needed before all coefficients are less than 10^{-20} in size. Figure 1 shows the number of terms for g_2, with the solid line representing $P = 4$, and the dotted line $P = 16$. The standard map is the limit of (2.8) as $b \to \infty$, giving 21 terms for $P = 4$, and 33 terms for $P = 16$. The minimum number of terms is 17 for $P = 4$, and 24 for $P = 16$. Similarly, Figure 2 (next page) shows the same information for g_3, with, here, the standard map being given by (2.9) with $c = 2P^2$. Again, the minimum number of coefficients is below that of the standard map.

We thus have to choose (a) a cutoff value P, (b) for this value of P, good values for b and c, with possibly different b-values for g_1 and g_2. For each possible combination of (P, b, c)-values we generate the Chebyshev coefficients by using Clenshaw's method [3]. This gives an infinite system of linear equations for the coefficients. By assuming that all coefficients beyond a certain point, which is called the zero coefficient cutoff point, are exactly zero, we derive a finite linear system which can be easily solved. To derive values for the parameters, we assumed 101 possibly nonzero coefficients, and performed the arithmetic using quadruple precision on a Prime 6350 (giving about 28 significant decimals).

A large amount of data is obviously generated. The decisions on reasonable values for P, b, c were based on this data. The value $P = 16$ was chosen as this gave a simple value, and approximately equal numbers of coefficients

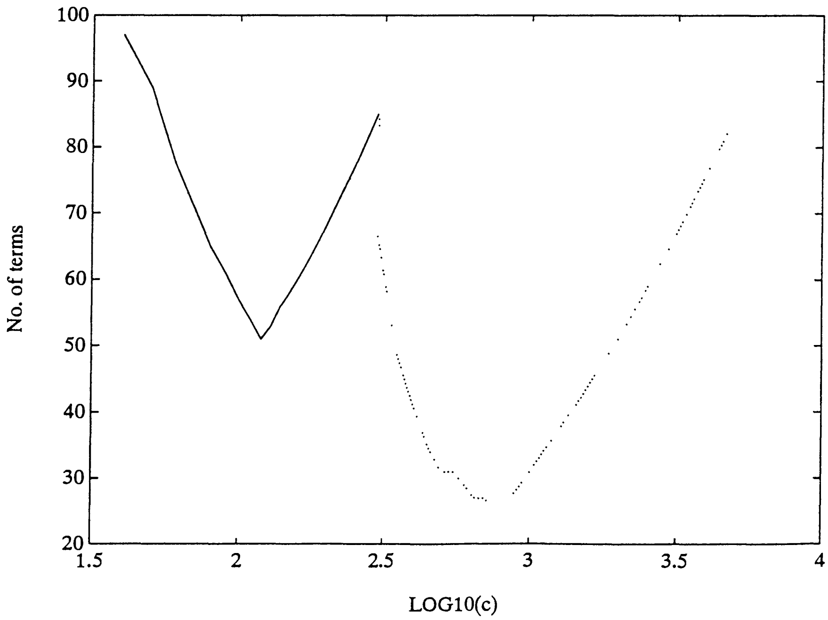

FIGURE 2. Plots for map (2.9)

for $0 \le x \le 16$ and $x > 16$. With this value of P, we varied b and c and found that, for b, the minimum number of coefficients occurred over a fairly wide range of values of b, while, for c, the minimum was more sensitive to values of the parameter. Since we only considered integral b and c, the range of b together with (2.8) enabled us to select a value of b giving simple exactly representable transformation coefficients. Integral c means that this also occurs in (2.9). The values chosen led to the following transformations:

$$\text{(2.10)} \qquad \mathbf{L}_0\colon \quad 0 \le x \le 16, \quad t = \frac{4x - 24}{x + 24},$$

$$\text{(2.11)} \qquad I_0 - \mathbf{L}_0\colon \quad 0 \le x \le 16, \quad t = \frac{6x - 40}{x + 40},$$

$$\text{(2.12)} \qquad I_0 - \mathbf{L}_0\colon \quad x > 16, \quad t = \frac{800 - x^2}{288 + x^2}.$$

With these transformations fixed, we repeated Clenshaw's method using a multiple-precision floating-point arithmetic package, written by the author. This performed calculations to about 75 significant decimals. The zero coefficient cutoff point was started at 60, and increased successively by 20, until the coefficients agreed to 40D. These coefficients $> 10^{-20}$ in size were then output, and are given in Tables 1–3.

TABLE 1. Coefficients for $g_1(x)$ for $0 \leq x \leq 16$

0	0.42127	45834	99799	24863	E	0
1	−0.33859	53639	12206	12188	E	0
2	0.21898	99481	27107	16064	E	0
3	−0.12349	48282	07131	85712	E	0
4	0.62142	09793	86695	8440	E	−1
5	−0.28178	06028	10954	7545	E	−1
6	0.11574	19676	63809	1209	E	−1
7	−0.43165	85743	06921	179	E	−2
8	0.14614	23499	07298	329	E	−2
9	−0.44794	21180	54614	78	E	−3
10	0.12364	74610	59437	61	E	−3
11	−0.30490	28334	79704	4	E	−4
12	0.66394	14015	21146		E	−5
13	−0.12553	83577	03889		E	−5
14	0.20073	44645	1228		E	−6
15	−0.25882	60170	637		E	−7
16	0.24114	37427	58		E	−8
17	−0.10159	67435	2		E	−9
18	−0.12024	30736			E	−10
19	0.26290	6137			E	−11
20	−0.15313	190			E	−12
21	−0.15747	60			E	−13
22	0.31563	5			E	−14
23	−0.4096				E	−16
24	−0.3620				E	−16
25	0.239				E	−17
26	0.36				E	−18
27	−0.4				E	−19

TABLE 2. Coefficients for $g_2(x)$ for $0 \le x \le 16$

0	0.52468	73679	14855	99138	E	0
1	−0.35612	46049	96505	86196	E	0
2	0.20487	20286	40099	27687	E	0
3	−0.10418	64052	04026	93629	E	0
4	0.46342	11095	54842	9228	E	−1
5	−0.17905	87192	40349	8630	E	−1
6	0.59796	86954	81143	177	E	−2
7	0.17177	75476	93565	429	E	−2
8	0.42204	65446	91714	22	E	−3
9	−0.87961	78522	09412	5	E	−4
10	0.15354	34234	86922	3	E	−4
11	−0.21978	07695	84743		E	−5
12	−0.24820	68393	6666		E	−6
13	−0.20327	06035	607		E	−7
14	0.90984	19842	1		E	−9
15	0.25617	93929			E	−10
16	−0.71060	9790			E	−11
17	0.32716	960			E	−12
18	0.23002	15			E	−13
19	−0.29210	9			E	−14
20	−0.3566				E	−16
21	0.1832				E	−16
22	−0.10				E	−18
23	−0.11				E	−18

TABLE 3. Coefficients for $g_3(x)$ for $x > 16$

0	2.00326	51024	11606	43125	E	0
1	0.19520	68515	76492	081	E	−2
2	0.38239	52356	99083	28	E	−3
3	0.75342	80817	05443	6	E	−4
4	0.14959	57655	89707	8	E	−4
5	0.29994	05312	10557		E	−5
6	0.60769	60482	2459		E	−6
7	0.12399	49554	4506		E	−6
8	0.25232	62552	649		E	−7
9	0.50463	48573	32		E	−8
10	0.97913	23623	0		E	−9
11	0.18389	11524	1		E	−9
12	0.33763	09278			E	−10
13	0.61117	9703			E	−11
14	0.10847	2972			E	−11
15	0.18861	271			E	−12
16	0.32803	45			E	−13
17	0.56564	7			E	−14
18	0.93300				E	−15
19	0.15881				E	−15
20	0.2791				E	−16
21	0.389				E	−17
22	0.70				E	−18
23	0.16				E	−18

3. Results for $\mathbf{L}_1$, $I_1 - \mathbf{L}_1$

The relevant expansions are

$$\mathbf{L}_1(x) = \frac{2x^2}{\pi}\left[\frac{1}{3} + \frac{x^2}{45} + \frac{x^4}{1575} + \cdots\right], \tag{3.1}$$

$$I_1(x) = \frac{x}{2} + \frac{x^3}{16} + \frac{x^5}{384} + \cdots, \tag{3.2}$$

$$I_1 - \mathbf{L}_1 \sim \frac{2}{\pi}\left[1 - \frac{1}{x^2} - \frac{3}{x^4} - \frac{45}{x^6} - \cdots\right]. \tag{3.3}$$

These expansions give rise to the approximating forms

$$\mathbf{L}_1 = \frac{2x^2}{3\pi} e^x g_4(x), \qquad 0 \le x \le P, \tag{3.4}$$

$$I_1 - \mathbf{L}_1 = \frac{x}{2} g_5(x), \qquad 0 \le x \le P, \tag{3.5}$$

$$I_1 - \mathbf{L}_1 = \frac{2}{\pi} g_6(x), \qquad x > P. \tag{3.6}$$

The functions g_4, g_5, g_6 are expanded in Chebyshev polynomials. Exactly the same investigation procedure as in §2 showed that $P = 16$ was again a reasonable choice, and the transformations could be taken from order 0 to the corresponding order 1 functions. Thus g_4, g_5, and g_6, use respectively (2.10), (2.11), and (2.12).

The 20D coefficients produced by the multiple-precision software are given in Tables 4–6.

4. Testing

The main test used was to write a Fortran program to evaluate g_1 to g_6 for various values of x (using quadruple precision), and to compare the results with values calculated in other ways.

For $0 \le x \le 16$, the various power series for $\mathbf{L}_0$, $\mathbf{L}_1$, $I_0 - \mathbf{L}_0$, $I_1 - \mathbf{L}_1$ were used to generate comparison values. In each unit interval, 1000 random values were generated and in all cases the maximum absolute error was less than $.5 \times 10^{-19}$.

For g_3 and g_6 in the range $x > 16$ we experienced more problems in testing. For x close to 16 the power series still gives sufficient accuracy. For large x, the asymptotic series was transformed into a continued fraction by the *q-d* method, and gave sufficient accuracy. There was, however, an interval from about $x = 25$ to $x = 50$ where we were unable to generate sufficiently accurate comparison values from either the power series or the continued fraction.

To get around this, we had to generate the comparison values from the power series using the multiple-precision package. The quadruple-precision argument x was transformed to multiple-precision form exactly by equivalencing it to eight 2-byte integers and decomposing these to give the exact binary representation. By using a base of 2^{12} in the package we preserve exactness. The results of the multiple-precision power series were then transformed back to quadruple-precision form. These results verified 20D absolute accuracy for g_3 and g_6.

TABLE 4. Coefficients for $g_4(x)$ for $0 \le x \le 16$

0	0.38996	02735	12295	38208	E	0
1	−0.33658	09610	19757	49366	E	0
2	0.23012	46791	25016	45616	E	0
3	−0.13121	59400	79608	32327	E	0
4	0.64259	22289	91284	6518	E	−1
5	−0.27500	32950	61663	5833	E	−1
6	0.10402	34148	63720	8871	E	−1
7	−0.35053	22949	36388	080	E	−2
8	0.10574	84984	21439	717	E	−2
9	−0.28609	42640	36665	58	E	−3
10	0.69257	08785	94220	8	E	−4
11	−0.14896	93951	12271	7	E	−4
12	0.28103	55825	97128		E	−5
13	−0.45503	87929	7776		E	−6
14	0.60901	71561	770		E	−7
15	−0.62354	37248	08		E	−8
16	0.38430	01206	7		E	−9
17	0.79054	3916			E	−11
18	−0.48982	4083			E	−11
19	0.46356	884			E	−12
20	0.68420	5			E	−14
21	−0.56974	8			E	−14
22	0.35324				E	−15
23	0.4244				E	−16
24	−0.644				E	−17
25	−0.21				E	−18
26	0.9				E	−19

TABLE 5. Coefficients for $g_5(x)$ for $0 \leq x \leq 16$

0	0.67536	36906	23505	76137	E	0
1	−0.38134	97109	72665	59040	E	0
2	0.17452	17077	51339	43559	E	0
3	−0.70621	05887	23502	5061	E	−1
4	0.25173	41413	55880	3702	E	−1
5	−0.78709	85616	06423	321	E	−2
6	0.21481	43686	51922	006	E	−2
7	−0.50862	19971	79062	36	E	−3
8	0.10362	60828	04423	30	E	−3
9	−0.17954	47212	05724	7	E	−4
10	0.25978	82745	15414		E	−5
11	−0.30442	40632	4667		E	−6
12	0.27202	39894	766		E	−7
13	−0.15812	61441	90		E	−8
14	0.18162	09172			E	−10
15	0.64796	7659			E	−11
16	−0.54113	290			E	−12
17	−0.30831	1			E	−14
18	0.30563	8			E	−14
19	−0.9717				E	−16
20	−0.1422				E	−16
21	0.84				E	−18
22	0.7				E	−19
23	−0.1				E	−19

TABLE 6. Coefficients for $g_6(x)$ for $x > 16$

0	1.99679	36189	67891	36501	E	0
1	−0.19066	32614	09686	132	E	−2
2	−0.36094	62241	01744	81	E	−3
3	−0.68418	47304	59982	0	E	−4
4	−0.12990	08228	50942	6	E	−4
5	−0.24715	21887	05765		E	−5
6	−0.47147	83969	1972		E	−6
7	−0.90208	19982	592		E	−7
8	−0.17304	58637	504		E	−7
9	−0.33232	36701	59		E	−8
10	−0.63736	42173	5		E	−9
11	−0.12180	23975	6		E	−9
12	−0.23173	46832			E	−10
13	−0.43906	8833			E	−11
14	−0.82847	110			E	−12
15	−0.15562	249			E	−12
16	−0.29131	12			E	−13
17	−0.54396	5			E	−14
18	−0.10117	7			E	−14
19	−0.18767				E	−15
20	−0.3484				E	−16
21	−0.643				E	−17
22	−0.118				E	−17
23	−0.22				E	−18
24	−0.4				E	−19
25	−0.1				E	−19

TABLE 7. Coefficients for relationship (4.6)

j	$r_{i,j}$	$s_{i,j}$
-3	$(3-i)/8i$	–
-2	$(9i-18)/2i$	$1400/i$
-1	$435(1-i)/8i$	$2800/i$
0	225	0
1	$-435(1+i)/8i$	$-2800/i$
2	$9(i+2)/21$	$-1400/i$
3	$-(3+i)/8i$	–

A completely different test can be performed using the relationships between the functions of different orders and properties of Chebyshev polynomials. We illustrate the method for $I_0 - \mathbf{L}_0$ and $I_1 - \mathbf{L}_1$ in the range $0 \le x \le 16$.

We have $I_0' = I_1$ and $\mathbf{L}_0' = \mathbf{L}_1 + \frac{2}{\pi}$ so

$$(I_0 - \mathbf{L}_0)' = (I_1 - \mathbf{L}_1) - 2/\pi, \tag{4.1}$$

giving

$$g_2' = -\frac{2}{\pi} + \frac{x}{2} g_5. \tag{4.2}$$

Since $t = (6x-40)/(x+40)$, standard algebra gives

$$(6-t)^3 \frac{dg_2}{dt} = 5600(1+t)g_5 - \frac{560}{\pi}(6-t). \tag{4.3}$$

Integrating gives

$$\begin{aligned}(216 - 108t + 18t^2 - t^3)g_2 &+ \int (108 - 36t + 3t^2)g_2 \\ &= 5600 \int (1+t)g_5 + \frac{280}{\pi}t^2 - \frac{3360}{\pi}t.\end{aligned} \tag{4.4}$$

Let

$$g_2 = \sum_{i=0}^{\infty}{}' c_i T_i(t), \qquad g_5 = \sum_{i=0}^{\infty}{}' 57 d_i T_i(t); \tag{4.5}$$

then, using standard Chebyshev relationships, we derive

$$\sum_{j=-3}^{3} r_{ij} c_{i+j} = \sum_{j=-2}^{+2} s_{ij} d_{i+j}, \qquad i = 3, 4, 5, \ldots, \tag{4.6}$$

where the coefficients r_{ij} $(j = -3, \ldots, 3)$ and s_{ij} $(j = -2, \ldots, 2)$ are given in Table 7. Applying these relationships to the coefficients, we get agreement to within acceptable rounding error. Similar techniques can be applied to relate the coefficients of $\mathbf{L}_0$ and $\mathbf{L}_1$, and the asymptotic forms of $I_0 - \mathbf{L}_0$ and $I_1 - \mathbf{L}_1$. Again acceptable agreement is found.

This mixture of tests leads us to accept the given coefficients as accurate.

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