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ZAREMBA'S CONJECTURE AND SUMS OF THE DIVISOR FUNCTION

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Dedicated to the memory of D. H. Lehmer

ABSTRACT. Zaremba conjectured that given any integer $m > 1$, there exists an integer $a < m$ with a relatively prime to m such that the simple continued fraction $[0, c_1, \dots, c_r]$ for a/m has $c_i \le B$ for $i = 1, 2, \dots, r$, where B is a small absolute constant (say $B = 5$). Zaremba was only able to prove an estimate of the form $c_i \le C \log m$ for an absolute constant C. His first proof only applied to the case where m is a prime; later he gave a very much more complicated proof for the case of composite m. Building upon some earlier work which implies Zaremba's estimate in the case of prime m, the present paper gives a much simpler proof of the corresponding estimate for composite m.

1. INTRODUCTION

Apparently, Zaremba [5, pp. 69 and 76] was the first to state the following:

Conjecture. *Given any integer $m > 1$, there is a constant B such that for some integer $a < m$ with a relatively prime to m the simple continued fraction $[0, c_1, \dots, c_r]$ for a/m has $c_i \le B$ for $i = 1, 2, \dots, r$.*

This conjecture is still unproved, though numerical evidence suggests that $B = 5$ would suffice. The best result known replaces the inequality in the conjecture by $c_i \le C \log m$ for some constant C; this was first proved by Zaremba [5, Theorem 4.6 with $s = 2$, p. 74] for prime values of m. Later, Zaremba [6] gave a very much more complicated proof for composite values of m.

As a byproduct of a more general investigation, I proved in an earlier paper [1, p. 154] that the inequality in the conjecture can be replaced by $c_i \le 4(m/\varphi(m))^2 \log m$, where $\varphi(m)$ is Euler's function. Of course, this implies $c_i \le C \log m$ if m is prime, but only gives $c_i \le C \log m (\log\log m)^2$ in general. In the present paper, I show how the argument of [1] can be refined to eliminate the $\log\log$ factors. The result is

Theorem 1. *Given any integer $m > 1$, there is an integer $a < m$ with a relatively prime to m such that the simple continued fraction $[0, c_1, \dots, c_r]$ for a/m has $c_i \le 3 \log m$ for $i = 1, 2, \dots, r$.*

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The proof is much simpler than the proof of the corresponding result in Zaremba [6]. I am grateful to Harald Niederreiter for suggesting that it would be worthwhile to publish this simpler proof.

2. Proof of Theorem 1

Let $\|x\|$ denote the distance from x to the nearest integer. We shall actually prove the following sharpening of the case $n = 2$ of the theorem in [1].

Theorem 2. *Given any integer* $m \geq 8$, *there exist integers* a_1, a_2 *relatively prime to* m *such that*

$$\prod_{i=1}^{2} \|ka_i/m\| > (3m \log m)^{-1} \quad \textit{for each } k, \ 1 \leq k < m.$$

As in [1], it is easy to deduce Theorem 1 from Theorem 2: We may assume $a_1 = 1$ and $a_2 = a$ in Theorem 2, since we may replace a_i by ba_i $(i = 1, 2)$, where $ba_1 \equiv 1 \bmod m$. Thus, Theorem 2 implies that for any $m \geq 8$ there exists an integer $a < m$ with a relatively prime to m such that

$$k\|ka/m\| > (3 \log m)^{-1} \quad \text{for each } k, \ 1 \leq k < m. \tag{1}$$

If $[0, c_1, \dots, c_r]$ is the simple continued fraction for a/m with convergents p_i/q_i $(0 \leq i \leq r)$, then we have $q_i\|q_i a/m\| < 1/c_{i+1}$ for $i = 0, 1, \dots, r-1$. Therefore, (1) implies Theorem 1. (For $m < 8$ it is easy to verify Theorem 1 by calculation.)

We begin the proof of Theorem 2 with some definitions taken from [1, p. 155]. Given any integer $m > 1$ and positive integers a_1, a_2, we let L denote a positive real number which we shall specify later. We say that the pair a_1, a_2 is *exceptional* (with respect to m and L) if

$$\prod_{i=1}^{2} \|ka_i/m\| > L^{-1} \quad \text{for each } k, \ 1 \leq k < m. \tag{2}$$

Obviously, the pair a_1, a_2 can be exceptional only if each a_i is relatively prime to m. If for some k, $1 \leq k < m$, the inequality in (2) is false, then we say that k *excludes* the pair a_1, a_2. We shall estimate the integer $J = J(k) = J(k, m, L) =$ number of pairs a_1, a_2 with each a_i relatively prime to m which are excluded by k and which satisfy $1 \leq a_1 < a_2 \leq m/2$. The requirement that a_1 and a_2 be different is convenient later on.

We first estimate $J(k, m, L)$ in the case where the greatest common divisor (k, m) is 1. Such a k excludes the pair a_1, a_2 if and only if 1 excludes the pair ka_1, ka_2; therefore.

$$J(k) = J(1) \quad \text{whenever } (k, m) = 1. \tag{3}$$

We shall prove

$$J(1) < \frac{\varphi(m)^2}{2L}(\log(m^2/L) + \log\log m). \tag{4}$$

In order to do this, we need to define the following sums $D(x, r, m)$ of the divisor function $d(n)$ (= the number of positive integer divisors of the positive integer n) over arithmetic progressions with difference m:

$$D(x, r, m) = \sum_{\substack{n \le x \\ n \equiv r \bmod m}} d(n).$$

A pair a_1, a_2 with $a_i \le m/2$ $(i = 1, 2)$ is excluded by $k = 1$ if

$$a_1 a_2 \le m^2/L. \tag{5}$$

The number of ways of writing any positive integer $n \le m^2/L$ as $a_1 a_2$ is just $d(n)$, and the factors are both relatively prime to m if and only if n is relatively prime to m. Hence, the number of pairs a_1, a_2 satisfying (5) and the additional conditions $(a_i, m) = 1$ $(i = 1, 2)$ and $1 \le a_1 < a_2 \le m/2$ does not exceed

$$\frac{1}{2} \sum_{\substack{n \le m^2/L \\ (n, m)=1}} d(n) = \frac{1}{2} \sum_{\substack{r=1 \\ (r, m)=1}}^{m} D(m^2/L, r, m)$$

(the factor of $\frac{1}{2}$ comes from the fact that $d(n)$ counts each factorization $n = a_1 a_2$ with distinct a_1 and a_2 twice; this is where our assumption that a_1 and a_2 are distinct is convenient). Thus, we have proved

$$J(1, m, L) \le \frac{1}{2} \sum_{\substack{r=1 \\ (r, m)=1}}^{m} D(m^2/L, r, m). \tag{6}$$

In order to estimate the sum in (6), we need some results of D. H. Lehmer [4] concerning the sums $H(x, r, m)$ defined by

$$H(x, r, m) = \sum_{\substack{n \le x \\ n \equiv r \bmod m}} 1/n.$$

Lehmer [4, p. 126] proved the existence of the generalized Euler constants $\gamma(r, m)$ defined for any integers r and $m > 0$ by

$$\gamma(r, m) = \lim_{x \to \infty} (H(x, r, m) - m^{-1} \log x). \tag{7}$$

Clearly, Euler's constant γ is $\gamma(0, 1)$, and $\gamma(r, m)$ is a periodic function of r with period m.

Lemma 1. *For any integers r, m with $m > 0$ and $0 \le r < m$, we have*

$$0 < H(x, r, m) - m^{-1} \log x - \gamma(r, m) < 1/x$$

for all $x \ge m$.

Proof. This follows easily from the proof of the existence of the limit in (7), as given by Lehmer [4, p. 126]. □

In order to state our next two lemmas, it is convenient to define the arithmetical functions $v(n)$ and $w(n)$ by

$$v(n) = -\sum_{d \mid n} \mu(d) d^{-1} \log d$$

(here, $\mu(d)$ is the Möbius function and the sum is taken over all positive integer divisors d of n) and

$$w(n) = nv(n)/\varphi(n) = \sum_{p|n} (\log p)/(p-1)$$

(here, the sum is taken over all prime divisors p of n).

Lemma 2. *For every positive integer* m,

$$\sum_{\substack{r=1 \\ (r,m)=1}}^{m} \gamma(r, m) = \varphi(m)m^{-1}(\gamma + w(m)).$$

Proof. This is equation (16) of Lehmer [4, p. 132]. □

Lemma 3. *For every integer* $m \geq 8$,

$$\gamma + w(m) < (m/\varphi(m)) \log\log m.$$

Proof. Theorem 5 of Davenport [2, p. 294] states

$$\limsup_{m\to\infty} v(m)/\log\log m = \tfrac{1}{4},$$

which implies the lemma for all large m. Some simple calculations (using $\gamma = .577\ldots$) gives the inequality as stated. □

Our final lemma gives an upper bound on the sum $D(x, r, m)$ when r is relatively prime to m.

Lemma 4. *For any integers* r, m *with* r *relatively prime to* m *and* $m \geq 8$, *we have*

$$D(x, r, m) < \varphi(m)m^{-2}x\log x + 2xm^{-1}\log\log m.$$

Proof. We adapt the standard proof of Dirichlet's theorem on summing $d(n)$ for $n \leq x$. The sum $D(x, r, m)$ is the number of lattice points (u, v) with $uv \equiv r \bmod m$ lying below the curve $uv = x$ in the first quadrant of the u, v plane. By using the symmetry in the line $u = v$, if we define $T = [x^{1/2}]$, then we have

$$D(x, r, m) < 2\sum_{i=1}^{T} F_i(x), \tag{8}$$

where $F_i(x)$ denotes the number of integers v such that $iv \equiv r \bmod m$ and $iv \leq x$; we have strict inequality here since we are double counting the lattice points in the square of side T formed by portions of the u- and v-axes. (For a more elaborate version of this argument, which leads to a O-estimate analogous to the one for the usual Dirichlet divisor problem, see Satz 2 of Kopetzky [3]. The simple inequality of Lemma 4 suffices for our purposes, since the more detailed argument does not affect the main term.) If r is relatively prime to m, then $iv \equiv r \bmod m$ is solvable if and only if i is also relatively prime to m, and in that case there is exactly one solution $v \bmod m$. It follows that $F_i(x) = 0$ unless i is relatively prime to m and that

$$F_i(x) \leq x(im)^{-1} \quad \text{for } (i, m) = 1. \tag{9}$$

Now (9) implies

$$\sum_{\substack{i=1\\(i,m)=1}}^{T} F_i(x) \le (x/m) \sum_{\substack{r=1\\(r,m)=1}}^{m} H(T, r, m).$$

Finally, Lemmas 1, 2, and 3 give the inequality in Lemma 4. □

It follows from (3), (6) and Lemma 4 that

$$(10) \qquad J(k, m, L) < \tfrac{1}{2}\varphi(m)^2 L^{-1} \log(m^2 L^{-1}) + m\varphi(m)L^{-1}\log\log m$$

holds for all k with k relatively prime to m. By the argument in [1, pp. 156–157], the inequality in (10) is still true if k is not relatively prime to m (indeed, in that case we can even insert a factor of 8/9 on the right-hand side of (10)).

We can now complete the proof of Theorem 2 (and so of Theorem 1) as in [1, p. 157]: Clearly, (2) holds if and only if the inequality in (2) is true for each $k \le m/2$. The total number of pairs a_1, a_2 with each a_i relatively prime to m and $1 \le a_1 < a_2 \le m/2$ is

$$\binom{\varphi(m)/2}{2} > \varphi(m)^2/8.$$

By (10) and the definition of $J(k, m, L)$, an exceptional pair a_1, a_2 certainly exists if

$$(11) \qquad \varphi(m)^2/8 > \tfrac{1}{2}m(\tfrac{1}{2}\varphi(m)^2 L^{-1}\log(m^2L^{-1}) + m\varphi(m)L^{-1}\log\log m).$$

Computation (using the well-known fact that $\limsup m(\varphi(m)\log\log m)^{-1} = e^\gamma = 1.781\ldots$) shows that (11) is true for $m \ge 8$ if $L \ge 3m\log m$. This completes the proof of Theorem 2.

3. Generalizations

It was pointed out in [1, pp. 154–155] that something like Theorem 2 can be proved in the case of n integers. The main result of [1] was

Theorem 3. *Given any integers $d > 4n$ and $n > 1$, there exist integers $a_1, \ldots, a_n$ relatively prime to m such that*

$$(12) \qquad \prod_{i=1}^{n} \|ka_i/m\| > 4^{-n}(\varphi(m)/m)^n(m\log^{n-1} m)^{-1} \quad \textit{for each } k, \ 1 \le k < m.$$

In view of the connection of Theorems 1 and 2 above, this can be regarded as an n-dimensional generalization of a weakened form of Zaremba's conjecture. In [1, p. 155], I proposed the following general conjecture; Zaremba's conjecture is the case $n = 2$.

Conjecture. For each $n \ge 2$, the lower bound in (12) can be replaced by $c(n)(m\log^{n-2} m)^{-1}$.

The proof of Theorem 2 above removed the factors $\varphi(m)/m$ in the case $n = 2$ of (12). One might hope to achieve the same result for arbitrary n by generalizing the proof of Theorem 2; this would require working with the generalized divisor functions $d_n(t) =$ the number of ways of writing the positive integer t as a product of n positive integer factors.

To conclude, I repeat another speculation from [1, p. 155]: It is possible that the lower bound in (12) could be replaced by $c(n)m^{-1}$ for $n = 3$, or even for all $n \geq 2$. A small amount of computer testing of this for $n = 3$ was reported in [1, p. 155]. Further computer experiments might be worthwhile.

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