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THE STRUCTURE OF THE PROJECTIVE INDECOMPOSABLE MODULES OF THE SUZUKI GROUP Sz(8) IN CHARACTERISTIC 2

GERHARD J. A. SCHNEIDER

ABSTRACT. This paper describes the socle series of the projective indecomposable modules and of tensor products of simple modules for the simple group Sz(8) in characteristic 2. The results have been obtained by computational means and the various steps are described. The main algorithm was modified to allow for parallel execution on a network of workstations. This made possible the effective handling of modules of degree 4030.

In this note we want to describe how to determine the Loewy series of the projective indecomposable modules of the smallest Suzuki group of order $29120 = 2^6 \cdot 5 \cdot 7 \cdot 13$ in characteristic 2. The computer will be the main tool used to achieve this goal, in particular, the CAYLEY system (see [1]) will be applied with the author's implementation of various representation-theoretic algorithms. These algorithms have been explained in detail in [6, 7]. This shows how the computer helps to gain insight into examples that seem to be difficult to analyze by purely theoretical means, once the required methods are available and implemented. The notation used in this paper is standard (see [6, 4] for further information).

For convenience, we include both the ordinary and 2-modular character table of this group in the appendix (see the Supplement section at the end of this issue). The tables are taken from [3, 5]. The group Sz(8) has eight modular characters. Seven lie in the principal block and are denoted by I, 4_a, 4_b, 4_c, 16_a, 16_b, and 16_c. The module 64 lies in a block of defect 0.

With the help of the information in the character tables the computation of the decomposition and Cartan matrix is a standard task, and the results are included in the appendix as well.

Since the computer is used to do most of the work, the representations have to be constructed explicitly. The four-dimensional representations are easy to obtain, as they are described in the ATLAS. The field GF(8) is the smallest field over which all representations can be realized. The 64-dimensional representation can be constructed from the permutation representation of Sz(8) on 65 letters (which is provided as part of the CAYLEY library facility), and thus it can be realized over GF(2). While the naming of the four-dimensional modules may be arbitrary, some care has to be used in distinguishing the 16-dimensional

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modules in order to have a naming convention consistent with the ATLAS. However, as the 16-dimensional representations may be obtained as tensor products, we can use the following proposition to overcome this difficulty.

Proposition 1. *There holds* $4_a \otimes 4_b =: 16_b$, $4_a \otimes 4_c =: 16_a$, *and* $4_b \otimes 4_c =: 16_c$.

Once the modules are known, we can easily determine their vertices and sources, using the standard CAYLEY procedures explained in [6]. In fact, the computer quickly realizes that all seven modules have a simple socle I when restricted to a Sylow-2 subgroup and to any subgroup of order 32; this proves the following proposition:

Proposition 2. *All seven modules in the principal block have a Sylow-2 subgroup* P *as a vertex. They remain indecomposable when restricted to a Sylow-2 subgroup. When restricted to* P, *the four-dimensional simple modules become uniserial, whereas the socle series of the* 16*-dimensional modules reads*

$$\begin{array}{c} \mathrm{I} \\ \mathrm{I\,I} \\ \mathrm{I\,I\,I} \\ \mathrm{I\,I\,I\,I} \\ \mathrm{I\,I\,I} \\ \mathrm{I\,I} \\ \mathrm{I} \end{array}$$

Let N denote the normalizer of a Sylow-2 subgroup in $\mathrm{Sz}(8)$; it is the stabilizer of a point in the permutation representation on 65 letters. It has order $448 = 2^6 \cdot 7$ and seven one-dimensional irreducible modules in the principal block. They are denoted by 1_a, 1_b, 1_c, 1_d, 1_e, 1_f, and I_N and are obtained as successive tensor products of 1_a with itself.

Proposition 3. *The simple modules of* $\mathrm{Sz}(8)$ *become their own Green correspondents when restricted to* N *and have the following socle series:*

$$\begin{array}{ccc} 4_a\downarrow_N & 4_b\downarrow_N & 4_c\downarrow_N \\ 1_f & 1_c & 1_e \\ 1_d & 1_b & 1_a \\ 1_c & 1_e & 1_f \\ 1_a & 1_d & 1_b \end{array}$$

$$\begin{array}{ccc} 16_a\downarrow_N & 16_b\downarrow_N & 16_c\downarrow_N \\ 1_d & 1_b & 1_a \\ \mathrm{I}\,1_b & \mathrm{I}\,1_a & \mathrm{I}\,1_d \\ 1_a\,1_e\,1_e & 1_d\,1_f\,1_f & 1_b\,1_c\,1_c \\ 1_a\,1_c\,1_d\,1_f & 1_b\,1_c\,1_d\,1_e & 1_a 1_b 1_e 1_f \\ 1_f\,1_b\,1_b & 1_a\,1_a\,1_c & 1_d\,1_d\,1_e \\ \mathrm{I}\,1_e & \mathrm{I}\,1_f & \mathrm{I}\,1_c \\ 1_c & 1_e & 1_f \end{array}$$

$$
\begin{array}{c}
64 \downarrow_n \\
\mathrm{I} \\
1_c\, 1_e\, 1_f \\
1_a\, 1_a\, 1_b\, 1_b\, 1_d\, 1_d \\
1_a\, 1_b\, 1_c\, 1_d\, 1_e\, 1_f\, \mathrm{IIII} \\
1_a\, 1_b\, 1_c\, 1_c\, 1_c\, 1_d\, 1_e\, 1_e\, 1_e\, 1_f\, 1_f\, 1_f \\
1_a\, 1_a\, 1_a\, 1_b\, 1_b\, 1_b\, 1_c\, 1_d\, 1_d\, 1_d\, 1_e\, 1_f \\
1_a\, 1_b\, 1_c\, 1_d\, 1_e\, 1_f\, \mathrm{IIII} \\
1_c\, 1_c\, 1_e\, 1_e\, 1_f\, 1_f \\
1_a\, 1_b\, 1_d \\
\mathrm{I}
\end{array}
$$

Since $64 \downarrow_N$ is the projective cover of the trivial module for N, the socle series of the other projective modules for N can easily be derived, using tensor products.

The determination of the restricted modules as well as their socle series is a straightforward application of the CAYLEY function RESTRICT in combination with the CAYLEY procedures given in [7]; user intervention is not required.

The process of inducing simple modules from N to Sz(8) is just as easy as the restriction. We obtain

Proposition 4. *The modules for* Sz(8) *that are obtained by inducting simple* N-*modules have the following socle series*:

$1_a \uparrow^{\mathrm{Sz}(8)}$	$1_b \uparrow^{\mathrm{Sz}(8)}$	$1_c \uparrow^{\mathrm{Sz}(8)}$	$1_d \uparrow^{\mathrm{Sz}(8)}$	$1_e \uparrow^{\mathrm{Sz}(8)}$	$1_f \uparrow^{\mathrm{Sz}(8)}$
4_a	4_c	16_a	4_b	16_b	16_c
I	I	4_a	I	4_b	4_c
4_c	4_b	I	4_a	I	I
I	I	4_b	I	4_c	4_a
4_b	4_a	I	4_c	I	I
16_b	16_a	4_a	16_c	4_b	4_c
4_b	4_a	I	4_c	I	I
I	I	4_c	I	4_a	4_b
4_c	4_b	16_c	4_a	16_a	16_b
I	I	4_c	I	4_a	4_b
4_a	4_c	I	4_b	I	I
I	I	4_a	I	4_b	4_c
4_c	4_b	I	4_a	I	I
16_c	16_b	4_b	16_a	4_c	4_a

The structure of the tensor products of the simple Sz(8)-modules may also be of interest. The determination of the series is yet another automated application of the CAYLEY procedures given in [7].

Proposition 5. *There holds* $4_a \otimes 4_b \otimes 4_c = 64$, *thus* $4_a \otimes 16_c = 64$, $4_b \otimes 16_c = 64$, *and* $4_c \otimes 16_b = 64$. *The modules that are obtained as tensor products of the four-dimensional modules with themselves have the socle series*:

$$\begin{array}{ccc} 4_a \otimes 4_a & 4_b \otimes 4_b & 4_c \otimes 4_c \\ I & I & I \\ 4_b & 4_c & 4_a \\ I & I & I \\ 4_c & 4_a & 4_b \\ I & I & I \\ 4_b & 4_c & 4_a \\ I & I & I \end{array}$$

The socle series of the remaining tensor products are listed in Table 1 (*in the Supplement section*).

The projective modules are also fairly easy to construct. Let Π_S denote the projective cover of the simple module S of $\mathrm{Sz}(8)$. The proof of the following facts is a simple application of the Nakayama relations in combination with Proposition 1 and 5.

Proposition 6. *We have*

$$\begin{array}{lll} 4_a \otimes 64 = \Pi_{16_c}, & 16_a \otimes 16_b = \Pi_{16_c}, & 16_c \otimes 64 = \Pi_{4_a} \oplus 64 \oplus 64, \\ 4_b \otimes 64 = \Pi_{16_a}, & 16_a \otimes 16_c = \Pi_{16_b}, & 16_a \otimes 64 = \Pi_{4_b} \oplus 64 \oplus 64, \\ 4_c \otimes 64 = \Pi_{16_b}, & 16_b \otimes 16_c = \Pi_{16_a}, & 16_b \otimes 64 = \Pi_{4_c} \oplus 64 \oplus 64, \end{array}$$

$$64 \otimes 64 = \Pi_I \oplus 2 \cdot \Pi_{16_a} \oplus 2 \cdot \Pi_{16_b} \oplus 2 \cdot \Pi_{16_e} \oplus 9 \cdot 64.$$

Proof. Let $[M, N]$ denote $\dim \mathrm{Hom}_{\mathrm{GF}(8)\,\mathrm{Sz}(8)}(M, N)$. Now,

$$\begin{aligned} [64 \otimes 64, 64] &= [4_a \otimes 16_c \otimes 64, 4_a \otimes 16_c] = [4_a \otimes 16_c \otimes 4_a, 64 \otimes 16_c] \\ &= [4_a \otimes 64, \Pi_{4_a} \oplus 64 \oplus 64] = [64, \Pi_{4_a} \otimes 4_a \oplus 4_a \otimes 64 \oplus 4_a \otimes 64] \\ &= [64, \Pi_{4_a} \otimes 4_a] = [64 \otimes 4_a, \Pi_{4_a}] = [\Pi_{16_c}, \Pi_{4_a}] = 9. \end{aligned}$$

The other statements can be proved in a similar way. □

Since the projective covers of the 16-dimensional simple modules are of small degree 256, we can directly apply the methods in [7] to these modules and obtain

Theorem 7. *The projective covers of the simple* 16*-dimensional modules of* $\mathrm{Sz}(8)$ *have socle series as listed in Table* 2 (*in the Supplement section*).

Once the modules are constructed, the computation can be carried out in the CAYLEY system, without any user intervention, merely by calling the appropriate procedures.

We now proceed to analyze the projective covers of the four-dimensional simple modules. Although the dimension of $16 \otimes 64$ is still within reach of our methods, the computation can be speeded up by first removing the two summands of dimension 64. This is best done by applying the MEATAXE. Since the MEATAXE is a random method, there is no guarantee that it will return a 64-dimensional simple module as a submodule or factor module when applied to $16 \otimes 64$. However, the CAYLEY implementation also returns the base change that the MEATAXE applies. Therefore, one can apply this process iteratively, until the desired modules are found, and then determine their preimages in the original module. Thus, the modules Π_{4_a}, Π_{4_b}, and Π_{4_c} can be constructed.

The proof of the following theorem may then be left to the computer.

Theorem 8. *The projective covers of the simple four-dimensional modules of* Sz(8) *have socle series as listed in Table* 3 (*in the Supplement section*).

The remaining projective module, the projective cover Π_I of the trivial module, presents some challenge. Not only is its degree 1984 too large for a direct application of the CAYLEY routines, but its construction involves a module of dimension 4096. Therefore, we will use the extended methods described in [7]. This means that we shall compute socle series of subquotients of the module until the series for the whole module is found. Rather than using the MEATAXE to compute some initial series of submodules, a good filtration of the module can be found by observing

$$\Pi_{I_N} \uparrow^{\mathrm{Sz}(8)} \cong I_N \uparrow^{\mathrm{Sz}(8)} \otimes 64 \cong \Pi_I \oplus 2 \cdot \Pi_{16_a} \oplus 2 \cdot \Pi_{16_a} \oplus 2 \cdot \Pi_{16_c} \oplus 10 \cdot 64.$$

Using the socle series from Proposition 3 for the module $64 \downarrow_N = \Pi_{I_N}$, we can choose a basis for the induced module such that the filtration given by

$$\mathrm{soc}_1(\Pi_{I_N}) \uparrow^{\mathrm{Sz}(8)} \leq \mathrm{soc}_2(\Pi_{I_N}) \uparrow^{\mathrm{Sz}(8)} \leq \cdots \leq \mathrm{soc}_9(\Pi_{I_N}) \uparrow^{\mathrm{Sz}(8)} \leq \mathrm{soc}_{10}(\Pi_{I_N}) \uparrow^{\mathrm{Sz}(8)}$$

can immediately be read off the representing matrices. Then we may compute the socle series of each of the quotients of these submodules. This yields an even finer filtration of the induced module with the property that each chief factor is indeed semisimple. We call this series a layer series of our module.

Without loosing information we can reduce the degree of the induced module by only inducing the heart of $\Pi_{I_N} \cong \Omega(I_N)/I_N$, as we then obtain the heart of Π_I plus projective modules:

$$\Omega(I_N)/I_N \uparrow^{\mathrm{Sz}(8)} \cong \Omega(I)/I \oplus 2 \cdot \Pi_{16_a} \oplus 2 \cdot \Pi_{16_b} \oplus 2 \cdot \Pi_{16_c} \oplus 8 \cdot 64.$$

The eight composition factors 64 are relatively easy to remove, as they are all in the nonprincipal block. We denote the remaining module by M; it is of dimension 3518. This module will be used for the actual computation.

Observe that we know most of the composition factors of M, as the number of composition factors isomorphic to the simple module S in a given layer of Π_I is equal to the number of trivial composition factors in the same layer of the projective Sz(8)-module Π_S, by [4, Lemma I.9.10]. It now happens that all composition factors isomorphic to a given S in a given socle layer of M are either all from $\Omega(I)/I$ or from the projective modules.

The following table describes the distribution of the composition factors of M as it is known at the beginning. The number of composition factors in socle layer i are listed in column i of the table. The number of factors coming from the projective modules are printed in italics.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
I	0	?	*6*	?	*6*	?	*12*	?	*6*	?	*12*	?	*18*	?
4_a	1	*2*	3	*2*	5	*4*	7	*4*	7	*6*	9	*6*	9	*6*
4_b	1	*2*	3	*2*	5	*4*	7	*4*	7	*6*	9	*6*	9	*6*
4_c	1	*2*	3	*2*	5	*4*	7	*4*	7	*6*	9	*6*	9	*6*
16_a	*2*	1	0	1	0	2	0	1	*4*	2	*2*	3	0	3
16_b	*2*	1	0	1	0	2	0	1	*4*	2	*2*	3	0	3
16_c	*2*	1	0	1	0	2	0	1	*4*	2	*2*	3	0	3

	15	16	17	18	19	20	21	22	23	24	25	26	27
I	*18*	?	*12*	?	*6*	?	*12*	?	*6*	?	*6*	0	0
4_a	9	*6*	7	*6*	7	*4*	4	*4*	3	*2*	1	*2*	0
4_b	9	*6*	7	*6*	7	*4*	4	*4*	3	*2*	1	*2*	0
4_c	9	*6*	7	*6*	7	*4*	4	*4*	3	*2*	1	*2*	0
16_a	0	2	*2*	1	*4*	2	0	1	0	1	0	0	*2*
16_b	0	2	*2*	1	*4*	2	0	1	0	1	0	0	*2*
16_c	0	2	*2*	1	*4*	2	0	1	0	1	0	0	*2*

Only the positions with a question mark have to be computed. Although this provides no shortcut to the main computation, we can use the information from this table as termination criteria for the following working algorithm:

– Compute a layer series of M, using the filtration that can be derived from the socle series of $\Omega(I_N)/I_N \uparrow^{\mathrm{Sz}(8)}$.

– Choose subquotients of M of reasonable size that contain only complete layers and start with odd-numbered layers of M. Compute their socle series and apply the resulting base change to M to get a new layer series for M.

– If the number of nontrivial composition factors in each layer is correct, and if the number of trivial modules in the odd-numbered layers is correct, then we have computed the complete socle series of M.

The actual computation involved the determination of the series of half the induced module, i.e., $\mathrm{soc}_5(\Pi_{I_N})/\mathrm{soc}_1(\Pi_{I_N}) \uparrow^{\mathrm{Sz}(8)}$, a module of dimension 2015. Since the result seems somewhat surprising, we give the details.

Proposition 9. *We have*

$$\mathrm{soc}_5(\Pi_{I_N})/\mathrm{soc}_1(\Pi_{I_N}) \uparrow^{\mathrm{Sz}(8)} \cong M_2 \oplus \Pi_{16_a} \oplus \Pi_{16_b} \oplus \Pi_{16_c} \oplus 4 \cdot 64 .$$

M_2 *has dimension* 1759 *and its socle series reads*:

$4_a\, 4_b\, 4_c$

III

$4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c$

$\mathrm{IIIIII}\; 16_a\, 16_b\, 16_c$

$4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c$

$\mathrm{IIIIIIIII}\; 16_a\, 16_b\, 16_c\, 16_a\, 16_b\, 16_c$

$4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c$

$\mathrm{IIIIIIIIIIII}\; 16_a\, 16_b\, 16_c$

$4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c$

$\mathrm{IIIIIIIIIIII}\; 16_a\, 16_b\, 16_c$

$4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c$

$\mathrm{IIIIIIIIIIII}\; 16_a\, 16_b\, 16_c$

$4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c$

$\mathrm{IIIIIIIIIIIII}\; 16_a\, 16_b\, 16_c\, 16_a\, 16_b\, 16_c$

$4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c$

$\mathrm{IIIIIIIII}\; 16_a\, 16_b\, 16_c$

$4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c\, 4_a\, 4_b\, 4_c$

$\mathrm{III}\; 16_a\, 16_b\, 16_c$

$4_a\, 4_b\, 4_c$

The computation can easily be parallelized and run on a network of workstations with a shared file system, as the socle series of various subquotients of M can be determined simultaneously. The workloads as well as different speeds and available user memory of the various machines have to be taken into account when setting up the jobs. These factors can be compensated by running subquotients of different sizes on the individual machines. Successful termination of a job on a machine is marked by a control file. Although this is a rather crude way of synchronizing parallel jobs, it worked well under the local conditions.

The result of the computations with Π_I can be summarized as follows:

Theorem 10. *The projective cover Π_I of the trivial module of* Sz(8) *has socle series as listed in Table* 4 *(in the Supplement section). The number of composition factors isomorphic to I in each socle layer can also be described by the following table*:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
I	1	0	3	0	9	0	13	0	18	0	18	0	18	0

	15	16	17	18	19	20	21	22	23	24	25	26	27
I	21	0	19	0	18	0	9	0	9	0	3	0	1

For convenience, we include the number of extensions between modules of Sz(8) in characteristic 2 (see also [8]). The following table has

$$\dim_{\mathrm{GF}(8)} \mathrm{Ext}^1_{\mathrm{GF}(8)\,\mathrm{Sz}(8)}(S_i, S_j)$$

in position (i, j), for any two simple Sz(8)-modules S_i, S_j:

	I	4_a	4_b	4_c	16_a	16_b	16_c	64
I	0	1	1	1	0	0	0	0
4_a	1	0	0	0	1	0	0	0
4_b	1	0	0	0	0	1	0	0
4_c	1	0	0	0	0	0	1	0
16_a	0	1	0	0	0	0	0	0
16_b	0	0	1	0	0	0	0	0
16_c	0	0	0	1	0	0	0	0
64	0	0	0	0	0	0	0	0

A final word should be said about the reliability of the results, as they have been achieved almost solely by means of a computer. Such an approach could give rise to additional nonmathematical errors, like programming bugs or compiler problems. However, the mathematical context that effectively controls the implementation of our algorithms is quite rigid: only eight different modules may show up in our computations and, as explained, the results can be cross-checked for consistency. Thus, any bug in the program will almost certainly yield a result that no longer conforms to the general mathematical context of the problem, like modules of dimension 0, etc. Since the results computed are consistent, this gives additional reassurance of the correctness of the theorems stated, although this argument is of course no proof.

The CPU requirements differed greatly for the various computations. While a lunch break was enough to do the series for Π_{16}, the series for Π_4 was computed over a weekend. The preparation and actual runs for the module Π_I (including the work on the module of dimension 4030) spread over several weeks.

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The computation with Π_{16_a} was carried out on an IBM 6150 at the Mathematical Sciences Research Institute in Berkeley. The analysis of the remaining modules was done on the two IBM RS6000 (models 320 and 540) and the three IBM 6150's at the Institut für Experimentelle Mathematik, Universität GH Essen, using the IBM implementation of CAYLEY Version 3.7.

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RECHENZENTRUM DER UNIVERSITÄT KARLSRUHE, ZIRKEL 2, D-7500 KARLSRUHE, GERMANY
E-mail address: schneider@dkauni2.bitnet
schneider@rz.uni-karlsruhe.de