Supplement to

A CONJECTURE IN ADDITION CHAINS
RELATED TO SCHOLZ'S CONJECTURE

WALTER AIELLO AND M. V. SUBBARAO

Appendix
Methods used for construction of short addition chains for
$2^n - 1$, n going from 2 to 128

```
NUM = 2**  2 - 1                    NUM = 2**  9 - 1
 l(n) =   1      S(NUM) =    2       l(n) =   4      S(NUM) =   12
 Methods used to form chain:        Methods used to form chain:
  TWO                                ONE, THREE


NUM = 2**  3 - 1                    NUM = 2** 10 - 1
 l(n) =   2      S(NUM) =    4       l(n) =   4      S(NUM) =   13
 Methods used to form chain:        Methods used to form chain:
  ONE                                ONE, TWO


NUM = 2**  4 - 1                    NUM = 2** 11 - 1
 l(n) =   2      S(NUM) =    5       l(n) =   5      S(NUM) =   15
 Methods used to form chain:        Methods used to form chain:
  TWO                                ONE, TWO


NUM = 2**  5 - 1                    NUM = 2** 12 - 1
 l(n) =   3      S(NUM) =    7       l(n) =   4      S(NUM) =   15
 Methods used to form chain:        Methods used to form chain:
  ONE, TWO                           ONE, TWO


NUM = 2**  6 - 1                    NUM = 2** 13 - 1
 l(n) =   3      S(NUM) =    8       l(n) =   5      S(NUM) =   17
 Methods used to form chain:        Methods used to form chain:
  ONE, TWO                           ONE, TWO


NUM = 2**  7 - 1                    NUM = 2** 14 - 1
 l(n) =   4      S(NUM) =   10       l(n) =   5      S(NUM) =   18
 Methods used to form chain:        Methods used to form chain:
  ONE                                ONE, TWO


NUM = 2**  8 - 1                    NUM = 2** 15 - 1
 l(n) =   3      S(NUM) =   10       l(n) =   5      S(NUM) =   19
 Methods used to form chain:        Methods used to form chain:
  TWO                                ONE, TWO, THREE
```

NUM = 2** 16 - 1
l(n) = 4   S(NUM) = 19
Methods used to form chain:
TWO

NUM = 2** 17 - 1
l(n) = 5   S(NUM) = 21
Methods used to form chain:
ONE, TWO

NUM = 2** 18 - 1
l(n) = 5   S(NUM) = 22
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 19 - 1
l(n) = 6   S(NUM) = 24
Methods used to form chain:
ONE, THREE

NUM = 2** 20 - 1
l(n) = 5   S(NUM) = 24
Methods used to form chain:
ONE, TWO

NUM = 2** 21 - 1
l(n) = 6   S(NUM) = 26
Methods used to form chain:
ONE, THREE

NUM = 2** 22 - 1
l(n) = 6   S(NUM) = 27
Methods used to form chain:
ONE, TWO

NUM = 2** 23 - 1
l(n) = 6   S(NUM) = 28
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 24 - 1
l(n) = 5   S(NUM) = 28
Methods used to form chain:
ONE, TWO

NUM = 2** 25 - 1
l(n) = 6   S(NUM) = 30
Methods used to form chain:
ONE, TWO, FIVE

NUM = 2** 26 - 1
l(n) = 6   S(NUM) = 31
Methods used to form chain:
ONE, TWO

NUM = 2** 27 - 1
l(n) = 6   S(NUM) = 32
Methods used to form chain:
ONE, THREE

NUM = 2** 28 - 1
l(n) = 6   S(NUM) = 33
Methods used to form chain:
ONE, TWO

NUM = 2** 29 - 1
l(n) = 7   S(NUM) = 35
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 30 - 1
l(n) = 6   S(NUM) = 35
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 31 - 1
l(n) = 7   S(NUM) = 37
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 32 - 1
l(n) = 5   S(NUM) = 36
Methods used to form chain:
TWO

NUM = 2** 33 - 1
l(n) = 6   S(NUM) = 38
Methods used to form chain:
TWO, Method for N = 33

NUM = 2** 34 - 1
l(n) = 6   S(NUM) = 39
Methods used to form chain:
ONE, TWO

NUM = 2** 35 - 1
l(n) = 7   S(NUM) = 41
Methods used to form chain:
ONE, TWO, SIX

NUM = 2** 36 - 1
l(n) = 6   S(NUM) = 41
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 37 - 1
l(n) = 7   S(NUM) = 43
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 38 - 1
l(n) = 7   S(NUM) = 44
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 39 - 1
l(n) = 7   S(NUM) = 45
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 40 - 1
l(n) = 6   S(NUM) = 45
Methods used to form chain:
ONE, TWO

NUM = 2** 41 - 1
l(n) = 7   S(NUM) = 47
Methods used to form chain:
ONE, TWO

NUM = 2** 42 - 1
l(n) = 7   S(NUM) = 48
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 43 - 1
l(n) = 7   S(NUM) = 49
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 44 - 1
l(n) = 7   S(NUM) = 50
Methods used to form chain:
ONE, TWO

NUM = 2** 45 - 1
l(n) = 7   S(NUM) = 51
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 46 - 1
l(n) = 7   S(NUM) = 52
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 47 - 1
l(n) = 8   S(NUM) = 54
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 48 - 1
l(n) = 6   S(NUM) = 53
Methods used to form chain:
ONE, TWO

NUM = 2** 49 - 1
l(n) = 7   S(NUM) = 55
n Methods used to form chain:
ONE, TWO

NUM = 2** 50 - 1
l(n) = 7   S(NUM) = 56
Methods used to form chain:
ONE, TWO, FIVE

NUM = 2** 51 - 1
l(n) = 7   S(NUM) = 57
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 52 - 1
l(n) = 7   S(NUM) = 58
Methods used to form chain:
ONE, TWO

NUM = 2** 53 - 1
l(n) = 8   S(NUM) = 60
Methods used to form chain:
ONE, TWO

NUM = 2** 54 - 1
l(n) = 7   S(NUM) = 60
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 55 - 1
l(n) = 8   S(NUM) = 62
Methods used to form chain:
ONE, TWO, FIVE

NUM = 2** 56 - 1
l(n) = 7   S(NUM) = 62
Methods used to form chain:
ONE, TWO

NUM = 2** 57 - 1
l(n) = 8   S(NUM) = 64
Methods used to form chain:
ONE, THREE

NUM = 2** 58 - 1
l(n) = 8   S(NUM) = 65
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 59 - 1
l(n) = 8   S(NUM) = 66
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 60 - 1
l(n) = 7   S(NUM) = 66
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 61 - 1
l(n) = 8   S(NUM) = 68
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 62 - 1
l(n) = 8   S(NUM) = 69
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 63 - 1
l(n) = 8   S(NUM) = 70
Methods used to form chain:
ONE, THREE

NUM = 2** 64 - 1
l(n) = 6   S(NUM) = 69
Methods used to form chain:
TWO

NUM = 2** 65 - 1
l(n) = 7   S(NUM) = 71
Methods used to form chain:
ONE, TWO

NUM = 2** 66 - 1
l(n) = 7   S(NUM) = 72
Methods used to form chain:
TWO, Method for N = 33

NUM = 2** 67 - 1
l(n) = 8   S(NUM) = 74
Methods used to form chain:
ONE, TWO, Method for N = 33

NUM = 2** 68 - 1
l(n) = 7   S(NUM) = 74
Methods used to form chain:
ONE, TWO

NUM = 2** 69 - 1
l(n) = 8   S(NUM) = 76
Methods used to form chain:
ONE, TWO, THREE, FOUR

NUM = 2** 70 - 1
l(n) = 8   S(NUM) = 77
Methods used to form chain:
ONE, TWO, SIX

NUM = 2** 71 - 1
l(n) = 9   S(NUM) = 79
Methods used to form chain:
ONE, TWO, FOUR, Method N = 33

NUM = 2** 72 - 1
l(n) = 7   S(NUM) = 78
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 73 - 1
l(n) = 8   S(NUM) = 80
Methods used to form chain:
ONE, TWO, THREE, Meth for N = 33

NUM = 2** 74 - 1
l(n) = 8   S(NUM) = 81
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 75 - 1
l(n) = 8   S(NUM) = 82
Methods used to form chain:
ONE, TWO, THREE, FIVE

NUM = 2** 76 - 1
l(n) = 8   S(NUM) = 83
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 77 - 1
l(n) = 8   S(NUM) = 84
Methods used to form chain:
ONE, TWO, Method for N = 77

NUM = 2** 78 - 1
l(n) = 8   S(NUM) = 85
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 79 - 1
l(n) = 9   S(NUM) = 87
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 80 - 1
l(n) = 7   S(NUM) = 86
Methods used to form chain:
ONE, TWO

NUM = 2** 81 - 1
l(n) = 8   S(NUM) = 88
Methods used to form chain:
ONE, THREE

NUM = 2** 82 - 1
l(n) = 8   S(NUM) = 89
Methods used to form chain:
ONE, TWO

NUM = 2** 83 - 1
l(n) = 8   S(NUM) = 90
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 84 - 1
l(n) = 8   S(NUM) = 91
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 85 - 1
l(n) = 8   S(NUM) = 92
Methods used to form chain:
ONE, TWO, FIVE

NUM = 2** 86 - 1
l(n) = 8   S(NUM) = 93
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 87 - 1
l(n) = 9   S(NUM) = 95
Methods used to form chain:
ONE, TWO, THREE, FOUR

NUM = 2** 88 - 1
l(n) = 8   S(NUM) = 95
Methods used to form chain:
ONE, TWO

NUM = 2** 89 - 1
l(n) = 9   S(NUM) = 97
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 90 - 1
l(n) = 8   S(NUM) = 97
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 91 - 1
l(n) = 9   S(NUM) = 99
Methods used to form chain:
ONE, TWO, THREE, SIX

NUM = 2** 92 - 1
l(n) = 8   S(NUM) = 99
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 93 - 1
l(n) = 9   S(NUM) = 101
Methods used to form chain:
ONE, TWO, THREE, FOUR

NUM = 2** 94 - 1
l(n) = 9   S(NUM) = 102
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2** 95 - 1
l(n) = 9   S(NUM) = 103
Methods used to form chain:
ONE, TWO, THREE, FIVE

NUM = 2** 96 - 1
l(n) = 7   S(NUM) = 102
Methods used to form chain:
ONE, TWO

NUM = 2** 97 - 1
l(n) = 8   S(NUM) = 104
Methods used to form chain:
ONE, TWO

NUM = 2** 98 - 1
l(n) = 8   S(NUM) = 105
Methods used to form chain:
ONE, TWO

NUM = 2** 99 - 1
l(n) = 8   S(NUM) = 106
Methods used to form chain, Method for N = 33
TWO, THREE, Method for N = 33

NUM = 2**100 - 1
l(n) = 8   S(NUM) = 107
Methods used to form chain:
ONE, TWO, FIVE

NUM = 2**101 - 1
l(n) = 9   S(NUM) = 109
Methods used to form chain:
ONE, TWO

NUM = 2**102 - 1
l(n) = 8   S(NUM) = 109
Methods used to form chain:
ONE, TWO, THREE

NUM = 2**103 - 1
l(n) = 9   S(NUM) = 111
Methods used to form chain:
ONE, TWO, THREE

NUM = 2**104 - 1
l(n) = 8   S(NUM) = 111
Methods used to form chain:
ONE, TWO

NUM = 2**105 - 1
l(n) = 9   S(NUM) = 113
Methods used to form chain:
ONE, TWO, THREE, SIX

NUM = 2**106 - 1
l(n) = 9   S(NUM) = 114
Methods used to form chain:
ONE, TWO

NUM = 2**107 - 1
l(n) = 9   S(NUM) = 115
Methods used to form chain:
ONE, TWO, FOUR

NUM = 2**108 - 1
l(n) = 8   S(NUM) = 115
Methods used to form chain:
ONE, TWO, THREE

NUM = 2**109 - 1
l(n) = 8   S(NUM) = 117
Methods used to form chain:
ONE, TWO, THREE

NUM = 2**110 - 1
l(n) = 9   S(NUM) = 118
Methods used to form chain:
ONE, TWO, FIVE

NUM = 2**111 - 1
l(n) = 9   S(NUM) = 119
Methods used to form chain:
ONE, TWO, THREE

NUM = 2** 33 - 1

Chain length: 38
Scholz length: 38
Chain PASSES

Methods used to form chain:
Method TWO
SPECIAL METHOD for N = 33

+ 8589934591
+ 4294967296
+ 4294967295
+ 4294901760
+ 2147450880
+ 1073725440
+ 536862720
+ 268431360
+ 134215680
+ 67107840
+ 33553920
+ 16776960
+ 8388480
+ 4194240
+ 2097120
+ 1048560
+ 524280
+ 262140
+ 131070
+ 65535
+ 65280
+ 32640
+ 16320
+ 8160
+ 4080
+ 2040
+ 1020
+ 510
+ 255
+ 240
+ 120
+ 60
+ 30
+ 15
+ 12
+ 6
+ 3
+ 2
+ 1

NUM = 2**112 - 1
l(n) = 8   S(NUM) = 119
Methods used to form chain:
    ONE, TWO

NUM = 2**113 - 1
l(n) = 9   S(NUM) = 121
Methods used to form chain:
    ONE, TWO

NUM = 2**114 - 1
l(n) = 9   S(NUM) = 122
Methods used to form chain:
    ONE, TWO, THREE

NUM = 2**115 - 1
l(n) = 9   S(NUM) = 123
Methods used to form chain:
    ONE, TWO, FOUR

NUM = 2**116 - 1
l(n) = 9   S(NUM) = 124
Methods used to form chain:
    ONE, TWO, FOUR

NUM = 2**117 - 1
l(n) = 9   S(NUM) = 125
Methods used to form chain:
    ONE, TWO, THREE

NUM = 2**118 - 1
l(n) = 9   S(NUM) = 126
Methods used to form chain:
    ONE, TWO, FOUR

NUM = 2**119 - 1
l(n) = 9   S(NUM) = 127
Methods used to form chain:
    ONE, TWO, SIX

NUM = 2**120 - 1
l(n) = 8   S(NUM) = 127
Methods used to form chain:
    ONE, TWO, THREE

NUM = 2**121 - 1
l(n) = 9   S(NUM) = 129
Methods used to form chain:
    ONE, TWO, THREE

NUM = 2**122 - 1
l(n) = 9   S(NUM) = 130
Methods used to form chain:
    ONE, TWO, THREE

NUM = 2**123 - 1
l(n) = 9   S(NUM) = 131
Methods used to form chain:
    ONE, TWO, THREE

NUM = 2**124 - 1
l(n) = 9   S(NUM) = 132
Methods used to form chain:
    ONE, TWO, FOUR

NUM = 2**125 - 1
l(n) = 9   S(NUM) = 133
Methods used to form chain:
    ONE, TWO, THREE, FIVE

NUM = 2**126 - 1
l(n) = 9   S(NUM) = 134
Methods used to form chain:
    ONE, TWO, THREE

NUM = 2**127 - 1
l(n) = 10   S(NUM) = 136
Methods used to form chain:
    ONE, TWO, FOUR

NUM = 2**128 - 1
l(n) = 7   S(NUM) = 134
Methods used to form chain:
    TWO

NUM = 2** 77 - 1

Chain length: 84
Scholz length: 84
Chain PASSES

Methods used to form chain:
Method ONE
Method TWO
SPECIAL METHOD for N = 77

+ 151115727451828646838271
+ 151115727451828646837760
+ 755578637259143234818880
+ 377789318629571617094940
+ 188894659314788080854720
+ 944473296573929042736 0
+ 472236648286964521360 0
+ 236118324143482260684 0
+ 118059162077411303420 0
+ 590295810358705651710
+ 295147905179352825855
+ 295147905162172956672
+ 147573952581086478336
+ 73786976290543239168
+ 36893488145271619584
+ 18446744072635809792
+ 9223372036317904896
+ 4611686018158952448
+ 2305843009079476224
+ 1152921504539738112
+ 576460752269869056
+ 288230376134934528
+ 144115188067467264
+ 72057594033733632
+ 36028797016866816
+ 18014398508433408
+ 9007199254216704
+ 4503599627108352
+ 2251799813554176
+ 1125899906777088
+ 562949953388544
+ 281474976694272
+ 140737488347136
+ 70368744173568
+ 35184372086784
+ 17592186043392
+ 8796093021696
+ 4398046510848
+ 2199023255424
+ 1099511627712
+ 549755813856
+ 274877906928
+ 137438953464
+ 68719476732
+ 34359738366
+ 17179869183
+ 17179738112

+ 8589869056
+ 429493528
+ 214746764
+ 1073733632
+ 536866816
+ 268433408
+ 134216704
+ 67108352
+ 33554176
+ 16777088
+ 8388544
+ 4194272
+ 2097136
+ 1048568
+ 524284
+ 262142
+ 131071
+ 130560
+ 65280
+ 32640
+ 16320
+ 8160
+ 4080
+ 2040
+ 1020
+ 511
+ 510
+ 255
+ 240
+ 120
+ 60
+ 30
+ 15
+ 12
+ 6
+ 3
+ 2
+ 1