23[65-02, 26C05, 33C45, 42C05, 65D99].—CLAUDE BREZINSKI, *Biorthogonality and Its Applications to Numerical Analysis*, Monographs and Textbooks in Pure and Applied Mathematics, Vol. 155, Dekker, New York, 1992, viii+166 pp., $23\frac{1}{2}$ cm. Price \$85.00.

The last decade has witnessed a quiet revolution in the theory and application of orthogonal polynomials. Of course, numerical analysts and practitioners of scientific computing have been always aware of the role of orthogonal polynomials in quadrature methods, spectral algorithms, numerical algebra, approximation theory, etc. Recent work, however, went a long way not just to emphasize the centrality of orthogonal polynomials to computation but to highlight a wide range of further applications—from quantum groups to dynamical systems, from coding theory to spectral properties of the Schrödinger equation, from group representation to signal processing... [7].

An important part of the unfolding orthogonal scene are 'exotic' concepts of orthogonality, e.g., convolution orthogonality [1] and orthogonality with respect to a Sobolev inner product [6]. A place of pride on this list belongs to biorthogonal functions, the subject of the book under review. And who can be a more natural expositor of this subject area than Professor Brezinski, one of the leading workers on both theory and applications of biorthogonality?!

There are several ways of introducing biorthogonal functions and the book follows the original framework of Davis [2]. Let E be an infinite-dimensional vector space and let $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots \in E$ be linearly independent. Moreover, suppose that $L_0, L_1, L_2, \ldots$ are linear functionals acting on E (i.e., elements of the dual space E^*) and that

$$\det \begin{bmatrix} L_0(\mathbf{x}_0) & L_1(\mathbf{x}_0) & \cdots & L_n(\mathbf{x}_0) \\ L_0(\mathbf{x}_1) & L_1(\mathbf{x}_1) & \cdots & L_n(\mathbf{x}_1) \\ \vdots & \vdots & & \vdots \\ L_0(\mathbf{x}_n) & L_1(\mathbf{x}_n) & \cdots & L_n(\mathbf{x}_n) \end{bmatrix} \neq 0$$

for all $n = 0, 1, 2, \ldots$. Then there exist unique linear combinations

$$L_n^* = \sum_{j=0}^{n} a_{n,j} L_j, \quad a_{n,n} \neq 0, \qquad \mathbf{x}_n^* = \sum_{j=0}^{n} b_{n,j} \mathbf{x}_j, \qquad n = 0, 1, 2, \ldots,$$

such that $L_n^*(\mathbf{x}_m^*) = \delta_{m,n}$. The set $\{L_n^*, \mathbf{x}_n^*\}_{n=0}^{\infty}$ is called a *biorthogonal family*.

Other definitions of biorthogonality require somewhat stricter frameworks. Nevertheless, they are highly illuminating and the loss of generality does not interfere with realistic applications. Thus, if E is a Hilbert space, then by the Riesz equivalence theorem biorthogonality can be expressed in terms of inner products. This is somewhat more 'symmetric', since the two sequences required for biorthogonality belong to the same space. Furthermore, if the inner product is selfadjoint, then the Daniel theory of integration implies that biorthogonality can be expressed in terms of Stieltjes-Lebesgue measures (or, for those at home with generalized functions, weight functions). The latter framework is particularly useful and transparent in the case of biorthogonal polynomials—more about it in the sequel.

The first part of the book is devoted to a formal exposition of biorthogonal functions and their theoretical features. It brings together a great deal of results, many quite recent, that have been scattered throughout the scientific literature. They are bound together with a generous measure of mathematical mortar, the cracks and gaps being filled by original and unpublished research. The formal functional-analytic approach does not make for an easy reading, but the extra effort required to master the contents and to follow its technical minutiae is worthwhile. It is, however, the sentiment of this reviewer that this part is needlessly formal and uncompromising in its attitude to the reader. Very seldom are we told the purpose of this or that construct at the point of embarkation. Instead, the exposition, as elsewhere in the book, is a Bourbaquiste progression of definition → proposition → theorem. The material of the book is *intrinisically* difficult and calls for mathematical maturity and ability to cope with subtle technicalities on the reader's part—why make it even more straightlaced and formal?

The surfeit of formalism occasionally hides gaps and ambiguities. Thus (pages 25–26, verbatim):

"3.5 The method of moments

This method, studied by Vorobyev [186] in a Hilbert space, is a particular case of Galerkin's method. We shall now extend it to an arbitrary vector space E and its dual E^*.

The method of moments consists in constructing a linear operator A_n on E_{n-1} such that

$$\begin{aligned} x_1 &= A_n x_0 \\ x_2 &= A_n x_1 \\ &\cdots \\ x_{n-1} &= A_n x_{n-2} \\ I_{n-1}(x_n) &= A_n x_{n-1} \end{aligned}$$

or

$$\begin{aligned} x_k &= A_n^k x_0 \qquad k = 0, \ldots, n-1 \\ I_{n-1}(x_n) &= A_n^n x_0 \text{."} \end{aligned}$$

We are told neither what is E_{n-1} (as distinct from E) nor whether the definition should be valid for just one $x_0 \in E_{n-1}$ or for all $x_0 \in E_{n-1}$. The interpolation operator I_{n-1} has been already defined in §3.2 in a different formalism—the reader should be able ultimately to work out its relevance in the present setting, but only with a wholly unnecessary extra effort. Most importantly, what is the purpose of the method of moments? Which problems is it supposed to solve? What are the advantages in studying it in arbitrary vector spaces?

To be fair, formalism has its rewards and, in Professor Brezinski's hands, it frequently becomes a powerful tool. An example, dear to this reviewer's heart, are Christoffel-Darboux identities for biorthogonal functions. Their derivation in [4] for the special case of biorthogonal polynomials required much effort (and, to be frank, much tedium), whereas the book presents a far-reaching generalization, based on its formalism—and does it, comprehensively, in a short page. Bravo!

In other instances, the quest for generality obscures important features of special cases. Thus, biorthogonal polynomials have been defined in [3] as follows: a nonzero nth-degree polynomial $p_n(\cdot; \mu_1, \mu_2, \ldots, \mu_n)$ is biorthogonal with respect to the parametrized Borel measure $d\varphi(x, \mu)$ if

$$\int_{-\infty}^{\infty} p_n(x; \mu_1, \mu_2, \ldots, \mu_n)\, d\varphi(x, \mu_\ell) = 0, \qquad \ell = 1, 2, \ldots, n.$$

This definition can be made to fit into the straightjacket of this book's formalism by defining the functionals

$$L_k^* f := \int_{-\infty}^{\infty} f(x) d\varphi(x, \mu_{k+1}), \qquad k = 0, 1, \ldots,$$

but there is a price to pay. Biorthogonal polynomials from [3] depend on their parameters $\mu_1, \ldots, \mu_n$ in a continuous (indeed, differentiable) manner. This is absolutely crucial to their application to trace loci of zeros of polynomial transformations [5].

The remainder of the book presents an exposition of a considerable number of applications of biorthogonality. The list is impressive: the Lanczos method, biconjugate gradients, rational approximation theory, acceleration of convergence, design of multistep methods for ordinary differential equations and least squares calculations. Claude Brezinski speaks with great authority on all these and has been a driving force in the implementation of biorthogonal techniques. Thus, I approached this part of the book with great anticipation but, alas, found the paucity of explanation and motivation a real stumbling block. Thus, on page 33 we are treated to the only explanation of what the Lanczos method and biconjugate gradients are all about:

> "In a Hilbert space it is well known that the method of moments gives rise to Lanczos' method and then to the conjugate and biconjugate gradient methods, see [17, pp. 79–91, 186–189]."

Well, an educated numerical analyst should have heard of the Lanczos method (and is unlikely to confuse it with the Lanczos τ method), although a brief reminder would have been welcome. But how many know of biconjugate gradients? Readers with plenty of commitment, motivation and spare time (to say nothing of a well-equipped library) may always consult the references—but what of the remaining 99%?

This is a book evidently written in a hurry. It is based on deep knowledge and scholarship and will be indispensable as a source for the small band of workers in the subject. I am, however, sceptical of its potential to popularize the important concept of biorthogonality in the broader numerical community.

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3. A. Iserles and S. P. Nørsett, *On the theory of biorthogonal polynomials*, Trans. Amer. Math. Soc. **306** (1988), 455–474.

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In the applied sciences one often faces the task of determining a numerical value for the limit of a slowly convergent sequence. In many situations the sequence is divergent, yet there is a commanding physical reason to attach a meaning—in the sense of limit—to the sequence. Techniques for summing slowly convergent, or divergent, sequences go by the generic name of *summation methods*. The idea is to transform the given sequence S_n into a sequence $\overline{S}_n$ by some kind of formula, $\overline{S}_n = F_n\{S_0, S_1, \dots, S_{k(n)}\}$, $n = 0, 1, 2, \dots$, so that $\overline{S}_n$ converges to the same limit, but more rapidly.

In such an undertaking the numerical analyst has to address several issues, partly philosophical in nature:

1) For a given sequence or class of sequences, which is the best technique to use?

2) What assurance does one have that the approximate limit will be close—arbitrarily close—to the true limit?

3) If the sequence is divergent, how can one know that the so-called limit calculated will reflect what the physical situation dictates?

We can easily dispose of the last dilemma. There can be *no* general assurance that the limit calculated is the "correct" one. In his book [3] on infinite series, Knopp gives an example to illustrate that several heuristically plausible "limits" can be assigned to a divergent sequence. Although textbook examples may be so concocted, reality seems gentler to the numerical analyst: it is a rule of thumb that in real-life situations one either gets no limit at all or the correct limit.

At a conference in January 1992 in Tenerife, E. J. Weniger presented some remarkable examples. The sequences in question were the partial sums, strongly divergent, of perturbation expansions for the ground state energies of the quartic, sextic, and octic anharmonic oscillators. The sequences posed challenging test problems for available summation methods, since their terms diverged, respectively, like $n!/n^{1/2}$, $(2n)!/n^{1/2}$, $(3n)!/n^{1/2}$. A favorite method—the Levin transformation—could not sum any of these sequences, and the failure was not an artifact of numerical instability or round-off error—a common pitfall of summation methods. Weniger performed the computations in Maple in 1000-digit precision; the failure of the Levin method was genuine. Another transformation *did* sum the sequences. It is interesting that in all the cases Weniger studied, the summation methods chosen either did not produce a convergent sequence