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24[40–01, 40–04, 65B10].—CLAUDE BREZINSKI & MICHELA REDIVO ZAGLIA, *Extrapolation Methods: Theory and Practice*, Studies in Computational Mathematics, Vol. 2, North-Holland, Amsterdam, 1991, x+464 pp. (includes floppy disk), $24\frac{1}{2}$ cm. Price $115.50/Dfl. 225.00.

In the applied sciences one often faces the task of determining a numerical value for the limit of a slowly convergent sequence. In many situations the sequence is divergent, yet there is a commanding physical reason to attach a meaning—in the sense of limit—to the sequence. Techniques for summing slowly convergent, or divergent, sequences go by the generic name of *summation methods*. The idea is to transform the given sequence S_n into a sequence $\overline{S}_n$ by some kind of formula, $\overline{S}_n = F_n\{S_0, S_1, \ldots, S_{k(n)}\}$, $n = 0, 1, 2, \ldots$, so that $\overline{S}_n$ converges to the same limit, but more rapidly.

In such an undertaking the numerical analyst has to address several issues, partly philosophical in nature:

1) For a given sequence or class of sequences, which is the best technique to use?

2) What assurance does one have that the approximate limit will be close—arbitrarily close—to the true limit?

3) If the sequence is divergent, how can one know that the so-called limit calculated will reflect what the physical situation dictates?

We can easily dispose of the last dilemma. There can be *no* general assurance that the limit calculated is the "correct" one. In his book [3] on infinite series, Knopp gives an example to illustrate that several heuristically plausible "limits" can be assigned to a divergent sequence. Although textbook examples may be so concocted, reality seems gentler to the numerical analyst: it is a rule of thumb that in real-life situations one either gets no limit at all or the correct limit.

At a conference in January 1992 in Tenerife, E. J. Weniger presented some remarkable examples. The sequences in question were the partial sums, strongly divergent, of perturbation expansions for the ground state energies of the quartic, sextic, and octic anharmonic oscillators. The sequences posed challenging test problems for available summation methods, since their terms diverged, respectively, like $n!/n^{1/2}$, $(2n)!/n^{1/2}$, $(3n)!/n^{1/2}$. A favorite method—the Levin transformation—could not sum any of these sequences, and the failure was not an artifact of numerical instability or round-off error—a common pitfall of summation methods. Weniger performed the computations in Maple in 1000-digit precision; the failure of the Levin method was genuine. Another transformation *did* sum the sequences. It is interesting that in all the cases Weniger studied, the summation methods chosen either did not produce a convergent sequence

or summed the sequence to its physically correct limit. Never did the methods deliver up a spuriously convergent sequence.

We can address the first and second issues in a more illuminating and productive way, and the present book presents an exhaustive survey of our current state of knowledge. One difficulty with working in this field is that crucial results are scattered throughout a voluminous technical literature, in technical reports, conference proceedings, published journals. Some remarkable theoretical and practical tools have emerged in the last decade. This book provides an account of them all. The senior author, Claude Brezinski, is a—perhaps *the*—leading authority on the subject, and the presentation reflects his expertise, experience, and organizational skills. One hat he wears is that of a mathematical historian, and the book is informed with a strong and compelling sense of the cultural primacy of numerical analysis, a subject that has fascinated the greatest mathematicians. This volume is a momentous contribution to the literature on computational mathematics.

Among very recent theoretical results are theorems describing those classes of convergent sequences that are accelerable, i.e., for which there is a summation method that will transform each sequence in the class into a more rapidly convergent sequence with the same limit. I asked in my book [6] on the subject—written in 1981 and, alas, quite out of date now—whether there was a universal method that would accelerate the convergence of all complex convergent sequences. The French school—consisting mostly of Claude and his present and past students—took a penetrating look at this question. The result was probably the most remarkable mathematics ever done in this area. The book quotes a beautiful theorem due to Jean Paul Delahaye (1988). First, a little notation. Let E be a metric space. For $M \subset E$ denote by M^* the set of accumulation points of M. Let $\mathbb{S}(E)$ be the class of convergent sequences in E with the property that for every sequence $\{S_n\} \subset \mathbb{S}(E)$ and every n_0, $S_n \neq \lim_{n\to\infty} S_n$ for some $n > n_0$.

Theorem. *A necessary and sufficient condition for* $\mathbb{S}(E)$ *to be accelerable is that* $(E^*)^* = \varnothing$.

This means, essentially, that if the class of its convergent sequences is to be accelerable, the metric space must be small, *very small*. The combined ranges of the sequences cannot even contain a perfect subset. The theorem is *general*. The older, classical summation methods are linear and generally offer only limited improvement in convergence. In the past 50 years or so, many highly effective nonlinear methods have appeared. The form of the method is irrelevant, however: the theorem makes no assumptions about the linearity of the sequence transformation used for the acceleration.

Researchers have discovered other striking facts. The union of two accelerable classes may be nonaccelerable. Also, it was a disappointment to those working in the area to find that the class of real monotone sequences is not accelerable. However, the class of complex linearly convergent sequences, i.e., convergent sequences with the property

$$\lim_{n\to\infty} \frac{S_{n+1} - S}{S_n - S} = \lambda, \qquad 0 < |\lambda| < 1, \ \lim_{n\to\infty} S_n = S,$$

is accelerable.

Current theoretical research has centered on defining more accurately the frontier between accelerability and nonaccelerabiltiy. What is the smallest non-accelerable subclass of convergent sequences? the largest accelerable subclass? The impossibility of finding a universal method has caused some authors to toy with the definitions of acceleration and transformation. Recent work in this area, due to researchers such as Wang, Jacobsen, and Germain-Bonne, has been fruitful.

Despite these intriguing theoretical observations, the book really aims for the problem reader, one with no specialized knowledge of summation methods, but with many, many obstinate sequences to sum. I have heard that scientists working in neutron diffusion problems encounter sequences consisting of vectors with tens of thousands of components. The attempt to find a numerical value of the limits of such a sequence will surely make a pragmatist of anyone.

The book starts out at square 1. Appropriately, the first section of the first chapter is entitled, *First steps*. To orient the reader, the authors at the outset quote two methods, one that of the arithmetic means (linear), and the other the Aitken Δ^2 method (nonlinear):

$$\overline{S}_n = \frac{S_n + S_{n+1}}{2}, \qquad n = 0, 1, 2, \ldots,$$

$$\overline{S}_n = \frac{S_n S_{n+2} - S_{n+1}^2}{S_{n+2} - 2S_{n+1} + S_n}, \qquad n = 0, 1, 2, \ldots.$$

These methods are old, and familiar to every numerical analyst, but they serve as paradigms for more modern approaches. The authors point out that the second process, rediscovered by Aitken, was known to the Japanese mathematician Seki Kowa (1642?–1708), who used it to accelerate the computation of π by the method of inscribed polygons.

Linear methods are the grandmothers of the subject: predictable, but tepid. Nonlinear methods are the bikers: robust, but erratic. Nonlinear methods are seldom regular, but they can sum vigorously divergent sequences.

Attempts to generalize and extend the two methods above have given rise to the contemporary literature on the subject. For instance, how can one extend the Aitken Δ^2 method to spaces where no inverses exist, for instance, topological vector spaces? Surprisingly, people have discovered such extensions. The methods, which employ the dual space, are ingenious.

We get in this chapter the basic definitions of *regularity* of an extrapolation algorithm (what I have been calling a summation method): $\overline{S}_n$ must converge to the same limit as S_n for any convergent S_n, of *accelativeness*—a bad word; is there a better one?—, of translativity, homogeneity, quasilinearity of algorithms. The *kernel* of a method is that subclass of sequences mapped identically into a constant sequence, the constant being the limit when the sequence is convergent. Constructing the kernel of an algorithm is obviously an important thing to do, since the kernel gives some sense of the types of sequences for which the algorithm will be effective. For the first process above, the kernel is the set of sequences of the form $S_n = S + a(-1)^n$. For the second it is the set of sequences of the form $S_n = S + a\lambda^n$.

In this chapter the authors establish a few essential principles: (i) One can express many noteworthy sequence transformations, for instance, the Shanks transformation and the Wynn ε-algorithm, based on it, as ratios of determi-

nants involving members of the original sequence. (ii) The numerical stability of a method is often a consequence of how one formulates the method. (iii) A knowledge of the asymptotic properties of real sequences, properties sometimes obscure and often new, is essential to an understanding of the success of extrapolation methods.

The authors also discuss the most popular scalar extrapolation methods in Chapter 1. Many algorithms go by the name of the mathematician who discovered, more often than not, *rediscovered*, them: the Shanks transformation, the Levin transformation, the Germain-Bonne transformation, the Thiele extrapolation process. Let me discuss briefly the Shanks transformation. Anyone not familiar with the subject will find this algorithm, and the ε-algorithm based on it, very mysterious. Assume that the original sequence S_n converges to its limit S in the following fashion:

$$S_n \sim S + c_1\lambda_1^n + c_2\lambda_2^n + c_3\lambda_3^n + \cdots + c_k\lambda_k^n .$$

In the parlance of electrical engineering, one assumes the sequence is its limit plus a finite number of exponential transients. If one replaces the "$\sim$" by "$=$" above, one may solve for S by eliminating the exponentials. This produces an approximation to S, which in reality is a transformed sequence $\overline{S}_n$:

$$e_k(S_n) = \overline{S}_n = \frac{\begin{vmatrix} S_n & \cdots & S_{n+k} \\ \Delta S_n & \cdots & \Delta S_{n+k} \\ \cdots & & \cdots \\ \Delta S_{n+k-1} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 \\ \Delta S_n & \cdots & \Delta S_{n+k} \\ \cdots & & \cdots \\ \Delta S_{n+k-1} & \cdots & \Delta S_{n+2k-1} \end{vmatrix}} . \tag{1}$$

A sensible tactic is to fix k, the number of transients, then allow n to get large and hope that $\overline{S}_n$ will converge more rapidly than S_n. For instance, $k = 1$ gives the Aitken Δ^2 method.[1] But what if the number of transients *and* n both get larger simultaneously? This speculation, a truly innovative one, caused a conceptual revolution in the subject. One generates a (n, k) table, in which one can travel downwards along an *arbitrary path*, hoping to select the path that will promote the most rapid convergence. But, of course, the determinants involved rapidly become unwieldy. Is there a better way of formulating the algorithm? There is, and it was discovered in 1956 by Peter Wynn [7].[2]

Define the sequence $\varepsilon_k^{(n)}$, $k, n = 0, 1, 2, \ldots$, by the following algorithm, sometimes called a *tableau*,

$$\varepsilon_{k+1}^{(n)} = \varepsilon_{k-1}^{(n+1)} + \frac{1}{\varepsilon_k^{(n+1)} - \varepsilon_k^{(n)}}, \qquad k, n = 0, 1, 2, \ldots, \tag{2}$$

with initial conditions $\varepsilon_1^{(n)} = 0$, $\varepsilon_0^{(n)} = S_n$, $n = 0, 1, 2, \ldots$.

[1] More history: The case $k = 2$ of the algorithm was used by James Clerk Maxwell, 1892. Schmidt [4] discovered the general case in 1941, Shanks only later [5, p. 39], in 1949.

[2] Any honest reader will admit to wishing that he or she had discovered this algorithm first. I do.

One can show that

$$\varepsilon_{2k}^{(n)} = e_k(S_n)\,. \tag{3}$$

How show? Alas, there seems to be no simple way. The proof Wynn gave depended on complicated determinant transformations known as Schweinsian identities.[3] More annoying is that one has to know that one wants to derive (3) to prove it. This is the Catch-22 of the mathematical enterprise. If only God gave us the formulas and we returned the proofs. Or vice versa. I have heard that Wynn based his supposition of (3) on a misreading of (1).

The ε-algorithm is dreadfully nonlinear, and thus, one might suppose, resistant to analysis. Yet, we now know a lot about it. A necessary and sufficient condition that a sequence S_n lie in the kernel of the ε-algorithm is that there exist $a_0, a_1, a_2, \ldots, a_k$ with $a_k \neq 0$ and $a_0 + a_1 + a_2 + a_k \neq 0$ such that

$$a_0(S_n - S) + a_1(S_{n+1} - S) + \cdots + a_k(S_{n+k} - S) = 0\,. \tag{4}$$

This beautiful result is due to Brezinski and Crouzeix (1970).

The reviewer and his coworkers have generalized the algorithm to abstract sequence spaces, and we can now completely characterize its effect on many complex sequences, at least for certain paths in the tableau. We have, for instance, the following result:

Let

$$S_n \sim S + \sum_{i=1}^{\infty} a_i(n+b)^{-i}, \qquad n \to \infty\,.$$

Then for k *fixed and* $n \to \infty$,

$$\varepsilon_{2k}^{(n)} \sim S + \frac{a_1}{(k+1)(n+b)}\,.$$

The authors display this result and many other tidbits in their Theorem 2.19.

The effect the ε-algorithm has on certain sequences is dramatic. For instance, it will sum the partial sums of the wildly divergent series

$$\sum_{n=0}^{\infty}(-1)^n n!\,,$$

in the sense that

$$\lim_{k\to\infty} \varepsilon_{2k}^{(0)} = .5963473623 = S\,.$$

S is the "correct" value, i.e.,

$$S = \int_0^{\infty} \frac{e^{-t}}{1+t}\,dt\,.$$

(Just expand $(1+t)^{-1}$ in its Taylor series and formally integrate term-by-term.)

Unfortunately, the algorithm is unstable—for instance, it must be expected that if the algorithm is to be effective, then when k is large, $\varepsilon_k^{(n)}$ and $\varepsilon_k^{(n+1)}$ will be close to each other during the computations. That is precisely what will produce a small denominator in the algorithm, and it tends to happen when S_n approaches its limit monotonically. Further, the algorithm is not

[3]There is a proof utilizing the theory of S-fractions [1], but it is not for the faint of heart (nor the unwary; there are crippling errors in text).

very flexible: it cannot accommodate itself to the shape of the sequence under consideration. Despite its elegance and theoretical importance—in the theory of Padé approximants, for instance—it is not today a significant numerical tool.

Numerical analysts often, but not universally, acknowledge that the two most effective algorithms are the Levin and the Θ-algorithm, the latter due to Brezinski. Problem readers needing a quick fix should read the material in Chapter 2, which introduces these algorithms and describes some of their properties. I caution the reader that the algorithm should depend on the application. If you want to accelerate the computation of definite integrals, think about the G-transformation. For monotone sequences, think about the Levin transformation. There is no universal algorithm. Statistical sequences, or sequences whose error sequences have erratic signs, will defeat any method.[4]

In applying any flexible extrapolation algorithm, it is very helpful to have an idea of the shape of the error sequence. The algorithm can then be tailored to that sequence. For many sequences, one can construct an asymptotic expansion of the error that will permit a more efficient use of an algorithm. Also, it may be possible to extract a subsequence from the given sequence and to accelerate the convergence of that subsequence rather than trying to tackle the original sequence. This is useful when the errors of the sequence have erratic behavior. The authors discuss these matters in Chapter 3. From their account of the large number of special devices one can employ, it is clear that the effective use of extrapolation algorithms is still much more of an art than a science.

As mentioned previously, vector sequences arise often in physics and engineering, usually in the numerical solution of partial differential equations by finite differences or the method of finite elements, and in matrix eigenvalue computations. Of course, one can simply use an extrapolation algorithm on each component of the vector sequence. Such an approach is neither effective nor intellectually satisfying. The theory of vector extrapolation is more sophisticated and has a connection with projection methods, which play an esteemed role in numerical analysis. I want to say some words about the vector ε-algorithm, since it has an intriguing theory and shows just how far ingenuity can take you in this business. If one attempts to apply the tableau (2) to vector sequences, the necessity of inverting vectors proves an impediment. Not to worry. Why not take as the inverse of a vector y the vector

$$y^{-1} = \frac{y}{(y, y)},$$

which gives the right thing when y is a scalar? The algorithm so defined is the *vector ε-algorithm.* McLeod and Graves-Morris studied the kernel of the vector ε-algorithm and discovered that the condition (4) is *sufficient* for a vector sequence to belong to the kernel. The characterization of necessity is much deeper, and the authors devote some space to it. No one knows whether determinant expressions similar to (1)–(3) hold for the algorithm. Another strategy is to *start* with a determinant representation, then formulate a recursive algorithm. The result is called the *topological ε-algorithm.* The problem is that

[4]In my book I give some methods due to myself and Bob Higgins for accelerating the convergence of statistical sequences. But now I see these methods as mired in meta-mathematical dubieties. What does it mean to accelerate the convergence of a sequence which only converges almost surely? At one time I thought I knew. Now I'm not so certain.

the formula (1) makes sense only when one is working in a field. The way of getting around this is really clever. Let y be an arbitrary vector. Interpret the determinant (1) as

$$e_k(S_n) = \frac{\begin{vmatrix} S_n & \cdots & S_{n+k} \\ \langle y, \Delta S_n \rangle & \cdots & \langle y, \Delta S_{n+k} \rangle \\ \cdots & & \cdots \\ \langle y, \Delta S_{n+k-1} \rangle & \cdots & \langle y, \Delta S_{n+2k-1} \rangle \end{vmatrix}}{\begin{vmatrix} 1 & \cdots & 1 \\ \langle y, \Delta S_n \rangle & \cdots & \langle y, \Delta S_{n+k} \rangle \\ \cdots & & \cdots \\ \langle y, \Delta S_{n+k-1} \rangle & \cdots & \langle y, \Delta S_{n+2k-1} \rangle \end{vmatrix}},$$

where $\langle\ ,\ \rangle$ is an inner product. Obviously, this formalism will work for any topological vector space $\mathscr{T}$: just pick y to be an element of the dual of $\mathscr{T}$.

It can be shown that the algorithm whose rules are

$$\varepsilon_{2k+1}^{(n)} = \varepsilon_{2k-1}^{(n+1)} + \frac{y}{\langle y, \varepsilon_{2k}^{(n+1)} - \varepsilon_{2k}^{(n)} \rangle}, \qquad k, n = 0, 1, \ldots,$$

$$\varepsilon_{2k+2}^{(n)} = \varepsilon_{2k}^{(n+1)} + \frac{\varepsilon_{2k}^{(n+1)} - \varepsilon_{2k}^{(n)}}{\langle \varepsilon_{2k+1}^{(n+1)} - \varepsilon_{2k+1}^{(n)}, \varepsilon_{2k}^{(n+1)} - \varepsilon_{2k}^{(n)} \rangle}, \qquad k, n = 0, 1, \ldots,$$

with initial conditions $\varepsilon_{-1}^{(n)} = 0$, $\varepsilon_0^{(n)} = S_n$, $n = 0, 1, 2, \ldots$, has the property that $\varepsilon_{2k}^{(n)} = e_k(S_n)$. I mentioned previously the lack of flexibility of the ε-algorithm. The same is true of the vector version, but there is a much more general algorithm, called the topological E-algorithm, that allows one to take advantage of information one may have about the shape of the sequence S_n.

In Chapter 6, the authors discuss the application of the material in the previous chapter to many problems in the applied sciences. It is here that this book is palpably stronger than any previous book. The applications are to summation of sequences and series, summation of double sequences, Chebyshev and Fourier series, continued fractions, vector sequences, systems of equations, projection methods, regularization and penalty techniques, nonlinear equations, continuation methods, eigenvalue and eigenvector computations, derivatives of eigensystems, integral and differential equations, implicit Runge-Kutta methods, boundary value problems, Laplace transform inversion, partial differential equations, interpolation and approximation, statistical procedures, in particular, Monte Carlo techniques, and numerical integration and differentiation. Many of the applications involve what are called mathematical *ill-posed problems*, that is, problems which are extremely sensitive to small perturbations in initial data.

Some numerical analysts consider sequence extrapolation to be a mere curiosity, hardly deserving of the energy its zealot admirers pour into it. No book on numerical analysis has an up-to-date account of the subject. The formulas (1), (2), (3) are among the most captivating in computational mathematics, and their theoretical and practical implications are considerable, yet how many numerical analysis books have an explanation of the ε-algorithm? I have long

suspected that the agenda of those who write books on numerical analysis is not to enlarge our set of tools for solving difficult and important problems but merely to, well, write still more books on numerical analysis.

This is an absurd state of affairs. It reflects a know-nothingness that is puzzling in an age in which our personal health and welfare may depend on the resolution of ill-posed problems, such as inverse scattering problems (medical imaging) and problems in prediction theory (transportation of atmospheric pollutants). I do not believe that some new, magical method will appear to rescue us from the challenges these problems present. Our best hope for getting the information we need may be to use extrapolation methods, cleverly tailored to the context. Monte Carlo methods are the numerical court of last resort for many ill-posed problems, but the convergence of Monte Carlo methods, $O(n^{-1/2})$, n being the sample size, is so woeful that the error can swamp the computations, or the computations may be too responsive to the vagaries of the random-number generating scheme. Our knowledge about accelerating the convergence of statistical sequences is in its infancy. The methods we have now are not good, and are tainted with philosophical paradox. I think we can develop effective techniques, though they probably will depend on our ability to accurately characterize the distribution of the class of sequences being studied. We need to do much more research.

A unique feature of this book is a floppy disk containing subroutines for the practical implementation of extrapolation algorithms. In the last chapter, the authors describe the programs on the disk. They have tried to present the application side of the subject as more than just a mindless set of recipes. They emphasize that extrapolation methods must be programmed with great care, since often cancellation of significant digits is an inevitable concomitant to a method. Occasionally, one can use algebraic tricks to minimize numerical instability. Sometimes one cannot. By the way, I mentioned previously that nature is always kind to the numerical analyst: it never deceives by presenting the analyst with a spurious limit. The authors in this chapter give an example that forces me to qualify that statement. Consider the sequence S_n defined by

$$S_{n+1} = e^{-S_n}, \qquad n = 0, 1, 2, \ldots, S_0 = 1.$$

The sequence converges, to $S = .567143\ldots$. We know the Aitken Δ^2 method accelerates the convergence of this sequence, at least *theoretically*. Yet, if one retains only a *fixed* number of significant figures in the computation, say, 7, the Δ^2 method gives the spurious limit of .5000000!

The resources the authors provide the reader through this disk are really impressive: 25 methods are here, the most important methods in their scalar, vector, and topological manifestations.

Who should buy this book? Well, not just applied mathematicians. Anyone who uses numerical computations in the analysis of mathematical models of physical phenomena should own a copy. I emphasize that this is not a theoretical book, which places it in a different category from books like Delahaye's [2] or my own [6], and in its practical attributes—its expanse and user-friendliness—it far surpasses previous books by the first author. The theoretical content energizes the book, but the book is more of a hands-on manual in the craft of obtaining

numerical data in onerous circumstances. There is nothing else available that does the job so well.

JET WIMP

Department of Mathematics and Computer Science
Drexel University
Philadelphia, PA 19104

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25[65–06, 65D30, 65D32].—TERJE O. ESPELID & ALAN GENZ (Editors), *Numerical Integration: Recent Developments, Software and Applications*, NATO ASI Series, Series C: Mathematical and Physical Sciences, Vol. 357, Kluwer, Dordrecht, 1992, xii+367 pp., $24\frac{1}{2}$ cm. Price \$115.00/Dfl.195.00.

Although it is well over 2,000 years ago that Archimedes (287–212 B.C.) started the subject of numerical integration by finding approximations to π, the interest in the subject appears to be far from waning, and this NATO Advanced Research Workshop on Numerical Integration, held in Bergen, Norway over five days in June 1991, attracted 38 delegates from around the world. In all, 34 papers were presented, twenty five appear in full in this volume together with three extended abstracts and one note. The aim of the workshop was to survey recent progress and show how theoretical results have been used in software development and practical applications. This aim has been well achieved. The papers have been subdivided into four sections: "Numerical Integration Rules", "Numerical Integration Error Analysis", "Numerical Integration Applications", and finally "Numerical Integration Algorithms and Software". A complete list of authors and papers is given at the end of this review.

To the reviewer's delight, he found that this volume is dedicated to James Lyness "on the occasion of his 60th birthday". James has been contributing to the subject of numerical integration since his first paper, with John Blatt and David Mustard, was published nearly 30 years ago in the Computer Journal [3]. This Conference did not find James lacking and he describes, in good anecdotal style, his experience involving quadrature over a triangle or quadrilateral when the integrand has a known singularity at a vertex. Under an affine transformation of the region one can get disastrous results. James describes his experiences with this problem.

It is neither possible nor desirable for me to attempt to review, however briefly, every paper, so I shall make a highly personalized selection of papers for brief comment. Let me start with the papers by those authors who were also writing on numerical integration when James Lyness published his first