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A B_2-SEQUENCE WITH LARGER RECIPROCAL SUM

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ABSTRACT. A sequence of positive integers is called a B_2-sequence if the pairwise differences are all distinct. The Mian-Chowla sequence is the B_2-sequence obtained by the greedy algorithm. Its reciprocal sum S^* has been conjectured to be the maximum over all B_2-sequences. In this paper we give a B_2-sequence which disproves this conjecture. Our sequence is obtained as follows: the first 14 terms are obtained by the greedy algorithm, the 15th term is 229, from the 16th term on, the greedy algorithm continues. The reciprocal sum of the first 300 terms of our sequence is larger than S^*.

1. INTRODUCTION

A sequence of positive integers $a_1 < a_2 < \cdots$ is called a B_2-sequence or a Sidon sequence if the pairwise differences are all distinct, or in other words, if all the sums $a_i + a_j$ $(i = j$ is permitted) are different. The Mian-Chowla sequence is the B_2-sequence obtained by the greedy algorithm; i.e., each term is the least integer greater than earlier terms which does not violate the distinctness of differences (or sums) condition.

If M is the maximum of reciprocal sums over all B_2-sequences and S^* is the sum of the reciprocals of the Mian-Chowla sequence, then $M \geq S^* > 2.156$. But Levine observes that

$$M \leq \sum_{n\geq 0} \frac{1}{1+\frac{n(n+1)}{2}} < 2.374$$

and would like to see a proof or disproof of $M = S^*$ [2, pp. 127–128].

The main task of this paper is to construct a B_2-sequence which disproves this conjecture. Most work is done on a personal computer IBM PC/XT. We state our results in the following two theorems:

Theorem 1. *We have* $S^* < 2.1596$.

Theorem 2. *We have* $M > 2.1597$.

In §2 we prove both theorems. The first 445 terms of the Mian-Chowla sequence $\{a_i : 1 \leq i \leq 445\}$ and the first 300 terms of our B_2-sequence $\{b_i :$

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$1 \le i \le 300\}$ which are used for proving the theorems are given in §3, and the search procedure used to find our sequence is given in §4.

2. Proofs of the theorems

Theorem 1. *We have* $S^* < 2.1596$.

Proof. Let

$$s_1 = \sum_{i=1}^{445} \frac{1}{a_i}, \quad s_2 = \sum_{i=446}^{11153} \frac{1}{a_i}, \quad s_3 = \sum_{i=11154}^{\infty} \frac{1}{a_i}.$$

Then by computer calculations we have

$$s_1 = 2.15828\ldots < 2.15829$$

and

$$\begin{aligned} s_2 &\le \frac{1}{a_{445} + d_1} + \frac{1}{a_{445} + d_1 + d_2} + \cdots + \frac{1}{a_{445} + d_1 + d_2 + \cdots + d_{10708}} \\ &= 0.00110\ldots < 0.00111, \end{aligned}$$

where $d_1 = 33$, $d_2 = 88, \ldots, d_{10708} = 16960$ are the first 10708 positive integers not belonging to the set $\{a_j - a_i : 1 \le i < j \le 445\}$. Put

$$h = a_{445} + d_1 + d_2 + \cdots + d_{10708} = 100005214 \quad \text{and} \quad d = d_{10708} = 16960.$$

Then

$$\begin{aligned} s_3 &\le \frac{1}{h+d+1} + \frac{1}{h+d+1+d+2} + \cdots \\ &= \sum_{k=1}^{\infty} \frac{1}{h + kd + k(k+1)/2} \\ &= \sum_{k=1}^{\infty} \frac{2}{k^2 + k(2d+1) + 2h} < \sum_{k=1}^{\infty} \frac{2}{(k+14000)(k+14001)} \\ &= 2\sum_{k=1}^{\infty} \left(\frac{1}{k+14000} - \frac{1}{k+14001} \right) \\ &= \frac{2}{14001} < 0.00015. \end{aligned}$$

Thus $S^* = s_1 + s_2 + s_3 < 2.15829 + 0.00111 + 0.00015 = 2.15955 < 2.1596$. This completes the proof. □

Theorem 2. *We have* $M > 2.1597$.

Proof. We construct our sequence $\{b_i\}$ as follows: the first 14 terms are obtained by the greedy algorithm. Thus, they are the same as the first 14 terms of the Mian-Chowla sequence. Let $b_{15} = b_{14} + 47 = 229$. Note that $a_{15} = a_{14} + 22 = 202$. From the 16th term on, the greedy algorithm continues. Since

$$\{b_{15} - b_i : 1 \le i \le 14\} \cap \{b_j - b_i : 1 \le i < j \le 14\} = \varnothing,$$

our sequence is a B_2-sequence. The reciprocal sum of the first 300 terms is greater than 2.1597. This completes the proof. □

3. Beginning terms of the two sequences

The first 445 terms of the Mian-Chowla sequence are as follows:

1	2	4	8	13	21	31	45	66	81
97	123	148	182	204	252	290	361	401	475
565	593	662	775	822	916	970	1016	1159	1312
1395	1523	1572	1821	1896	2029	2254	2379	2510	2780
2925	3155	3354	3591	3797	3998	4297	4433	4779	4851
5123	5243	5298	5751	5998	6374	6801	6925	7460	7547
7789	8220	8503	8730	8942	9882	10200	10587	10898	11289
11614	11876	12034	12931	13394	14047	14534	14901	15166	15688
15972	16619	17355	17932	18845	19071	19631	19670	20722	21948
22526	23291	23564	23881	24596	24768	25631	26037	26255	27219
28566	29775	30094	31311	32217	32620	32912	34277	35330	35469
36204	38647	39160	39223	39943	40800	41882	42549	43394	44879
45907	47421	47512	48297	50064	50902	52703	52764	54674	55307
56663	58425	59028	60576	60995	62205	63129	64488	66999	67189
68512	68984	70170	71365	75618	76793	77571	79047	80309	83179
84345	87016	87874	88566	89607	91718	92887	93839	95103	97974
99583	101337	102040	103626	104554	106947	107205	108622	111837	112800
113949	114642	116291	117177	121238	125492	126637	129170	130986	131697
134414	134699	136635	139964	143294	144874	146605	147499	148593	150146
152318	152834	156836	157150	160782	163010	163502	164868	170984	172922
174171	177853	180249	182071	185403	188314	190726	190894	193477	196832
199646	201472	202699	205325	206811	208748	214435	217182	218011	225350
226682	229163	231694	233570	234619	235152	238727	240814	247822	253857
254305	260433	261620	262317	266550	269195	271511	274250	274753	280180
284289	290005	293034	295037	296506	298414	302663	305782	308841	317739
321173	323672	324806	329181	331018	336642	340901	343359	347001	348110
348899	362520	366119	368235	370696	371542	377450	380366	382012	382245
384957	387479	390518	391462	399174	403920	411847	412671	416880	417991
422453	433973	434773	440619	441148	443779	446065	456289	458426	462402
470670	474668	475800	481476	482868	498435	501084	508193	511258	514644
524307	527197	535369	536903	538331	542020	555275	564016	566106	567408
572027	582478	583407	585871	593257	596837	598426	599784	607794	610404
621790	624574	627703	633442	640047	648858	659179	663558	667337	672815
673522	686013	691686	693169	694279	696931	703162	711364	723249	729860
731008	739958	740124	744403	753293	768134	770113	773912	779917	787407
794900	797567	800658	813959	814414	827123	829129	839728	847430	850695
851627	862856	880796	884725	889285	896691	897160	904970	909586	915254
922852	935695	937825	938876	959937	961353	964857	970227	976356	980581
986799	1008106	1009835	1016906	1020306	1028612	1033242	1036012	1042818	1050881
1051783	1060844	1086402	1092043	1096162	1103456	1123464	1134057	1136410	1144080
1145152	1147774	1156687	1164278	1166255	1174751	1187057	1195316	1201262	1207345
1212654	1218610	1225019	1227887	1240777	1247071	1258235	1265462	1274089	1279515
1288613	1298980	1306248	1326918	1333809	1341190	1343482	1367480	1372734	1374779
1384952	1388147	1394240	1395346	1409612	1417336	1418943	1423296	1446209	1448494
1462599	1468933	1474698	1496110	1502217					

Remark. We see that 22 does occur as a difference, in fact $22 = a_{15} - a_{14}$. This answers a question of Erdős and Graham [1]. But we do not know if 33 occurs as a difference.

The first 300 terms of our new sequence are:

1	2	4	8	13	21	31	45	66	81
97	123	148	182	229	257	290	312	381	419
467	507	621	721	770	864	927	1050	1178	1289
1457	1561	1615	1774	1907	2090	2164	2263	2309	2539
2800	2938	3035	3310	3380	3738	4043	4239	4439	4726
4851	5016	5169	5289	5490	5760	6646	6843	7015	7442
7674	7986	8284	8506	8772	9240	9778	9996	10344	10431
11614	12263	12687	12984	13516	13628	14462	15089	15409	15651
15843	16248	17803	17988	18943	19118	19925	20583	21013	21052
21639	21972	23418	23896	24375	24557	25511	26162	26504	26827
26997	27787	30183	30655	31111	32618	33304	33394	33597	36231
36635	37572	38579	39307	40172	42490	42945	44117	44712	45918
46451	47209	48109	50129	51876	52603	53226	53978	54634	55209
56172	58002	59622	60226	61723	62429	64083	65305	66454	67232
68376	68965	69467	69834	71342	72357	74931	75408	78410	78804
80465	80977	81899	85789	86303	88765	90296	91867	92825	94246
95038	97985	100549	101393	102896	106706	107722	109182	110300	110826
112294	114453	115132	117807	119010	121309	123560	124680	128649	129208
130147	131841	135031	136835	139607	144905	145300	146501	147326	147546
152056	153457	156069	157702	160454	162156	165033	168582	168846	172201
174579	177225	180299	182838	183432	187888	190545	193228	194415	196471
200123	203926	205545	208826	211391	214065	215912	219163	220080	222866
225828	228312	231729	234870	240374	240715	242989	244055	247540	249272
252053	254674	255953	258048	262213	265401	271626	274063	278059	281693
284081	285408	288025	294965	300778	303944	305303	306944	312834	314028
322154	322889	326892	328809	331976	338379	342067	346155	347540	350057
351529	352261	353556	358624	360002	361672	369690	370786	371556	378880
387973	392160	393680	398659	399841	409682	414057	422868	423870	426168
427830	432720	437763	446086	446941	453707	455386	462236	466551	469613
470241	474466	480984	485330	489829	494840	498160	501210	511039	517520

4. Search procedure

We first use a search procedure to find a finite B_2-sequence $1 = b_1 < b_2 < \cdots < b_{120}$ with reciprocal sum as large as possible. The procedure contains six main steps:

(1) For a given integer u $(2 \leq u \leq 20)$, use the greedy algorithm to get the first u terms $1 = b_1 < b_2 < \cdots < b_u$.

(2) Let $d_1 < d_2 < \cdots$ be the consecutive integers which can be a candidate for $b_{u+1} - b_u$ so that the set $\{b_1, b_2, \ldots, b_u, b_{u+1}\}$ does not violate the distinctness of differences condition.

(3) For a given integer v $(2 \leq v \leq 5)$, let $b_{u+1} = b_u + d_v$.

(4) Continue the greedy algorithm to get the last $119 - u$ terms: $b_{u+2} < b_{u+3} < \cdots < b_{120}$.

(5) Calculate $F_{u,v} = \sum_{i=1}^{120} 1/b_i$, then go to (3) for another choice of v until $v > 5$.

(6) Go to (1) for another choice of u until $u > 20$.

We find that $F_{14,4} = 2.15848\ldots \geq F_{u,v}$ for $2 \leq u \leq 20$, $2 \leq v \leq 5$.

Note that the reciprocal sum of the first 120 terms of the Mian-Chowla sequence is $2.15686\ldots$. Thus, it is reasonable to conjecture that if we extend the sequence with $u = 14$, $v = 4$ ($b_{14} = 182$, $d_4 = 47$, and $b_{15} = 229$) to an infinite sequence, we will get a B_2-sequence with larger reciprocal sum than that of the Mian-Chowla sequence. To prove this we must have enough terms of both sequences as indicated in §§2 and 3. The Pascal programs for proving Theorems 1 and 2 ran about 5.5 hours in total on an IBM PC/XT.

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