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SQUARES IN QUADRATIC PROGRESSION

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ABSTRACT. The sequence of consecutive integer squares has constant second difference 2. We list every such sequence of squares containing a term less than 1000^2.

0. INTRODUCTION

We call a sequence a *quadratic progression* if the second difference is constant. The sequence of consecutive integer squares is certainly a quadratic progression with second difference equal to 2. A number of families of nontrivial four-term integer square progressions with second difference 2 are known (Buell [2]) but no such five-term quadratic progression of integer squares is known. Leech asked whether there is such a sequence starting at 0 and we answer this in the negative by listing all four-term progressions with an element less than 1000 and showing that none includes 0. We also find that none of these progressions can be extended to a fifth square term.

1. QUADRATIC PROGRESSIONS WITH GIVEN SECOND TERM

Suppose that

$$\begin{aligned} x^2, \qquad d^2, \qquad y^2 &= 2d^2 - x^2 + 2, \\ z^2 &= 3d^2 - 2x^2 + 6 \end{aligned} \tag{1.1}$$

is a four-term progression with constant second difference 2. We regard the second term d^2 as given. Then we have

$$\begin{aligned} y^2 + x^2 &= 2d^2 + 2, \\ z^2 + 2x^2 &= 3d^2 + 6 \end{aligned} \tag{1.2}$$

with obvious solutions $\pm x = d \pm 1$ corresponding to the trivial quadratic progression of consecutive squares. For given d, it is easy to tabulate the finite set of solutions x to each of the individual equations in this pair of equations. The results for $d \leq 999$ are included in Table 1.

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TABLE 1. Table of solutions

t	x	y	z	t	x	y	z
6	23	32	39	570	7879	11128	13623
16	87	122	149	584	2257	3138	3821
39	32	23	6	601	4832	6807	8326
39	70	91	108	651	580	499	402
51	148	203	246	651	778	887	984
59	228	317	386	849	718	557	324
59	630	889	1088	856	1537	1998	2371
79	242	333	404	883	25566	36145	44264
83	516	725	886	862	1713	2264	2705
108	91	70	39	886	725	516	83
108	157	194	225	916	26605	37614	46063
108	707	994	1215	984	887	778	651
108	6643	9394	11505	984	1145	1286	1413
147	302	401	480	1088	889	630	59
149	122	87	16	1215	994	707	108
177	878	1229	1500	1226	1017	752	311
225	194	157	108	1247	1048	801	430
225	296	353	402	1411	1180	891	442
240	839	1162	1413	1413	1162	839	240
246	203	148	51	1413	1286	1145	984
287	11838	16739	20500	1500	1229	878	177
311	752	1017	1226	2371	1998	1537	856
324	557	718	849	2561	2112	1537	514
334	3693	5212	6379	2705	2264	1713	862
386	317	228	59	2853	2348	1699	510
386	6237	8812	10789	3821	3138	2257	584
402	353	296	225	6150	5029	3572	477
402	499	580	651	6379	5212	3693	334
404	333	242	79	8326	6807	4832	601
419	11020	15579	19078	10789	8812	6237	386
430	801	1048	1247	11505	9394	6643	108
442	891	1180	1411	13623	11128	7879	570
477	3572	5029	6150	19078	15579	11020	419
480	401	302	147	20500	16739	11838	287
510	1699	2348	2853	44264	36145	25566	883
514	1537	2112	2561	46063	37614	26605	916

2. Quadratic progressions with given first term

Suppose that

$$(2.1)\qquad c^2, \qquad x^2, \qquad y^2 = 2x^2 - c^2 + 2, \qquad z^2 = 3x^2 - 2c^2 + 6$$

is a four-term quadratic progression with constant second difference 2. We regard the first term c^2 as given. Then we have a pair of simultaneous Pellian equations

$$(2.2)\qquad y^2 - 2x^2 = 2 - c^2, \qquad z^2 - 3x^2 = 6 - 2c^2$$

with obvious solutions $\pm x = c \pm 1$ corresponding to the trivial quadratic progression of consecutive squares. For given c, each Pellian equation has infinitely many solutions x. We shall show that in practice a simple search can be used to identify small common solutions, and that Baker's method of linear forms in logarithms can be used to determine whether or not the set of solutions found in this way is complete.

We illustrate the method on the case $c = 39$, which exhibits all the features of interest. We have

$$y^2 - 2x^2 = -1519, \qquad z^2 - 3x^2 = -3036.$$

The first equation is solved by working in the field $\mathbb{Q}(\sqrt{2})$, which has class number 1 and fundamental unit $1 + \sqrt{2}$. We have

$$\pm y \pm x\sqrt{2} = \delta_i \eta^n$$

for some integer n and $i = 1$, 2 or 3, where $\eta = 3 + 2\sqrt{2}$ is the fundamental totally positive unit and $\delta_1 = 7 + 28\sqrt{2}$, $\delta_2 = 23 + 32\sqrt{2}$, $\delta_3 = 37 + 38\sqrt{2}$. The possible values of $\pm x$ therefore fall into three binary recurrence sequences $A_n^{(i)}$, each with recurrence relation

$$A_{n+1} = 6A_n - A_{n-1},$$

which has the minimal equation for η as auxiliary polynomial, and initial values $A_0^{(1)} = 28$, $A_1^{(1)} = 98$; $A_0^{(2)} = 32$, $A_1^{(2)} = 142$; $A_0^{(3)} = 38$, $A_1^{(3)} = 188$, respectively.

We solve the second equation similarly in $\mathbb{Q}(\sqrt{3})$, which has class number 1 and fundamental unit $2 + \sqrt{3}$. We have

$$\pm z \pm x\sqrt{3} = \beta_i \varepsilon^m$$

for some integer m and $j = 1$ or 2, where $\varepsilon = 2 + \sqrt{3}$ is the fundamental totally positive unit and $\beta_1 = 6 + 32\sqrt{3}$, $\beta_2 = 36 + 38\sqrt{3}$. The possible values of $\pm x$ therefore fall into two binary recurrence sequences $B_m^{(j)}$, each with recurrence relation

$$B_{m+1} = 4B_m - B_{m-1},$$

which has the minimal equation for ε as auxiliary polynomial, and initial values $B_0^{(1)} = 32$, $B_1^{(1)} = 70$; $B_0^{(2)} = 38$, $B_1^{(2)} = 112$, respectively.

The solutions x to the original pair of equations are therefore the common values between the sequences $A_n^{(i)}$ and $B_m^{(j)}$. Computing the values of the sequences with m, n from -10 to $+10$, we find that $A_{-1}^{(1)} = B_1^{(1)} = 70$, $A_0^{(2)} = B_0^{(1)} = 32$, $A_0^{(3)} = B_0^{(2)} = 38$, and $A_{-1}^{(3)} = B_{-1}^{(2)} = 40$. We shall show that these are in fact the only occurrences of common values between the sequences $A^{(i)}$ and $B^{(j)}$.

We show first that there is no term in common between the pairs of sequences $A^{(1)}$ and $B^{(2)}$, between $A^{(2)}$ and $B^{(2)}$, or between $A^{(3)}$ and $B^{(1)}$. We do so by considering the (finite) sets of values of $A^{(1)}$ and $B^{(2)}$ modulo 41, $A^{(2)}$ and $B^{(2)}$ modulo 408, $A^{(3)}$ and $B^{(1)}$ modulo 315, and observing that in each case the sets of A-values and B-values are disjoint.

We now show that the only values in common between the pairs of sequences $A^{(1)}$ and $B^{(1)}$; between $A^{(2)}$ and $B^{(1)}$; and between $A^{(3)}$ and $B^{(2)}$ are the values given above. By Theorem 2.6 of [3] any solution to $y + x\sqrt{2} = \delta_1 \eta^n$, $z + x\sqrt{3} = \beta_1 \varepsilon^m$, must have m, n less than $\exp(63.11)$. We apply Algorithm 3.1 of [3] to the sequences $A^{(1)}$ and $B^{(1)}$ with List K the set of primes up to 953 and List L the primes less than 241 and find that if $A^{(1)}_n = B^{(1)}_m$, then $n \equiv -1 \bmod N$ and $m \equiv 1 \bmod M$ for moduli N, M with $\log N \geq 79.05$ and $\log M \geq 100.00$, which shows that in fact we must have $n = -1$ and $m = 1$ as required. A similar analysis holds for the pair of sequences $A^{(2)}$ and $B^{(1)}$.

To show that the sequences $A^{(3)}$ and $B^{(2)}$ have only the two values 38 and 40 in common, we split $A^{(3)}$ up into two subsequences $C^{(1)}$ and $C^{(2)}$ of alternate terms, having the recurrence relations $C_{n+1} = 34C_n - C_{n-1}$ with initial values $C^{(1)}_0 = A^{(3)}_0 = 38$, $C^{(1)}_1 = A^{(3)}_2 = 1090$; $C^{(2)}_0 = A^{(3)}_{-1} = 40$, $C^{(2)}_1 = A^{(3)}_1 = 188$. Again computing the values of $C^{(i)}_n$ for i between -10 and $+10$, we find that the only common values are $C^{(1)}_0 = B^{(2)}_0 = 38$ and $C^{(2)}_0 = B^{(2)}_{-1} = 40$. We now apply Algorithm 3.1 of [3] to each pair of sequences in turn as before to show that these are indeed the unique common values.

Theorem 1. *The only nontrivial four-term quadratic progressions of integer squares with a term less than* 1000^2 *are those given by Table* 1 *or their reversal.*

Proof. Since reversing a quadratic progression with constant second difference 2 gives another such, it is sufficient to consider those for which the first or the second term is less than 1000. Those for which the second term is less than 1000 are dealt with by the remarks of §1. The calculation illustrated above for the case $c = 39$ is simple to automate and was programmed in Algol-68C. In each case where sequences appeared to have no common term, it was possible to show this by taking values to a modulus at most 965. In each case where sequences had a term in common, the bound on the exponents m and n was less than $\exp(63.11)$, and the algorithm was applied with List K being the primes up to 953 and List L the primes up to 241. Each application of the algorithm produced moduli in excess of the bound, showing that each pair of sequences with apparently only one term in common did indeed have only that term as a common value. The whole table was computed and verified completely on a Sun 3/60 workstation, in about eleven hours of CPU time. □

5. Other second differences

Similar results can be obtained for any fixed constant second difference δ. We replace (1.2) by

$$y^2 + x^2 = 2d^2 + \delta, \qquad z^2 + 2x^2 = 3d^2 + 3\delta,$$

and (2.2) by

$$y^2 - 2x^2 = \delta - c^2, \qquad z^2 - 3x^2 = 3\delta - 2c^2, \tag{2.2}$$

and it is clear that the same techniques apply.

The problem of finding squares in quadratic progression with constant but unrestricted second difference is equivalent to finding a quadratic polynomial with consecutive square values. This problem was considered by Allison [1], who obtains the sequence $53^2, 173^2, 217^2, 233^2, 227^2, 197^2, 127^2$ of seven distinct squares with constant second difference -9960. He shows that there are infinitely many quadratic polynomials taking eight consecutive square values symmetric about the turning point if this falls midway between integers, the first giving the progression

$$17^2, 53^2, 67^2, 73^2, 73^2, 67^2, 53^2, 17^2$$

with constant second difference -840. In [4] it is shown that there are no such polynomials if the turning point is at an integer. In each of these "symmetric" cases the argument reduces to analyzing the group of rational points on an elliptic curve given by simultaneous Pell equations.

Bibliography

1. Allison, *On square values of quadratics*, Math. Proc. Cambridge Philos. Soc. **99** (1986), 381-383.
2. D. Buell, *Integer squares with constant second difference*, Math. Comp. **49** (1987), 635–644.
3. R. G. E. Pinch, *Simultaneous Pellian equations*, Math. Proc. Cambridge Philos. Soc. **103** (1988), 35–46.
4. ——, *Square values of quadratic polynomials*, (to appear).

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