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ON A PROBLEM OF ERDŐS CONCERNING PRIMITIVE SEQUENCES

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Dedicated to Paul Erdős on the occasion of his 80th birthday

ABSTRACT. A sequence $A = \{a_i\}$ of positive integers $a_1 < a_2 < \cdots$ is said to be primitive if no term of A divides any other. Let $\Omega(a)$ denote the number of prime factors of a counted with multiplicity. Let $p(a)$ denote the least prime factor of a and $A(p)$ denote the set of $a \in A$ with $p(a) = p$. The set $A(p)$ is called *homogeneous* if there is some integer s_p such that either $A(p) = \varnothing$ or $\Omega(a) = s_p$ for all $a \in A(p)$. Clearly, if $A(p)$ is homogeneous, then $A(p)$ is primitive. The main result of this paper is that if A is a positive integer sequence such that $1 \notin A$ and each $A(p)$ is homogeneous, then

$$\sum_{a \le n,\, a \in A} \frac{1}{a \log a} \le \sum_{p \le n,\, p \text{ prime}} \frac{1}{p \log p} \quad \text{for } n > 1.$$

This would then partially settle a question of Erdős who asked if this inequality holds for any primitive sequence A.

1. INTRODUCTION

A sequence $A = \{a_i\}$ of positive integers $a_1 < a_2 < \cdots$ is said to be primitive if no term of A divides any other (cf. [3] or [5]). We denote by p_m the mth prime, by p a variable prime and by $p(a)$ the least prime factor of a. We define the degree of an integer a, denoted by $\Omega(a)$, to be the number of prime factors of a counted with multiplicity. The degree of an integer sequence A, denoted by $d^\circ(A)$, is defined as the maximum degree of its terms. We take $d^\circ(A) = 0$ if $A = \{1\}$ or $\varnothing$.

For a primitive sequence A with $d^\circ(A) > 0$ we define

$$f(A) = \sum_{a \in A} 1/(a \log a).$$

We take $f(A) = 0$ if $d^\circ(A) = 0$. Erdős [1] proved that there exists an absolute constant C such that $f(A) \le C$ for any primitive sequence A. Recently he [2] has asked if the inequality

$$(1) \qquad \sum_{a \le n,\, a \in A} \frac{1}{a \log a} \le \sum_{p \le n,\, p \text{ prime}} \frac{1}{p \log p} \quad \text{for } n > 1$$

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is always true for any primitive sequence A. Zhang [8] proved that if A is primitive with $d^\circ(A) \le 4$, then the inequality is true. Erdős and Zhang [4] proved that $f(A) < 1.84$ for any primitive sequence A, and gave a necessary and sufficient condition for the inequality (1), namely $\sum_{b\in B} 1/(b\log b) \le \sum 1/(p\log p)$ for any primitive sequence B. Clearly, if (1) is true then $C = \sum 1/(p\log p) < 1.64$.

In this paper we partially settle this question of Erdős in another direction. To give our result, we need some more notation and concepts. Let $A(p)$ denote the set of $a \in A$ with $p(a) = p$. A sequence B is called *homogeneous* if either $B = \varnothing$ or $\Omega(b) = d^\circ(B)$ for all $b \in B$. Clearly, if B is homogeneous, then B is primitive. Now we state our main result as the following

Theorem. *If A is a positive integer sequence such that $1 \notin A$ and each $A(p)$ is homogeneous, then the inequality* (1) *is true.*

The basic idea for proving the theorem is the same as that used in [8]; i.e., we consider the least prime factors of the terms of A. The key point of this paper is to prove that, for a given prime p, if $B = B(p)$ is homogeneous and nonempty, then

$$\sum_{b\in B} \frac{1}{b\log b} \le \frac{1}{p\log p}. \tag{2}$$

It is clear that (2) immediately implies the theorem. In fact we have the stronger result where "$a \le n$" is replaced in (1) with "$(a, n!) > 1$".

2. Proof of the theorem

We first define two functions:

$$w(s, m) = \sum_{\Omega(a)=s-1,\, p(a)\ge p_{m+1}} \frac{1}{a\log(p_{m+1}a)}$$

for integers $s \ge 2$, $m \ge 0$, and

$$h(m) = \sum_{i>m} \frac{1}{p_i \log(i-1)}$$

for integers $m \ge 2$.

We need nine lemmas.

Lemma 1. *We have $p_n > n\log n$ for $n \ge 1$ and $p_n < n(\log n + \log\log n)$ for $n \ge 6$.*

These results may be found in [6] and [7].

Lemma 2. *We have $h(m) < 1/\log m$ for $m \ge 2$.*

Proof. Note that for each $i \ge 3$, we have

$$\frac{1}{i\log i\log(i-1)} < \frac{\log(i/(i-1))}{\log i\log(i-1)} = \frac{1}{\log(i-1)} - \frac{1}{\log i}.$$

Thus, from Lemma 1,

$$h(m) < \sum_{i>m} \frac{1}{i\log i\log(i-1)} < \sum_{i>m}\left(\frac{1}{\log(i-1)} - \frac{1}{\log i}\right) = \frac{1}{\log m}. \quad \square$$

In the following we define $i(a) = i$ if the largest prime factor of a is p_i.

Lemma 3. *For $m \geq 2$, $s \geq 1$, we have*

$$\sum_{p(a)>p_m, \Omega(a)=s} \frac{1}{a\log(i(a)-1)} \leq h(m) < \frac{1}{\log m}.$$

Proof. We proceed by induction on s. If $s = 1$, then this is just Lemma 2. Assume the lemma for s. For the $s+1$ case, we have, by Lemma 2,

$$\begin{aligned}\sum_{p(a)>p_m,\ \Omega(a)=s+1} &\frac{1}{a\log(i(a)-1)} \\ &= \sum_{p(b)>p_m,\ \Omega(b)=s} \frac{1}{b} \sum_{j\geq i(b)} \frac{1}{p_j \log(j-1)} \\ &< \sum_{p(b)>p_m,\ \Omega(b)=s} \frac{1}{b\log(i(b)-1)} \leq h(m) < \frac{1}{\log m}. \quad \Box\end{aligned}$$

Lemma 4. *For $i \geq 2$, $B \geq 2$, we have*

$$\begin{aligned}\sum_{j>i} \frac{1}{p_j \log(Bp_j)} &< \frac{\log(1+\log B/\log i)}{\log B} \\ &\leq \min\left\{\frac{1}{\log i}, \frac{1}{e\log i} + \frac{1}{e\log B}\right\},\end{aligned}$$

where $e = 2.718\cdots$ is the base of the natural logarithms.

Proof. We have, by Lemma 1,

$$\begin{aligned}\sum_{j>i} \frac{1}{p_j \log(Bp_j)} &< \int_i^\infty \frac{dx}{x\log x \log(Bx)} \\ &= \frac{\log(1+\log B/\log i)}{\log B} \leq \min\left\{\frac{1}{\log i}, \frac{1}{e\log i} + \frac{1}{e\log B}\right\},\end{aligned}$$

observing that the last inequality follows from

$$\log(1+x) < x \quad \text{and} \quad \log x = 1 + \log(1+(x-e)/e) \leq x/e$$

for all $x > 0$. $\Box$

Lemma 5. *For $m \geq 2$, $B \geq 2$, $s \geq 2$, we have*

$$\begin{aligned}\sum_{p(u)>p_m,\ \Omega(u)=s} &\frac{1}{u\log(Bu)} \\ &< (e^{-1} + \cdots + e^{1-s})h(m) + e^{1-s} \sum_{j>m} \frac{1}{p_j \log(Bp_j)}.\end{aligned}$$

Proof. We proceed by induction on s. If $s = 2$, then we have, by Lemma 4,

$$\begin{aligned}\sum_{p(u)>p_m,\ \Omega(u)=2} \frac{1}{u\log(Bu)} &= \sum_{j>m} \frac{1}{p_j} \sum_{k\geq j} \frac{1}{p_k \log(Bp_jp_k)} \\ &< e^{-1}h(m) + e^{-1} \sum_{j>m} \frac{1}{p_j \log(Bp_j)}.\end{aligned}$$

For the $s+1$ case, we have, by Lemmas 3 and 4 and the s case,

$$\begin{aligned}\sum_{p(u)>p_m,\ \Omega(u)=s+1} \frac{1}{u\log(Bu)} &= \sum_{p(b)>p_m,\ \Omega(b)=s} \frac{1}{b} \sum_{j\ge i(b)} \frac{1}{p_j \log(Bbp_j)} \\ &< \sum_{p(b)>p_m,\ \Omega(b)=s} \frac{e^{-1}}{b}\left(\frac{1}{\log(i(b)-1)} + \frac{1}{\log(Bb)}\right) \\ &< (e^{-1}+\cdots+e^{-s})h(m) + e^{-s}\sum_{j>m}\frac{1}{p_j\log(Bp_j)}. \quad \square\end{aligned}$$

Lemma 6. *For $m \ge 5$, $s \ge 2$, we have $w(s, m) < 1/\log p_{m+1}$.*

Proof. We have, by Lemmas 2, 4, and 5,

$$w(s, m) < W(s, m),$$

where

$$W(s, m) = \frac{e^{-1}+\cdots+e^{1-s}}{\log m} + \frac{e^{1-s}}{\log p_{m+1}}.$$

By Lemma 1 we have

$$\begin{aligned}\frac{\log p_{m+1}}{\log m} &< \frac{\log(m+1) + \log(\log(m+1) + \log\log(m+1))}{\log m} \\ &\le \frac{\log 6 + \log(\log 6 + \log\log 6)}{\log 5} = 1.65\cdots < e-1.\end{aligned}$$

Thus,

$$W(s, m) - W(s+1, m) = e^{-s}\left(\frac{e-1}{\log p_{m+1}} - \frac{1}{\log m}\right) > 0$$

for $m \ge 5$, $s \ge 2$. Therefore,

$$\begin{aligned}w(s, m) < W(s, m) \le W(2, m) &= \frac{1}{e\log m} + \frac{1}{e\log p_{m+1}} \\ &< \frac{e-1}{e\log p_{m+1}} + \frac{1}{e\log p_{m+1}} = \frac{1}{\log p_{m+1}}. \quad \square\end{aligned}$$

Lemma 7. *For $0 \le m \le 4$, we have $w(2, m) < 1/\log p_{m+1}$.*

Proof. We have, by Lemma 4,

$$w(2, m) < w(m) \quad \text{for } 0 \le m \le 4,$$

where

$$\begin{aligned}w(m) = &\frac{1}{p_{m+1}\log(p_{m+1}^2)} + \frac{1}{p_{m+2}\log(p_{m+1}p_{m+2})} \\ &+ \frac{1}{\log p_{m+1}}\log\left(1 + \frac{\log p_{m+1}}{\log(m+2)}\right) \quad \text{for } 1 \le m \le 4\end{aligned}$$

and

$$w(0) = \frac{1}{2\log 4} + \frac{1}{3\log 6} + \frac{1}{5\log 10} + \frac{1}{\log 2}\log\left(1 + \frac{\log 2}{\log 3}\right).$$

By calculation we have Table 1.

TABLE 1

m	$w(m)$	p_{m+1}	$1/\log p_{m+1}$
4	0.388...	11	0.417...
3	0.464...	7	0.513...
2	0.581...	5	0.621...
1	0.856...	3	0.910...
0	1.339...	2	1.442...

Thus, $w(2, m) < w(m) < 1/\log p_{m+1}$ for $0 \le m \le 4$. □

Lemma 8.1. *For $s \ge 3$, $2 \le m \le 4$, we have $w(s, m) < 1/\log p_{m+1}$.*

Proof. For a fixed m, put

$$\gamma_s = (e^{-1} + \cdots + e^{2-s})h(m) + e^{2-s}w(m),$$

where $w(m)$ is the upper bound of $w(2, m)$, defined in the proof of Lemma 7. Then by Lemma 5 we have for $s \ge 3$ that

$$w(s, m) < (e^{-1} + \cdots + e^{2-s})h(m) + e^{2-s}w(2, m) < \gamma_s.$$

If $h(m)/w(m) < e - 1$ and $m \le 4$, then we have, from Table 1,

$$\gamma_s < ((e^{-1} + \cdots + e^{2-s})(e - 1) + e^{2-s})w(m) = w(m) < 1/\log p_{m+1}.$$

For $m = 4$, we have, by Lemma 2,

$$h(4) = \sum_{i=5}^{10} \frac{1}{p_i \log(i-1)} + h(10) < 0.6442,$$

using $h(10) < 1/\log 10$. Thus, $h(4)/w(4) < 1.7 < e - 1$, so that the case $m = 4$ is done.

For $m = 3$ we have

$$h(3) = 1/(7\log 3) + h(4) < 0.7743, \quad \text{and} \quad h(3)/w(3) < 1.7 < e - 1.$$

Thus the $m = 3$ case is done.

For $m = 2$, since

$$h(2) = 1/(5\log 2) + h(3) < 1.063,$$

we use the upper bound $H = 1.063$ for $h(2)$ and we see that

$$H/w(2) > e - 1.$$

However, we then have

$$\gamma_s < (e^{-1} + \cdots + e^{2-s})H + e^{2-s}\frac{H}{e-1} = \frac{H}{e-1} < 0.62 < 1/\log 5,$$

so that the $m = 2$ case is done. □

Lemma 8.2. *We have $w(s, 1) < 1/\log p_2$ for $s \ge 3$.*

Proof. We have $w(s, 1) = u(s) + v(s)$, where

$$u(s) = \frac{1}{3} \sum_{\substack{\Omega(b)=s-2 \\ p(b)\ge p_2}} \frac{1}{b\log(9b)} \quad \text{and} \quad v(s) = \sum_{\substack{\Omega(b)=s-1 \\ p(b)\ge p_3}} \frac{1}{b\log(3b)}.$$

Taking

$$h(2) < \sum_{i=3}^{25} \frac{1}{p_i \log(i-1)} + \frac{1}{\log 25} < 1.0396$$

and

$$\sum_{i>2} \frac{1}{p_i \log(3p_i)} < \sum_{i=3}^{25} \frac{1}{p_i \log(3p_i)} + \frac{1}{\log 25} < 0.5779 < \frac{1.0396}{e-1},$$

we have, by Lemma 5,

$$v(s) < 1.0396(e^{-1} + \cdots + e^{2-s}) + 0.5779e^{2-s} < \frac{1.0396}{e-1} < 0.6051 < \frac{2/3}{\log 3}.$$

Since $w(2, 1) < 1/\log 3$ by Lemma 7 and $u(s) < w(s-1, 1)/3$, we have, for $s \geq 3$,

$$w(s, 1) < w(s-1, 1)/3 + v(s) < (1/3)/\log 3 + (2/3)/\log 3 = 1/\log 3. \quad \square$$

Lemma 8.3. *We have* $w(s, 0) < 1/\log 2$ *for* $s \geq 3$.

Proof. Put

$$u_i(s) = \frac{1}{p_i} \sum_{\Omega(b)=s-2,\, p(b)\geq p_i} \frac{1}{b \log(2p_i b)} \quad \text{for } 1 \leq i \leq 9$$

and

$$v_i(s) = \sum_{\Omega(b)=s-1,\, p(b)\geq p_i} \frac{1}{b \log(2b)} \quad \text{for } 1 \leq i \leq 10.$$

Then for $1 \leq i \leq 9$, we have

$$v_i(s) = u_i(s) + v_{i+1}(s) \tag{3}$$

and

$$u_i(s) < \frac{v_i(s-1)}{p_i}. \tag{4}$$

Let $N = 800$. Put

$$h = \sum_{i=10}^{N} \frac{1}{p_i \log(i-1)} + \frac{1}{\log N} < 0.403693$$

and

$$g = \sum_{i=10}^{N} \frac{1}{p_i \log(2p_i)} + \frac{1}{\log N} < 0.306441.$$

Then

$$h(9) < h \quad \text{and} \quad \sum_{i>9} \frac{1}{p_i \log(2p_i)} < g.$$

We have, by Lemma 5, $v_{10}(s) < V_{10}(s)$, where

$$V_{10}(s) = (e^{-1} + \cdots + e^{2-s})h + e^{2-s}g.$$

By calculation we get the upper bounds of $V_{10}(s)$, for $3 \leq s \leq 9$, listed in Table 2, which serve as upper bounds of $v_{10}(s)$ for $3 \leq s \leq 9$.

By Lemma 4 we have

$$u_i(3) < \frac{1}{p_i}\left(\sum_{j=i}^{N}\frac{1}{p_j\log(2p_ip_j)} + \frac{1}{\log N}\right).$$

By calculation we get the upper bounds of $u_i(3)$, for $1 \le i \le 9$, listed in Table 2.

Since we now have upper bounds for $v_{10}(3)$ and $u_i(3)$, we can, by equation (3), get upper bounds of $v_i(3)$ for $i = 9, 8, \ldots, 2, 1$. Then, by equation (3), inequality (4) and the upper bounds of $v_{10}(s)$, we can get upper bounds of $v_i(s)$ for $i = 9, 8, \ldots, 2, 1$; $s = 4, 5, \ldots, 9$.

In this way we get upper bounds (listed in Table 2) of

$$w(s, 0) = v_1(s) < 1/\log 2 \quad \text{for } 3 \le s \le 9.$$

In the above calculations we also get the upper bounds of $v_i(9)$, for $1 \le i \le 10$, listed in Table 2.

Let $k_1 = 1/\log 2$ and

$$k_i = \frac{\prod_{j=1}^{i-1}(1 - 1/p_j)}{\log 2} \quad \text{for } 2 \le i \le 10.$$

We list the values of k_i, for $1 \le i \le 10$, in Table 2.

TABLE 2. Upper bounds of $V_{10}(s)$, $u_i(3)$, $w(s, 0) = v_1(s)$ and $v_i(9)$; and values of k_i

s or i	$V_{10}(s)$	$u_i(3)$	$w(s, 0) = v_1(s)$	$v_i(9)$	k_i
1		0.4264		1.4412	1.4426...
2		0.1885		0.7204	0.7213...
3	0.2613	0.0843	1.1049	0.4795	0.4808...
4	0.2447	0.0512	1.2814	0.3835	0.3847...
5	0.2385	0.0287	1.3787	0.3286	0.3297...
6	0.2363	0.0228	1.4224	0.2987	0.2997...
7	0.2355	0.0164	1.4380	0.2757	0.2767...
8	0.2352	0.0141	1.4417	0.2595	0.2604...
9	0.2351	0.0112	1.4412	0.2458	0.2467...
10				0.2351	0.2360...

We see that

$$(5) \qquad v_i(9) < k_i \quad \text{for } 1 \le i \le 10.$$

Since $V_{10}(9) < k_{10}$ and $V_{10}(s+1) - V_{10}(s) = e^{1-s}(h - (e-1)g) < 0$, we have

$$(6) \qquad v_{10}(s) < V_{10}(s) < k_{10} \quad \text{for } s \ge 9.$$

For $i = 9$ down to 1, for $s = 9, 10, \ldots,$ we have, by (3), (4), (5), and (6),

$$v_i(s+1) < \frac{v_i(s)}{p_i} + v_{i+1}(s+1) < \frac{k_i}{p_i} + k_{i+1} = k_i.$$

Thus, $w(s, 0) = v_1(s) < k_1 = 1/\log 2$ for $s \ge 9$. □

Combining Lemmas 8.1, 8.2, and 8.3, we have the following

Lemma 8. *We have* $w(s, m) < 1/\log p_{m+1}$ *for* $s \geq 3$, $0 \leq m \leq 4$.

Lemma 9. *For a given prime* p, *if* $B = B(p)$ *is homogeneous and nonempty, then*

$$\sum_{b \in B} \frac{1}{b \log b} \leq \frac{1}{p \log p}.$$

Proof. This follows from Lemmas 6, 7, and 8. □

As we have seen above, Lemma 9 immediately implies the theorem.

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