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CONTINUED FRACTIONS AND LINEAR RECURRENCES

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Dedicated to the memory of D. H. Lehmer

ABSTRACT. We prove that the numerators and denominators of the convergents to a real irrational number θ satisfy a linear recurrence with constant coefficients if and only if θ is a quadratic irrational. The proof uses the Hadamard Quotient Theorem of A. van der Poorten.

Let θ be an irrational real number with simple continued fraction expansion $[a_0, a_1, a_2, \dots]$. Define the numerators and denominators of the *convergents* to θ as follows:

$$p_{-2} = 0; \quad p_{-1} = 1; \quad p_n = a_n p_{n-1} + p_{n-2} \quad \text{for } n \ge 0; \tag{1}$$

$$q_{-2} = 1; \quad q_{-1} = 0; \quad q_n = a_n q_{n-1} + q_{n-2} \quad \text{for } n \ge 0. \tag{2}$$

By the classical theory of continued fractions (see, for example, [2, Chapter X]), we have

$$\frac{p_n}{q_n} = [a_0, a_1, \dots, a_n].$$

In this note, we consider the question of when the sequences $(p_n)_{n\ge0}$ and $(q_n)_{n\ge0}$ can satisfy a linear recurrence with constant coefficients. If, for example, $\theta = \sqrt{3}$, then $\theta = [1, 1, 2, 1, 2, 1, 2, \dots]$, and it is easy to verify that $q_{n+4} = 4q_{n+2} - q_n$ for all $n \ge 0$. Our main result shows that this exemplifies the situation in general.

Theorem 1. *Let θ be an irrational real number. Let its simple continued fraction expansion be $\theta = [a_0, a_1, \dots]$, and let (p_n) and (q_n) be the sequence of numerators and denominators of the convergents to θ, as defined above. Then the following four conditions are equivalent:*

(a) $(p_n)_{n\ge0}$ *satisfies a linear recurrence with constant complex coefficients;*
(b) $(q_n)_{n\ge0}$ *satisfies a linear recurrence with constant complex coefficients;*
(c) $(a_n)_{n\ge0}$ *is an ultimately periodic sequence;*
(d) θ *is a quadratic irrational.*

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Our proof is simple, but uses a deep result of van der Poorten known as the Hadamard Quotient Theorem. We do not know how to give a short proof of the implication (b) $\Longrightarrow$ (c) from first principles.

Proof. The equivalence (c) $\Longleftrightarrow$ (d) is classical. We will prove the equivalence (b) $\Longleftrightarrow$ (c); the equivalence (a) $\Longleftrightarrow$ (c) will follow in a similar fashion.

(c) $\Longrightarrow$ (b): It is easy to see (cf. Frame [1]) that

$$\begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = \begin{bmatrix} a_0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \cdots \begin{bmatrix} a_n & 1 \\ 1 & 0 \end{bmatrix}. \tag{3}$$

Now if the sequence $(a_n)_{n\geq 0}$ is ultimately periodic, then there exists an integer $r \geq 0$, and r integers $b_0, b_1, \ldots, b_{r-1}$, and an integer $s \geq 1$ and s positive integers $c_0, c_1, \ldots, c_{s-1}$ such that

$$\theta = [b_0, b_1, \ldots, b_{r-1}, c_0, c_1, \ldots, c_{s-1}, c_0, c_1, \ldots, c_{s-1}, \ldots].$$

Now for each integer i modulo s, define

$$M_i = \prod_{0\leq j<s} \begin{bmatrix} c_{i+j} & 1 \\ 1 & 0 \end{bmatrix}.$$

Then for all $n \geq r$, we have, by equation (3),

$$\begin{bmatrix} p_{n+s} & p_{n+s-1} \\ q_{n+s} & q_{n+s-1} \end{bmatrix} = \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} M_{n-r}. \tag{4}$$

Since for all pairs (i, j) it is possible to find matrices A, B such that $M_i = AB$ and $M_j = BA$, and since $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$, it readily follows that $t = \mathrm{Tr}(M_i)$ does not depend on i. Hence the characteristic polynomial of each M_i is $X^2 - tX + (-1)^s$. Since every matrix satisfies its own characteristic polynomial, we see that $M_{n-r}^2 - tM_{n-r} + (-1)^s I$ is the zero matrix. Combining this observation with equation (4), we get

$$\begin{bmatrix} p_{n+2s} & p_{n+2s-1} \\ q_{n+2s} & q_{n+2s-1} \end{bmatrix} - t \begin{bmatrix} p_{n+s} & p_{n+s-1} \\ q_{n+s} & q_{n+s-1} \end{bmatrix} + (-1)^s \begin{bmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{bmatrix} = 0.$$

Therefore, $q_{n+2s} - tq_{n+s} + (-1)^s q_n = 0$ for all $n \geq r$, and hence the sequence $(q_n)_{n\geq 0}$ satisfies a linear recurrence with constant integral coefficients.

(b) $\Longrightarrow$ (c): The proof proceeds in two stages. First we show, by means of a theorem of van der Poorten, that if $(q_n)_{n\geq 0}$ satisfies a linear recurrence, then so does $(a_n)_{n\geq 0}$. Next we show that the a_n are bounded because otherwise the q_n would grow too rapidly. The periodicity of $(a_n)_{n\geq 0}$ then follows immediately.

Let us recall a familiar definition: if the sequence of complex numbers $(u_n)_{n\geq 0}$ satisfies a linear recurrence with constant complex coefficients

$$u_n = \sum_{1\leq i\leq d} e_i u_{n-i}$$

for all n sufficiently large, and d is chosen to be as small as possible, then $X^d - \sum_{1\leq i\leq d} e_i X^{d-i}$ is said to be the *minimal polynomial* for the linear recurrence. Also recall that a sequence of complex numbers $(u_n)_{n\geq 0}$ satisfies a linear recurrence with constant coefficients if and only if the formal series $\sum_{n\geq 0} u_n X^n$ represents a rational function of X.

Define the two formal series $F = \sum_{n\geq 0}(q_{n+2} - q_n)X^n$ and $G = \sum_{n\geq 0} q_{n+1}X^n$. Clearly F and G represent rational functions. We now use the following theorem of van der Poorten [4, 5, 6]:

Theorem 2 (Hadamard Quotient Theorem). *Let $F = \sum_{i\geq 0} f_i X^i$ and $G = \sum_{i\geq 0} g_i X^i$ be formal series representing rational functions in $\mathbf{C}(X)$. Suppose that the f_i and g_i are complex numbers such that $g_i \neq 0$ and f_i/g_i is an integer for all $i \geq 0$. Then $\sum_{i\geq 0}(f_i/g_i)X^i$ also represents a rational function.*

Since $q_{n+2} = a_{n+2}q_{n+1} + q_n$ for all $n \geq 0$, it follows from this theorem that $\sum_{n\geq 0} a_{n+2}X^n$ represents a rational function, and hence the sequence of partial quotients $(a_n)_{n\geq 0}$ also satisfies a linear recurrence with constant coefficients.

We now require the following lemma:

Lemma 3. *Suppose that $(y_n)_{n\geq 0}$ and $(z_n)_{n\geq 0}$ are sequences of complex numbers, each satisfying a linear recurrence, with the property that the minimal polynomial of $(z_n)_{n\geq 0}$ divides the minimal polynomial of $(y_n)_{n\geq 0}$. Let d denote the degree of the minimal polynomial of $(y_n)_{n\geq 0}$. Then there exist constants $c > 0$ and n_0 such that for all $n \geq n_0$ we have*

$$\max(|y_{n-d+1}|, |y_{n-d+2}|, \dots, |y_n|) > c|z_n|.$$

Proof. Put $Y = \sum_{n\geq 0} y_n X^n = f/g$ with $\gcd(f, g) = 1$ and $\deg g = d$, and $Z = \sum_{n\geq 0} z_n X^n = h/g$; here $f, g, h \in \mathbf{C}[X]$. Since $\gcd(f, g) = 1$, we can find a polynomial $k = \sum_{0\leq i<d} k_i X^i$ of degree $< d$ such that $kf \equiv h \pmod{g}$. Then $Z = kY + m$, for a polynomial m, and $z_n = \sum_{0\leq i<d} k_i y_{n-i}$ for $n > n_0 = \deg m$. It follows that

$$|z_n| \leq \left(\sum_{0\leq i<d} |k_i|\right) \max(|y_{n-d+1}|, |y_{n-d+2}|, \dots, |y_n|),$$

and the lemma follows, with $c = (1 + \sum_{0\leq i<d} |k_i|)^{-1}$. $\square$

Since $(a_n)_{n\geq 0}$ satisfies a linear recurrence, we may express a_n as a generalized power sum

$$a_n = \sum_{1\leq i\leq d} A_i(n)\alpha_i^n,$$

for all n sufficiently large. Here the α_i are distinct nonzero complex numbers (the "characteristic roots") and the $A_i(n)$ are polynomials in n.

Now take $y_n = a_n$ and $z_n = n^l\alpha^n$, where $\alpha = \alpha_i$ and $l = \deg A_i$ for some i. Then the hypothesis of Lemma 3 holds, and we conclude that at least one of $a_{n-d+1}, a_{n-d+2}, \dots, a_n$ is greater than $cn^l|\alpha|^n$, for all n sufficiently large. Then, using equation (2), we have

$$q_{dm} \geq \prod_{1\leq j\leq dm} a_j > c' \cdot c^m \cdot d^{lm} \cdot (m!)^l \cdot (|\alpha|^d)^{m(m+1)/2}$$

for some positive constant c' and all $m \geq 1$. But $(q_n)_{n\geq 0}$ satisfies a linear recurrence, and therefore $\log q_{dm} = O(dm)$. It follows that $|\alpha_i| \leq 1$ for all i, and further that $\deg A_i = 0$ for those i with $|\alpha_i| = 1$. Hence the sequence $(a_n)_{n\geq 0}$ is bounded. But a simple argument using the pigeonhole principle (see, for example, [3, Part VIII, Problem 158]) shows that any bounded integer sequence satisfying a linear recurrence is ultimately periodic. This completes the proof. □

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