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Supplement to
ON THE DISTRIBUTION OF k-DIMENSIONAL VECTORS FOR SIMPLE AND COMBINED TAUSWORTHE SEQUENCES

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6. Examples

In this section, a LS2 (or Tausworthe) generator g with multiplier a and modulus m will be denoted by $g = (a, m)$, and our use of Theorem 2 will always be with $h = k$ and $r = -l$.

6.1. A Combination of Two or Three Toy Generators. As a first illustration, we examine in detail the (low-dimensional) behavior of the three simple "toy" generators $g_1 = (x,\ x^3+x+1)$, $g_2 = (x^2,\ x^4+x+1)$, and $g_3 = (x^3,\ x^5+x^2+1)$, as well as the combination g_{23} of the last two, and the combination g_{123} of all three. Since $\gcd(2^5-1,\ 2^4-1) = \gcd(31, 15) = 1$, the period of g_{23} is $31 \times 15 = 465$. Similarly, since $\gcd(2^3 - 1,\ 465) = 1$, the period of g_{123} is $465 \times 7 = 3255$.

TABLE 7
Dimensions associated with g_1, g_2, g_3, and their combinations, for $k = 2$.

l	d	d_{12}	d_{23}	d_{13}	d_1	d_2	d_3	D
1	10	5	7	6	1	2	3	2
2	8	3	5	4	0	0	1	3
3	6	2	3	2	0	0	0	2
4	4	1	2	1	0	0	0	1
5	2	0	1	0	0	0	0	0
6	0	0	0	0	0	0	0	0

TABLE 8
Values of $\varphi_l(n)$ for g_1, g_2, and g_3, for $k = 2$.

g_1		
l	n	$\varphi_l(n)$
1	2	3
	1	1
	0	0
2	1	7
	0	9

g_2		
l	n	$\varphi_l(n)$
1	4	3
	3	1
	0	0
2	1	15
	0	1

g_3		
l	n	$\varphi_l(n)$
1	8	3
	7	1
	0	0
2	2	15
	1	1
	0	0
3	1	31
	0	33

TABLE 9
Values of $\varphi_l(n)$ for g_{23} and g_{123}, for $k = 2$.

g_{23}		
l	n	$\varphi_l(n)$
1	117	1
	116	3
	0	0
2	30	1
	29	15
	0	0
3	8	24
	7	33
	6	7
	0	0
4	4	84
	3	41
	2	3
	0	128
5	2	210
	1	45
	0	769

g_{123}		
l	n	$\varphi_l(n)$
1	814	3
	813	1
	0	0
2	204	7
	203	9
	0	0
3	53	16
	52	16
	51	3
	50	5
	49	20
	48	4
	0	0
4	16	48
	14	64
	13	4
	12	60
	11	33
	10	35
	9	10
	8	2
	0	0
5	4	504
	3	246
	2	228
	1	45
	0	1
6	1	3255
	0	841

Table 7 gives all the kernel dimensions referred to in Tables 1–6, for $k = 2$. These have been computed using Theorem 2 and Lemma 3. From these values, we have computed the $\varphi_l(n)$'s for different values of l, for the generators g_1, g_2, g_3, g_{23}, and g_{123}. These are given in Tables 8 and 9. One can see that g_1, g_2, and g_3 reach the best possible resolution considering their period length, for all l. But their periods are very small. For the combination g_{23}, one has $d = p - lk$ (and maximum resolution) only for $l \leq 3$. For $l = 4$ and $l = 5$, there are empty cells and also cells that contain more than one point. Note that for $l = 1$ and 2, the number of cells with 2^d points turns out to be zero. That kind of situation happens quite frequently. For g_{123}, $d = p - lk$ holds for l up to 6. After that, any cell will contain either 0 or 1 point. Observe that the values of d, d_i, or d_{ij} never increase when l increases. But this does not necessarily hold for D.

TABLE 10
Dimensions d, d_2, d_3, and values of $\varphi_l(n)$ for g_{23}, for $k = 3$.

l	d	d_2	d_3
1	6	1	2
2	3	0	0
3	0	0	0

g_{23}		
l	n	$\varphi_l(n)$
1	59	1
	58	7
	0	0
2	8	24
	7	33
	6	7
	0	0
3	1	465
	0	47

Table 10 gives the values that correspond to g_{23} in dimension $k = 3$. One has $d = p - lk$ for $l \leq 3$. Despite that, for $l = 3$, there are 47 empty cells, owing to the fact that in this case, $n = 0$ for lines 2, 3, and 5 in Table 4.

Figure 1 shows all the points produced by the generator g_{23}, in dimension $k = 2$. This illustrates the results of the left-hand part of Table 9. For example, the grid on the figure partitions the square into $2^6 = 64$ cells, which corresponds to $l = 3$. As indicated by Table 9, 24 cells contain 8 points, 33 cells contain 7 points, and 7 cells contain 6 points. If the grid were refined to partition the square into $2^8 = 256$ cells (i.e., $l = 4$), then, as indicated by Table 9, there would be 128 empty cells while the other cells would contain either 2, 3, or 4 points.

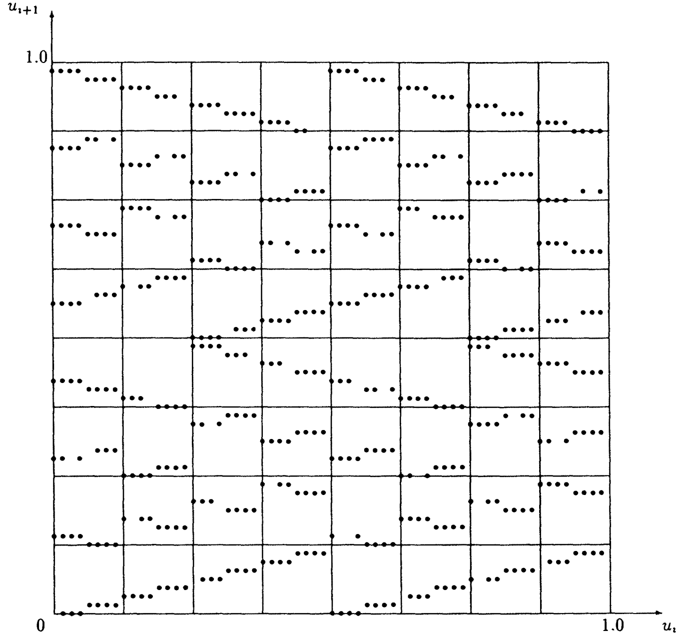

FIGURE 1
The pairs (u_i, u_{i+1}), produced by the combined Tausworthe generator g_{23}.

6.2. A Simple Generator From André, Mullen, and Niederreiter. We now examine a simple Tausworthe generator based on a polynomial of degree $p = 32$, which has been obtained by André et al. [1] and was called "universally optimal". This generator is $g_A = (x^{32},\ x^{32} + x^{31} + x^{30} + x^{28} + x^{27} + x^{26} + x^{24} + x^{22} + x^{21} + x^{12} + x^{11} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$. Table 11 gives the values of d for all dimensions $k \le 10$, and all l. For $l > 16$ and for the entries marked "-", one has $d = 0$. There are only three nonzero entries for which $d > p - lk$, i.e., for which one does not have k-distribution with resolution l, and these are the three entries with $d = 1$. Table 12 shows what happens with the values of Table 1 for each of these three cases: there are 2^{31} empty cells and $2^{31} - 1$ cells that contain two points each.

TABLE 11
The values of d in dimensions $k = 2$ to 10, for generator g_A.

	k								
l	2	3	4	5	6	7	8	9	10
1	30	29	28	27	26	25	24	23	22
2	28	26	24	22	20	18	16	14	12
3	26	23	20	17	14	11	8	5	2
4	24	20	16	12	8	4	1	-	-
5	22	17	12	7	2	-	-	-	-
6	20	14	8	2	-	-	-	-	-
7	18	11	4	-	-	-	-	-	-
8	16	8	-	-	-	-	-	-	-
9	14	5	-	-	-	-	-	-	-
10	12	2	-	-	-	-	-	-	-
11	10	1	-	-	-	-	-	-	-
12	8	-	-	-	-	-	-	-	-
13	6	-	-	-	-	-	-	-	-
14	4	-	-	-	-	-	-	-	-
15	2	-	-	-	-	-	-	-	-
16	1	-	-	-	-	-	-	-	-

TABLE 12
Values of $\varphi_l(n)$ for g_A in dimensions $k = 2$, 3, and 8.

$k = 2$		
l	n	$\varphi_l(n)$
14	16	$2^{28} - 1$
	15	1
15	4	$2^{30} - 1$
	3	1
16	2	$2^{31} - 1$
	1	1
	0	2^{31}

$k = 3$		
l	n	$\varphi_l(n)$
9	32	$2^{27} - 1$
	31	1
10	4	$2^{30} - 1$
	3	1
11	2	$2^{31} - 1$
	1	1
	0	2^{31}

$k = 8$		
l	n	$\varphi_l(n)$
2	2^{16}	$2^{16} - 1$
	$2^{16} - 1$	1
3	2^8	$2^{24} - 1$
	$2^8 - 1$	1
4	2	$2^{31} - 1$
	1	1
	0	2^{31}

6.3. A Simple Generator From Mullen and Niederreiter. Here, we look at another so-called "optimal polynomial", suggested in Mullen and Niederreiter [9]. This one has degree 64 and yields the generator $g_M = (x^{64},\ x^{64} + x^{63} + x^{60} + x^{59} + x^{58} + x^{54} + x^{49} + x^{32} + 1)$. This polynomial was designed to be optimal only relative to the 2-dimensional distribution. So, it is not necessarily expected to behave well in higher dimensions. Table 13 gives the values of d for all dimensions $k \le 5$, and all $l \ge 11$. For $k = 3$, one has $d > p - lk$ for all $l \ge 17$. Also, $d > p - lk$ for $k = 5$ and $l = 13$. Table 14 shows what happens with $\varphi_l(n)$ in dimension $k = 3$. For example, with $l = 21$, one has approximately 2^{63} empty cells and 2^{53} cells that contain 2^{11} points each. This is not very good.

TABLE 13
The values of d in dimensions $k = 2$ to 5, for generator g_M.

	k			
l	2	3	4	5
11	42	31	20	9
12	40	28	16	4
13	38	25	12	1
14	36	22	8	-
15	34	19	4	-
16	32	16	-	-
17	30	15	-	-
18	28	14	-	-
19	26	13	-	-
20	24	12	-	-
21	22	11	-	-
22	20	10	-	-
23	18	9	-	-
24	16	8	-	-
25	14	7	-	-
26	12	6	-	-
27	10	5	-	-
28	8	4	-	-
29	6	3	-	-
30	4	2	-	-
31	2	1	-	-
32	-	-	-	-

TABLE 14
Values of $\varphi_l(n)$ for g_M in dimension $k = 3$.

$k = 3$		
l	n	$\varphi_l(n)$
16	2^{16}	$2^{48} - 1$
	$2^{16} - 1$	1
17	2^{15}	$2^{49} - 1$
	$2^{15} - 1$	1
	0	$2^{51} - 2^{49}$
18	2^{14}	$2^{50} - 1$
	$2^{14} - 1$	1
	0	$2^{54} - 2^{50}$
19	2^{13}	$2^{51} - 1$
	$2^{13} - 1$	1
	0	$2^{57} - 2^{51}$
20	2^{12}	$2^{52} - 1$
	$2^{12} - 1$	1
	0	$2^{60} - 2^{52}$
21	2^{11}	$2^{53} - 1$
	$2^{11} - 1$	1
	0	$2^{63} - 2^{53}$

6.4. A Combined Tausworthe Generator Taken From SUPER-DUPER. Our last example is a combined Tausworthe generator, which is itself a component of the generator Super-Duper proposed by Marsaglia et al. [8]. This generator is given by $g = (x^{32},\ x^{32} + x^{15} + 1)$. Note that $M(x) = x^{32} + x^{15} + 1$ is not irreducible and can be written as $M(x) = (x^{21} + x^{19} + x^{15} + x^{13} + x^{12} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^4 + x^2 + 1)(x^{11} + x^9 + x^7 + x^2 + 1)$. So, this generator can be regarded as a combined Tausworthe generator. The maximum possible period is $(2^{21} - 1)(2^{11} - 1)$ and thereby almost all initial values give the maximum period. Table 15 gives the values of d, d_1, and d_2 for all l, in dimensions 2 to 4. From that, one can use Table 4 to compute $\varphi_l(n)$. The results are given in Table 16. Here, the values for which maximum resolution ($d = p - lk$) is not attained are $l = 16$ for $k = 2$, and all $l \ge 3$ for $k = 3$ and 4. Therefore, bad behavior is to be expected in dimensions 3 and 4. Table 16 confirms that. For example, in dimension 3 and with $l = 6$, there are 245760 empty cells, 2047 cells that contain 262015 points each, and 14337 cells that contain 262016 points each.

TABLE 15
Values of d, d_1, and d_2 for the component of Super-Duper.

	$k = 2$		
l	d	d_1	d_2
1	30	19	9
2	28	17	7
3	26	15	5
4	24	13	3
5	22	11	1
6	20	9	0
7	18	7	0
8	16	5	0
9	14	3	0
10	12	1	0
11	10	0	0
12	8	0	0
13	6	0	0
14	4	0	0
15	2	0	0
16	1	0	0

	$k = 3$		
l	d	d_1	d_2
1	29	18	8
2	26	15	5
3	24	13	3
4	22	11	1
5	20	9	0
6	18	7	0
7	16	5	0
8	14	3	0
9	12	1	0
10	10	0	0

	$k = 4$		
l	d	d_1	d_2
1	28	17	7
2	24	13	3
3	22	11	1
4	20	9	0
5	18	7	0
6	16	5	0
7	14	3	0
8	12	1	0
9	10	0	0
10	8	0	0

TABLE 16

Values of $\varphi_l(n)$ for the component of Super-Duper.

$k = 2$		
l	n	$\varphi_l(n)$
14	16	$2^{28} - 2^{21} - 2^{11} + 16$
	15	$2^{21} + 2^{11} - 2^5 + 1$
	14	15
15	4	$2^{30} - 2^{21} - 2^{11} + 4$
	3	$2^{21} + 2^{11} - 7$
	2	3
16	2	$2^{31} - 2^{21} - 2^{11} + 2$
	1	$2^{21} + 2^{11} - 3$
	0	$2^{31} + 1$

$k = 3$		
l	n	$\varphi_l(n)$
2	$2^{26} - 2^{15} - 2^5$	$2^6 - 1$
	$2^{26} - 2^{15} - 2^5 + 1$	1
3	$2^{24} - 2^{13} - 2^3$	$2^8 - 1$
	$2^{24} - 2^{13} - 2^3 + 1$	1
	0	2^8
4	$2^{22} - 2^{11} - 2$	$2^{10} - 1$
	$2^{22} - 2^{11} - 1$	1
	0	$2^{12} - 2^{10}$
5	$2^{20} - 2^9$	$2^{12} - 2^{11} + 1$
	$2^{20} - 2^9 - 1$	$2^{11} - 1$
	0	$2^{15} - 2^{12}$
6	$2^{18} - 2^7$	$2^{14} - 2^{11} + 1$
	$2^{18} - 2^7 - 1$	$2^{11} - 1$
	0	$2^{18} - 2^{14}$

$k = 4$		
l	n	$\varphi_l(n)$
2	$2^{24} - 2^{13} - 2^3$	$2^8 - 1$
	$2^{24} - 2^{13} - 2^3 + 1$	1
3	$2^{22} - 2^{11} - 2$	$2^{10} - 1$
	$2^{22} - 2^{11} - 1$	1
	0	$2^{12} - 2^{10}$
4	$2^{20} - 2^9$	$2^{12} - 2^{11} + 1$
	$2^{20} - 2^9 - 1$	$2^{11} - 1$
	0	$2^{16} - 2^{12}$
5	$2^{18} - 2^7$	$2^{14} - 2^{11} + 1$
	$2^{18} - 2^7 - 1$	$2^{11} - 1$
	0	$2^{20} - 2^{14}$
6	$2^{16} - 2^5$	$2^{16} - 2^{11} + 1$
	$2^{16} - 2^5 - 1$	$2^{11} - 1$
	0	$2^{24} - 2^{16}$

APPENDIX

In this appendix, we derive a technical result that was used in the proof of Lemma 3 in §5.3. Let $V = V_1 + V_2 + V_3$ be a direct sum of vector spaces and $W \subset V$ a subspace. For each $i \neq j$, let $W_i = W \cap V_i$, $W_{ij} = W \cap (V_i + V_j)$, $d = \dim(W)$, $d_i = \dim(W_i)$, and $d_{ij} = \dim(W_{ij})$. Let

$$\tilde{W} = W_{12} + W_{13} + W_{23} \tag{22}$$

and $\tilde{d} = \dim(\tilde{W})$. Let $D = \dim(((V_1 + W) \cap (V_2 + W) \cap (V_3 + W))/W)$. Our aim is now to express D as a function of quantities defined above.

Let $W_0 = W_1 + W_2 + W_3$, and for each subspace E of V, let $\tilde{E}$ denote the image of E by the canonical mapping $V \to V/W_0$. We will perform a reduction of everything modulo W_0. The development will then be easier in space $\tilde{V}$, owing to the fact that $\tilde{W} \cap \tilde{V}_i = \tilde{W}_i = \{0\}$ for each i. For each i and $j \neq i$, one has the following:

$$\begin{aligned} \tilde{V} &= \tilde{V}_1 + \tilde{V}_2 + \tilde{V}_3 \quad \text{(direct sum)}, \\ \tilde{W}_i &= \tilde{W} \cap \tilde{V}_i = \{0\}, \\ \tilde{W}_{ij} &= \tilde{W} \cap (\tilde{V}_i + \tilde{V}_j), \\ \tilde{d}_{ij} &\overset{\text{def}}{=} \dim(\tilde{W}_{ij}) = \dim(\tilde{W} \cap (\tilde{V}_i + \tilde{V}_j)) = \dim(W_{ij}) - \dim(W_{ij} \cap W_0) \\ &= \dim(W_{ij}) - \dim(W_i + W_j) = d_{ij} - d_i - d_j, \\ \tilde{d} &\overset{\text{def}}{=} \dim(\tilde{W}) = \dim(W) - \dim(W_0) = d - d_1 - d_2 - d_3. \end{aligned} \tag{23}$$

One has $\tilde{\tilde{W}} \overset{\text{def}}{=} \sum_{i \neq j} \tilde{W}_{ij} = \tilde{\tilde{W}}$ and

$$\tilde{\tilde{d}} \overset{\text{def}}{=} \dim(\tilde{\tilde{W}}) = \tilde{d} - d_1 - d_2 - d_3. \tag{24}$$

Finally,

$$\dim((V_1+W)\cap(V_2+W)\cap(V_3+W)) = \dim((\tilde{V}_1+\tilde{W})\cap(\tilde{V}_2+\tilde{W})\cap(\tilde{V}_3+\tilde{W}))+d_1+d_2+d_3. \tag{25}$$

Let $H_1 = \tilde{V}_1 \cap (\tilde{V}_2 + \tilde{W}_{12}) \cap (\tilde{V}_3 + \tilde{W}_{13})$.

Lemma 4. *One has*

$$(\tilde{V}_1 + \tilde{W}) \cap (\tilde{V}_2 + \tilde{W}) \cap (\tilde{V}_3 + \tilde{W}) = H_1 + \tilde{W} \tag{26}$$

and $D = \dim(H_1)$.

Proof. Let $v_1 + w_1 = v_2 + w_2 = v_3 + w_3$ be a common element to the three spaces that intersect on the left in equation (26). One then has $v_1 = v_2 + (w_2 - w_1) = v_3 + (w_3 - w_1)$, where $w_2 - w_1 \in \tilde{W}_{12}$ and $w_3 - w_1 \in \tilde{W}_{13}$. Therefore, v_1 belongs to H_1 and the set on the left is a subset of the one on the right. The inclusion in the other direction is immediate.

Since $H_1 \subseteq \tilde{V}_1$, we have that $H_1 \cap \tilde{W}$ is a subset of $\tilde{W}_1$ and is therefore $\{0\}$. This means that the sum $H_1 + \tilde{W}$ is direct. Then, from (26) and (25), one has

$$\begin{aligned} D &= \dim((\tilde{V}_1 + \tilde{W}) \cap (\tilde{V}_2 + \tilde{W}) \cap (\tilde{V}_3 + \tilde{W})) + d_1 + d_2 + d_3 - d \\ &= \dim(H_1) + \dim(\tilde{W}) + d_1 + d_2 + d_3 - d \\ &= \dim(H_1). \ \square \end{aligned}$$

Let π_i denote the canonical projection $\tilde{V} \mapsto \tilde{V}_i$. Since $\tilde{W}_i = \{0\}$, for each $v_1 \in H_1$ there are unique elements $w_{12} \in \tilde{W}_{12}$ and $w_{13} \in \tilde{W}_{13}$ such that $\pi_1(w_{12}) = \pi_1(w_{13}) = v_1$. We can then define a linear mapping $\mu : H_1 \mapsto \tilde{W}_{23}$ by $\mu(v_1) = \pi_2(w_{12}) - \pi_3(w_{13})$ ($= w_{12} - w_{13}$, since $\pi_1(w_{12} - w_{13}) = 0$).

Lemma 5. *The mapping* μ *is one-to-one and* $\mu(H_1) = (\tilde{W}_{12} + \tilde{W}_{13}) \cap \tilde{W}_{23}$.

Proof. If $\mu(v_1) = 0$, then $w_{12} = w_{13} \in (\tilde{V}_1 + \tilde{V}_2) \cap (\tilde{V}_1 + \tilde{V}_3) = \tilde{V}_1$, so that $w_{12} = w_{13} = 0$ and $v_1 = \pi_1(w_{12}) = 0$. This implies that μ is one-to-one. By construction, we have $\mu(H_1) \subset (\tilde{W}_{12} + \tilde{W}_{13}) \cap \tilde{W}_{23}$, and it remains to show the reverse inclusion. Let $w_{23} = w_{12} - w_{13} \in (\tilde{W}_{12} + \tilde{W}_{13}) \cap \tilde{W}_{23}$ and $v = \pi_1(w_{12})$. Since $\pi_1(w_{23}) = 0$, we have $\pi_1(w_{13}) = \pi_1(w_{12}) = v$ and $v \in H_1$. Then, from the definition of μ, $\mu(v) = w_{23}$, and this completes the proof. $\square$

Lemma 6. *We have*

$$\begin{aligned} D = \dim(H_1) &= \tilde{d}_{12} + \tilde{d}_{13} + \tilde{d}_{23} - \dim(\tilde{\tilde{W}}) \\ &= d_{12} + d_{13} + d_{23} - d_1 - d_2 - d_3 - \bar{d}. \end{aligned} \tag{27}$$

Proof. Keeping in mind that the sum $\tilde{W}_{12} + \tilde{W}_{13}$ is direct because each $\tilde{W}_i$ is $\{0\}$, and using the previous lemma, one has

$$\begin{aligned} \dim(\tilde{\tilde{W}}) &= \dim(\tilde{W}_{12} + \tilde{W}_{13} + \tilde{W}_{23}) \\ &= \dim(\tilde{W}_{23}) + \dim(\tilde{W}_{12} + \tilde{W}_{13}) - \dim((\tilde{W}_{12} + \tilde{W}_{13}) \cap \tilde{W}_{23}) \\ &= \tilde{d}_{23} + \tilde{d}_{12} + \tilde{d}_{13} - \dim(H_1). \end{aligned}$$

This gives the middle equality. The first equality is already contained in Lemma 4, while the last one follows from (23) and (24). $\square$

The following example shows that knowing d, the d_i's, and d_{ij}'s is not sufficient in general to compute D.

Example. For $i = 1, 2, 3$, let $\dim(V_i) = 2$ and let $\{v_i, v_i'\}$ be a basis for V_i. We consider two cases. In the first case, suppose that $W = \mathbb{F}_2 \cdot (v_1 + v_2') + \mathbb{F}_2 \cdot (v_2 + v_3') + \mathbb{F}_2 \cdot (v_3 + v_1')$, where $\mathbb{F}_2 \cdot v$ means the space $\{0, v\}$. Then, $W_i = W \cap V_i = \{0\}$ for each i and $H_1 = V_1 \cap (V_2 + W_{12}) \cap (V_3 + W_{13}) = V_1 \cap (V_2 + \mathbb{F}_2 \cdot (v_1 + v_2')) \cap (V_3 + \mathbb{F}_2(v_3 + v_1')) = V_1 \cap (V_2 + \mathbb{F}_2 \cdot v_1) \cap (V_3 + \mathbb{F}_2 \cdot v_1') = (\mathbb{F}_2 \cdot v_1) \cap (\mathbb{F}_2 \cdot v_1') = \{0\}$, so that $D = \dim(H_1) = 0$. In the second case, suppose that $W = \mathbb{F}_2 \cdot (v_1 + v_2) + \mathbb{F}_2 \cdot (v_2 + v_3) + \mathbb{F}_2 \cdot (v_1' + v_2' + v_3')$. Then, $W \cap V_i = \{0\}$ for each i and $H_1 = V_1 \cap (V_2 + W_{12}) \cap (V_3 + W_{13}) = V_1 \cap (V_2 + \mathbb{F}_2 \cdot (v_1 + v_2)) \cap (V_3 + \mathbb{F}_2(v_1 + v_3)) = V_1 \cap (V_2 + \mathbb{F}_2 \cdot v_1) \cap (V_3 + \mathbb{F}_2 \cdot v_1) = \mathbb{F}_2 v_1$, so that $D = \dim(H_1) = 1$. In both cases, $d = 3$, $d_{ij} = 1$, and $d_i = 0$, but the two cases have different values of $\bar{d}$, namely $\bar{d} = 3$ in the first case and $\bar{d} = 2$ in the second. So, by Lemma 6, they have different values of D.