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Supplement to

FOURIER ANALYSIS OF MULTIGRID METHODS FOR GENERAL SYSTEMS OF PDES

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Appendix A

A lemma referred to in the proof of Theorem 2.1 and an auxiliary lemma are proved in this appendix. The first lemma is concerned with the smoothing iterations. Here, u denotes a smooth function, $u \in S$, and g with indices denotes a bounded function, $g \in B$.

LEMMA A.1. *Let the Assumptions* 2.1 *and* 2.2 *be fulfilled. Then for* $0 \le l \le L$,

$$S_l^r u = (I - r\Delta t_l Q_l)u + h_l \Delta t_l g_s,\ r \ge 1.$$

Proof. The proof is by induction. It follows from Assumptions 2.1i and 2.2i that

$$S_l u = (I - \Delta t_l P)u + h_l \Delta t_l g. \tag{A.1}$$

Our induction hypothesis is

$$S_l^r u = (I - r\Delta t_l Q_l)u + h_l \Delta t_l g_s,\quad r \ge 1 \tag{A.2}$$

Then by Assumption 2.1i, 2.2i, (A.1) and (A.2),

$$\begin{aligned} S_l^{r+1}u &= S_l(I - r\Delta t_l P)u + h_l \Delta t_l S_l g_s' \\ &= (I - \Delta t_l P)(I - r\Delta t_l P)u + h_l \Delta t_l(g' + S_l g_s). \end{aligned} \tag{A.3}$$

Since S_l is bounded, the last term in (A.3) can be written $h_l \Delta t_l g''$. The term $\Delta t_l^2 P^2 u$ is bounded. By Assumption 2.1i and (2.7),

$$S_l^{r+1}u = (I - (r+1)\Delta t_l Q_l)u + h_l \Delta t_l \tilde{g}_s.$$

Since (A.2) is valid for $r = 1$, see (A.1), the induction argument is complete.□

The following lemma is the main result of this appendix.

LEMMA A.2. *Let the Assumptions* 2.1-2.4 *be fulfilled. Then for* $0 \le l \le L$,

$$M_l u = (I - c_l \Delta t_l Q_l)u + h_l \Delta t_l g_l,\quad c_l = (p+q)\sum_{k=0}^{l} \alpha_k/\alpha_l, \tag{A.4}$$

$$M_l \textit{ is a bounded operator.} \tag{A.5}$$

Proof. The proof is by induction. The hypothesis is (A.4) and (A.5).
Rewrite (A.4) such that

$$M_l u = S_l^{p+q} u - S_l^q\, p_l (I - M_{l-1}) Q_{l-1}^{-1}\, r_l\, Q_l\, S_l^p\, u\,. \quad (A.6)$$

It follows from Lemma A.1, Assumptions 2.1i, 2.2ii, 2.3i and 2.3ii that

$$Q_{l-1}^{-1}\, r_l\, Q_l\, S_l^p\, u = Q_{l-1}^{-1}\, r_l\, Q_l\, S_l((I - (p-1)\Delta t_l\, P)u + h_l\, \Delta t_l\, g_s) = v + h_l\, \Delta t_l\, \tilde{g}_s\,, \quad (A.7)$$

where $v \in S$. Denote the last term of (A.6) by $R_l u$ and insert (A.7) and (A.4) with $l-1$ to obtain

$$\begin{aligned} R_l u &= S_l^q\, p_l (I - M_{l-1})(v + h_l\, \Delta t_l\, \tilde{g}_s) \\ &= c_{l-1}\, \Delta t_{l-1} S_l^q\, p_l (Q_{l-1} v - h_{l-1}\, g_{l-1}) + h_l\, \Delta t_l\, S_l^q\, p_l (I - M_{l-1}) \tilde{g}_s\,. \end{aligned} \quad (A.8)$$

By Assumptions 2.2ii and 2.4ii and (A.5) with $l-1$, $R_l u$ in (A.8) can be written

$$R_l u = c_{l-1}\, \Delta t_{l-1}\, S_l^q\, p_l\, Q_{l-1}\, v + h_l\, \Delta t_l\, g_R\,. \quad (A.9)$$

Insert v from (A.7) into (A.9) and use the boundedness of S_l, $\Delta t_{l-1}\, Q_{l-1}$ and p_l. Then

$$R_l u = c_{l-1}\, \Delta t_{l-1}\, S_l^q\, p_l\, r_l\, Q_l\, S_l^p\, u + h_l\, \Delta t_l\, \tilde{g}_R\,. \quad (A.10)$$

The first term in (A.10) is rewritten by simplifying

$$\begin{aligned} \Delta t_{l-1}\, S_l^q\, p_l\, r_l\, Q_l\, S_l^p\, u &= \Delta t_{l-1}\, S_l^q\, p_l\, r_l\, Q_l (I - p\, \Delta t_l\, P)u + h_l\, \Delta t_l\, g_1 \\ &= \Delta t_{l-1}\, S_l^q\, p_l\, r_l\, Pu + h_l\, \Delta t_l\, g_2 = \Delta t_{l-1}\, S_l^q\, Pu + h_l\, \Delta t_l\, g_3 \\ &= \Delta t_{l-1} (I - q\, \Delta t_l\, Q_l) Pu + h_l\, \Delta t_l\, g_4 = \Delta t_{l-1}\, Q_l\, u + h_l\, \Delta t_l\, g_5\,. \end{aligned} \quad (A.11)$$

When deriving the equalities in (A.11), the boundedness of the operators is used frequently without mentioning it explicitly. The first equality is obtained from Lemma A.1 and Assumption 2.1i. The second equality follows from the smoothness of Pu, Assumption 2.1i and (2.7). Assumption 2.4i gives the third equality. The fourth equality is derived from Lemma A.1. The last equality follows from Assumption 2.1i and (2.7).

Introduce (A.11) into (A.10), (A.10) into (A.6) and use Lemma A.1 to obtain

$$M_l u = (I - (p+q)\Delta t_l\, Q_l)u - c_{l-1}\, \Delta t_{l-1}\, Q_l u + h_l\, \Delta t_l\, g_l\,. \quad (A.12)$$

Let

$$c_l = (p+q) + c_{l-1}\, \alpha_{l-1}/\alpha_l \quad (A.13)$$

in (A.12), and we have (A.4). Since

$$M_0 u = S_0^{p+q}\, u = (I - (p+q)\Delta t_0\, Q_0)u + h_0\, \Delta t_0\, g_0\,, \quad (A.14)$$

(A.4) is valid for $l = 0$. The explicit expression for c_l is given by (A.13) and c_0 in (A.4) and (A.14),

$$c_l = (p+q) \sum_{k=0}^{l} \alpha_k/\alpha_l\,.$$

The first part of the lemma is proved.

The second part (A.5) is proved by (2.5) and Assumptions 2.2ii, 2.3ii, 2.4ii and (A.5) with $l-1$. Furthermore, M_0 in (A.14) is bounded by Assumption 2.2ii. The induction proof is complete. □

Appendix B

This appendix contains two lemmas and their proofs, and the proof of Theorem 3.1. The first lemma is:

LEMMA B.1. *Let the Assumptions* 3.1, 3.2, 3.3 *and* 3.5 *be satisfied. If* $\xi \in D_{00}$, $\eta_l = h_l(\xi_1,\xi_2)^T$ *and* $\|\eta_l\|$ *is sufficiently small, then for* l *such that* $1 \le l \le L$ *and the partitioning* (3.20),

$$\tilde{M}_{00} = I - (p+q) \sum_{j=0}^{l} \Delta t_j \hat{H}_j + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)),$$
$$\tilde{M}_{I0} = O(\Delta t_0 \|\eta_0\|). \tag{B.1}$$

If also Assumption 3.4 *is satisfied, then*

$$\tilde{M}_{0I} = O(\Delta t_0),$$
$$\tilde{M}_{II} \text{ is bounded.} \tag{B.2}$$

If

$$\xi = (\xi_1, 2\pi/h_0 - \xi'_2)^T \in D^0_{01}\ , \quad \eta_l = h_l(\xi_1, \xi'_2)^T,$$

or

$$\xi = (2\pi/h_0 - \xi'_1, \xi_2)^T \in D^0_{10}\ , \quad \eta_l = h_l(\xi'_1, \xi_2)^T,$$

or

$$\xi = (2\pi/h_0 - \xi'_1, 2\pi/h_0 - \xi'_2)^T \in D^0_{11}\ , \quad \eta_l = h_l(\xi'_1, \xi'_2)^T,$$

and $\|\eta_l\|$ *is sufficiently small, then* (B.1) *and* (B.2) *are valid also in* $D^0_{01} \cup D^0_{10} \cup D^0_{11}$.

Proof. The proof is based on an induction argument. Assume that the matrices at level l - 1 have certain properties. Then show that level l shares these properties. All matrices involved in the proof are partitioned in the same way as $\tilde{M}$. We begin by studying ξ in D_{00}, and then generalize to C_0^*.

The Fourier symbol matrix $\tilde{M}$ satisfies the recursion (see (3.16))

$$\tilde{M} = \tilde{S}^q(I - \tilde{P}\, \tilde{B}_{l-1} \tilde{R}\, \tilde{Q})\tilde{S}^p,$$
$$\tilde{B}_{l-1} = (I - \tilde{M}_{l-1})\tilde{Q}_{l-1}^{-1}\ , \quad l \ge 0\ ,$$
$$\tilde{M}_{-1} = I\ . \tag{B.3}$$

The proof of the first part of the lemma is divided into 6 steps in which parts of $\tilde{M}$ in (B.3) are investigated successively. The induction hypothesis is that $\tilde{M}_{l-1}$ satisfies claim (B.1) in the lemma.

1. $\tilde{W} = \tilde{R}\tilde{Q}$.

The definition of $\tilde{R}$ and $\tilde{Q}$ in § 3.1 gives

$$\tilde{W}_{00} = \hat{r}_{10}\, \hat{r}_{20}\, \hat{Q}_{00}\ , \tag{B.4}$$

and

$$\tilde{W}_{I0} = 0\ . \tag{B.5}$$

2. $\tilde{F} = \tilde{B}_{l-1}\, \tilde{W}$.

The partitioning of $\tilde{B}_{l-1}$ is also such that $(\hat{B}_{l-1})_{00} \in C^s \times C^s$. It follows from (B.4) and (B.5) that

$$\tilde{F}_{00} = (\tilde{B}_{l-1})_{00}\tilde{W}_{00} = (I - (\tilde{M}_{l-1})_{00})\tilde{C}_{00}\ ,$$
$$\tilde{F}_{I0} = (\tilde{B}_{l-1})_{I0}\, \tilde{W}_{00} = -(\tilde{M}_{l-1})_{I0}\, \tilde{C}_{00}\ , \tag{B.6}$$

where

$$\tilde{C}_{00} = (\tilde{Q}_{l-1})_{00}^{-1}\, \hat{r}_{10}\, \hat{r}_{20}\, \hat{Q}_{00}\ .$$

According to Assumptions 3.1 and 3.2,

$$\tilde{C}_{00} = (1 + O(\|\eta_l\|))(I + O(\|\eta_{l-1}\|)) = I + O(\|\eta_{l-1}\|)\ . \tag{B.7}$$

Thus, by (B.6), (B.7) and the induction hypothesis,

$$\tilde{F}_{00} = \gamma \sum_{j=0}^{l-1} \Delta t_j\, \hat{H}_j + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)), \quad \gamma = p + q\ ,$$
$$\tilde{F}_{I0} = -(\tilde{M}_{l-1})_{I0}(I + O(\|\eta_{l-1}\|)) = O(\Delta t_0 \|\eta_0\|). \tag{B.8}$$

3. $\tilde{G} = \tilde{P}\,\tilde{F}$.

The partitioning of $\tilde{P}$ is

$$\tilde{P} = \begin{bmatrix} \hat{p}_{10}\hat{p}_{20}I_s & \\ \hat{p}_{10}\hat{p}_{2,m/2}I_s & \\ \hat{p}_{1,m/2}\hat{p}_{20}I_s & 0 \\ \hat{p}_{1,m/2}\hat{p}_{2,m/2}I_s & \\ 0 & \tilde{P}_{II} \end{bmatrix}, \quad \tilde{P}_{II} \in C^{s(m^2-4)} \times C^{s(m^2/4-1)} .$$

The conclusion from (B.8) and Assumption 1 is

$$\tilde{G}_{00} = \hat{p}_{10}\,\hat{p}_{20}\,\tilde{F}_{00} = (1 + O(\|\eta_l\|)) \cdot (\gamma \sum_{j=0}^{l-1} \Delta t_j\, \hat{H}_j + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)) \tag{B.9}$$
$$= \gamma \sum_{j=0}^{l-1} \Delta t_j\, \hat{H}_j + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)) .$$

For the column below G_{00} we have

$$\tilde{G}_{10} = \begin{bmatrix} \hat{p}_{10}\hat{p}_{2,m/2}\tilde{F}_{00} \\ \hat{p}_{1,m/2}\hat{p}_{20}\tilde{F}_{00} \\ \hat{p}_{1,m/2}\hat{p}_{2,m/2}\tilde{F}_{00} \\ \tilde{P}_{II}\tilde{F}_{10} \end{bmatrix} = \begin{bmatrix} (1+O(\eta_{1l}))O(\Delta t_0\eta_{2l}) \\ (1+O(\eta_{2l}))O(\Delta t_0\eta_{1l}) \\ O(\eta_{1l}\eta_{2l}\Delta t_0) \\ O(\Delta t_0\|\eta_0\|) \end{bmatrix} = O(\Delta t_0\|\eta_0\|). \tag{B.10}$$

4. The smoothing iteration $\tilde{S}^p$.

Since $\tilde{S}$ is block diagonal, we infer

$$\tilde{S}^p_{00} = (\hat{S}_{00})^p .$$

Hence, by Assumption 3.3,

$$\tilde{S}^p_{00} = I - p\Delta t\, \hat{H} + O(\Delta t(\Delta t + \|\eta\|)) . \tag{B.11}$$

5. The final assembly.

The multigrid matrix can be written

$$\tilde{M} = \tilde{S}^q(I - \tilde{G})\tilde{S}^p .$$

Therefore, by (B.9) and (B.11),

$$\tilde{M}_{00} = \tilde{S}^q_{00}(I - \tilde{G}_{00})\tilde{S}^p_{00} = (I - \gamma\Delta t\, \hat{H} + O(\Delta t(\Delta t + \|\eta\|)) \cdot \tag{B.12}$$
$$(I - \gamma \sum_{j=0}^{l-1} \Delta t_j\, \hat{H}_j + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)) = I - \gamma \sum_{j=0}^{l} \Delta t_j\, \hat{H}_j + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)).$$

Since $\tilde{S}$ is bounded, we conclude from the estimate in (B.10) that

$$\tilde{M}_{I0} = -\tilde{S}^q_{II}\,\tilde{G}_{I0}\,\tilde{S}^p_{00} = O(\Delta t_0\|\eta_0\|) . \tag{B.13}$$

We assumed that $\tilde{M}_{l-1}$ has the properties in (B.1) of the lemma and have shown in steps 1–5 that $\tilde{M}_l$ also has these properties. The order of the wave numbers is preserved between level $l-1$ and l such that the upper left corner of the matrices always corresponds to the lowest wave number.

6. The case $l = 1$.

We must consider the matrices $\tilde{M}_0$, $\tilde{B}_0$ and $\tilde{G}_1$. For the multigrid matrix, the following holds:

$$\tilde{M}_0 = (\tilde{M}_0)_{00} = (\tilde{S}_0)^{\gamma}_{00} = I - \gamma\,\Delta t_0\, \hat{H}_0 + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)).$$

Thus, $\tilde{B}_0 \equiv (\tilde{B}_0)_{00}$ and $(\tilde{C}_1)_{00}$ has the property in (B.7) and $(\tilde{B}_0)_{I0} = \emptyset$. Equation (B.8) is valid for $(\tilde{F}_1)_{00}$ and $(\tilde{F}_1)_{I0} = \emptyset$. The relations (B.9) and (B10) are satisfied by $\tilde{G}_1$. The first part of the lemma is true for $l = 1$. The induction argument is complete and the first part of the lemma is proved for $\xi \in D_{00}$.

Assume that $(\tilde{M}_{l-1})_{0I}$ has the property in the second claim (B.2). At level l ,

$$\tilde{F} = (I - \tilde{M}_{l-1})\tilde{Q}^{-1}_{l-1}\,\tilde{R}\,\tilde{Q} . \tag{B.14}$$

Since $\tilde{Q}_k$, $k = l-1, l,$ is block diagonal, the matrix $\tilde{C}$ defined by

$$\tilde{C} = \tilde{Q}^{-1}_{l-1}\,\tilde{R}\,\tilde{Q}$$

has the same structure as $\tilde{R}$. A diagonal block $\tilde{C}_{jk}$ of $\tilde{C}$ is

$$\tilde{C}_{jk} = (\tilde{Q}_{l-1})^{-1}_{jk}(\hat{r}_{1j}\hat{r}_{2k}\hat{Q}_{jk}, g_2\hat{r}_{1j}\hat{r}_{2,k+m/2}\hat{Q}_{j,k+m/2},$$
$$g_1\hat{r}_{1,j+m/2}\hat{r}_{2k}\hat{Q}_{j+m/2,k}, g_1g_2\hat{r}_{1,j+m/2}\hat{r}_{2,k+m/2}\hat{Q}_{j+m/2,k+m/2})$$
$$= (f_{jk}, f_{j,k+m/2}, f_{j+m/2,k}, f_{j+m/2,k+m/2}). \tag{B.15}$$

By Assumption 3.4 and (B.15) it follows that $\tilde{C}$ is bounded when $\|\eta\| < \delta$.

For $\tilde{F}$ in (B.14) we have

$$\tilde{F}_{0I} = (I - (\tilde{M}_{l-1})_{00})\tilde{C}_{0I} - (\tilde{M}_{l-1})_{0I}\,\tilde{C}_{II}\,. \tag{B.16}$$

The boundedness of $\tilde{C}$, the induction hypothesis, (B.12) and (B.16) yield

$$\tilde{F}_{0I} = O(\Delta t_0)\,. \tag{B.17}$$

It follows from Assumption 3.1, (B.17) and the structure of $\tilde{P}$ that

$$\tilde{G}_{0I} = \hat{p}_{10}\,\hat{p}_{20}\,\tilde{F}_{0I} = O(\Delta t_0)\,. \tag{B.18}$$

Since the diagonal blocks of $\tilde{S}$ are bounded, we conclude from step 5 and (B.18) that

$$(\tilde{M}_l)_{0I} = O(\Delta t_0)\,. \tag{B.19}$$

The induction hypothesis is verified also at level l. Furthermore, $(\tilde{M}_l)_{II}$ is bounded, since all matrices involved are bounded.

At $l = 1$ we have $(\tilde{M}_0)_{0I} = \varnothing$ in (B.16). The relation (B.17) is still true, and therefore also (B.19). The induction proof of the second claim (B.2) in the lemma for $\xi \in D_{00}$ is complete. The properties of $\tilde{M}(\xi)$ when $\xi \in D^0_{01} \cup D^0_{10} \cup D^0_{11}$ are proved in the same manner, taking Assumption 3.5 and (3.19) into account. □

The second lemma is concerned with the eigenvalues and eigenvectors of $\tilde{M}$.

LEMMA B.2. *Let the Assumptions* 3.1 - 3.8 *be satisfied. The eigenvector matrices of* $\hat{H}$ *and* $\tilde{M}_{II}$ *are denoted by* T_0 *and* V_0, *respectively. Choose* δ *sufficiently small and let*

$$\delta t = (p + q)\sum^{l} \Delta t_i\,.$$

Then the eigenvalues of $M(\xi)$, $\xi \in C^*_0$, *are*

$$I - \delta t\,\lambda_i(\hat{H}) + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)),\quad i = 1,2,\ldots,s\,,$$
$$\mu_k + O(\Delta t_0\|\eta_0\|),\quad k = 1,2,\ldots,s(m^2-1)\,.$$

The eigenvector matrix of $\tilde{M}$ *is*

$$\begin{pmatrix} T & C\Delta t_0 \\ D\Delta t_0\|\eta_0\| & V \end{pmatrix},$$

where

$$T = T_0 + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)),$$
$$V = V_0 + O(\Delta t_0\|\eta_0\|),$$
$$C = O(1),\quad D = O(1).$$

<u>*Proof.*</u> It follows from Lemma B.1, the assumption on $\hat{H}$ and the theorem on eigenvalues and eigenvectors of perturbed matrices [13,7.7.1] that the eigenvalues of $\tilde{M}_{00}$ are simple and

$$\lambda_i(\tilde{M}_{00}) = 1 - \delta t\,\lambda_i(\hat{H}) + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)),$$

if δ is sufficiently small. Since the eigenvalues of $\tilde{M}_{00}$ and $\tilde{M}_{II}$ are separated when δ is sufficiently small, all eigenvalues of

$$\tilde{M}_* = \begin{pmatrix} \tilde{M}_{00} & \tilde{M}_{0I} \\ 0 & \tilde{M}_{II} \end{pmatrix}$$

are simple. The eigenvalues of $\tilde{M}_*$ are

$$\lambda_i(\tilde{M}_{00}),\quad i = 1,2,\ldots,s\,,$$
$$\mu_k(\tilde{M}_{II}),\quad k = 1,2,\ldots,s(m^2-1)\,.$$

The submatrix C' in the eigenvector matrix of $\tilde{M}_*$ satisfies

$$\begin{pmatrix} \tilde{M}_{00} & \tilde{M}_{0I} \\ 0 & \tilde{M}_{II} \end{pmatrix} \begin{pmatrix} T_1 & C' \\ 0 & V_0 \end{pmatrix} = \begin{pmatrix} T_1 & C' \\ 0 & V_0 \end{pmatrix} \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix},$$

$$\Lambda_1 = \mathrm{diag}(\lambda_i), \quad \Lambda_2 = \mathrm{diag}(\mu_k).$$

By the perturbation theorem [13,7.7.1] the eigenvectors of $\tilde{M}_{00}$ fulfill

$$T_1 = T_0 + O(\Delta t_0(\Delta t_0 + \|\eta_0\|)).$$

The equation satisfied by C' in (B.20) is

$$\tilde{M}_{00}\, C' - C'\, \Lambda_2 = -\tilde{M}_{0I}\, V_0. \tag{B.21}$$

Since $\tilde{M}_{00}$ and Λ_2 have no eigenvalues in common, (B.21) has a unique solution C' [13,8.5.1]. The scaling of V_0 is such that $V_0 = O(1)$. We conclude from Lemma B.1 and (3.23) that

$$C' = O(\Delta t_0).$$

Since

$$\tilde{M} = \tilde{M}_* + \begin{pmatrix} 0 & 0 \\ \tilde{M}_{I0} & 0 \end{pmatrix},$$

the statements on the eigenvalues and eigenvectors of $\tilde{M}$ follow from Lemma 4.1 and [13, 7.7.1].□

Proof of Theorem 3.1. The matrices $\hat{H}$ and $\tilde{M}_{II}$ are diagonalizable for δ sufficiently small. Thus, T_0^{-1} and V_0^{-1} exist and are bounded independent of Δt_0 and η_0. Let

$$\varepsilon_1 = \Delta t_0, \quad \varepsilon_2 = \Delta t_0 \|\eta_0\|, \quad \varepsilon_3 = \Delta t_0 + \|\eta_0\|.$$

It follows from Lemma B.2 that

$$T^{-1} = T_0^{-1} + O(\varepsilon_1\varepsilon_3), \qquad V^{-1} = V_0^{-1} + O(\varepsilon_2). \tag{B.22}$$

The inverse of the eigenvector matrix of $\tilde{M}$,

$$\begin{pmatrix} T & \varepsilon_1 C \\ \varepsilon_2 D & V \end{pmatrix},$$

is

$$\begin{pmatrix} T^{-1} + \varepsilon_1\varepsilon_2 T' & \varepsilon_1 C' \\ \varepsilon_2 D' & V^{-1} + \varepsilon_1\varepsilon_2 V' \end{pmatrix}. \tag{B.23}$$

The matrices with primes in (B.23) are of $O(1)$. Taking (B.22) into account, we can rewrite (B.23) as

$$\begin{pmatrix} T_0^{-1} + \varepsilon_1\varepsilon_3 T'' & \varepsilon_1 C' \\ \varepsilon_2 D' & V_0^{-1} + \varepsilon_2 V'' \end{pmatrix},$$

where T'' and V'' are of $O(1)$. Then

$$\tilde{M}^n = \begin{pmatrix} T & \varepsilon_1 C \\ \varepsilon_2 D & V \end{pmatrix} \begin{pmatrix} \Lambda_1^n & 0 \\ 0 & \Lambda_2^n \end{pmatrix} \begin{pmatrix} T_0^{-1} + \varepsilon_1\varepsilon_3 T'' & \varepsilon_1 C' \\ \varepsilon_2 D' & V_0^{-1} + \varepsilon_2 V'' \end{pmatrix}$$
$$= \begin{pmatrix} T_0 \Lambda_1^n T_0^{-1} + O(\varepsilon_1\varepsilon_3) & O(\varepsilon_1) \\ O(\varepsilon_2) & V_0 \Lambda_2^n V_0^{-1} + O(\varepsilon_2) \end{pmatrix}. \tag{B.24}$$

The eigenvalue matrices Λ_β, $\beta = 1,2$, in (B.24) are defined by

$$\Lambda_{1i} = 1 - \delta t\, \lambda_i(\hat{H}) + O(\varepsilon_1\varepsilon_3), \quad i = 1,2,\ldots,s,$$
$$\Lambda_{2k} = \mu_k(\tilde{M}_{II}) + O(\varepsilon_2), \quad k = 1,2,\ldots,s(m^2-1).$$

Hence, the diagonal entries of Λ_1^n are

$$\Lambda_{1i}^n = (1 - \delta t\, \lambda_i(\hat{H}))^n + O(\varepsilon_3 n \Delta t_0), \tag{B.25}$$

since $\hat{H}$ is bounded by Assumption 3.3. Premultiply Λ_1^n by T_0 and postmultiply by T_0^{-1} to obtain the upper left corner of $\tilde{M}^n$ in the theorem.

Then define

$$\psi_k = 1 - \delta t\, \lambda_k(\hat{H}).$$

Since

$$\log(1 - \delta t\, \lambda_k) = -\delta t \cdot \lambda_k + O(\delta t^2),$$

we obtain

$$\psi_k^n = \exp n(\log(1 - \delta t\, \lambda_k(\hat{H}))) = |\psi_k|^n \exp(-in\, \delta t\, \mathrm{Im}\, \lambda_k(\hat{H})) + O(\delta t).$$

For the upper left corner in (B.24) we have

$$\Lambda_{1k}^n = (\psi_k + O(\varepsilon_1 \varepsilon_3))^n = \psi_k^n + O(\varepsilon_3)\ .$$

The property of $\tilde{M}_{II}^n$ follows directly from Lemma B.2. The theorem is proved.□