

**Supplement to**  
**NUMERICAL COMPUTATIONS CONCERNING THE ERH**  
**ROBERT RUMELY**

Section 5. Tables and Graphs.

Table 5.1: STATISTICS ABOUT L-SERIES, BY MODULUS

#	ERH	SUM	LEAST	LEAST	GREATEST	GREATEST	LEAST	GREATEST	
Q	L-SER	HEIGHT	1/ R	ROOT	MAX	MAX	S(T)	GAP	
			BOUND						
1	1	10029.021	4.322	14.13473	0.003967	16.90615	1.44847	0.04210	2.48351
3	1	10133.702	5.672	8.03974	0.004529	15.04223	1.34760	0.05368	2.59871
4	1	10248.864	6.092	6.02095	0.001965	11.86942	1.37010	0.03333	2.52035
5	3	10072.973	19.100	4.13290	0.001475	19.62612	1.48385	0.03512	2.62789
7	5	10017.500	34.278	2.50937	0.000316	23.50929	1.52513	0.01357	2.75128
8	2	10021.581	14.056	3.57615	0.002663	13.59221	1.37205	0.04961	2.48191
9	4	10157.746	28.916	2.90199	0.000869	18.22939	1.47636	0.02545	2.70072
11	9	10019.363	68.002	1.23119	0.000502	27.66859	1.61542	0.02522	2.81486
12	1	10026.176	7.573	3.80463	0.001977	9.30459	1.28586	0.03403	2.41170
13	11	10019.119	86.079	0.88396	0.000096	28.54738	1.65258	0.00899	2.83365
15	3	2524.086	17.654	2.73460	0.000932	13.13095	1.29729	0.02518	2.22265
16	4	2520.832	24.138	1.58558	0.002081	11.48749	1.33572	0.03984	2.42998
17	15	2748.166	94.573	0.39131	0.000748	23.89895	1.61490	0.02081	2.69313
19	17	2512.243	107.915	0.01896	0.002810	23.24676	1.57042	0.04142	2.82688
20	3	2604.865	18.965	2.35893	0.006685	9.64142	1.28374	0.05928	2.22985
21	5	2544.293	31.877	1.61202	0.001400	14.41247	1.37781	0.02660	2.47479
23	21	2542.055	138.599	0.59543	0.001209	24.61406	1.57408	0.03103	2.79958
24	2	2612.653	13.220	1.97719	0.001093	7.96155	1.26778	0.02606	2.15096
25	16	2506.712	106.934	0.39645	0.000123	21.44001	1.47188	0.01258	2.60182
27	12	2541.274	81.797	0.40516	0.000391	17.43803	1.47291	0.02043	2.65296
28	5	2503.605	33.848	1.20418	0.001316	11.61433	1.32874	0.02480	2.40888
29	27	2512.351	186.749	0.28629	0.001480	25.70754	1.60159	0.02479	2.84417
31	29	2515.510	203.685	0.00694	0.000809	25.58984	1.63640	0.02346	2.80325
32	8	2646.407	57.191	0.33600	0.002206	13.37724	1.46624	0.04030	2.48466
33	9	2530.468	63.576	0.50072	0.001538	16.73714	1.40489	0.02966	2.46203
35	15	2510.826	106.872	0.58660	0.000182	20.25735	1.43655	0.00878	2.53116
36	4	2585.494	28.858	0.93469	0.002588	9.34404	1.25067	0.03808	2.34660
37	35	2520.072	254.727	0.12353	0.001259	28.98442	1.61969	0.03535	2.93011
39	11	2524.507	80.053	0.65151	0.001633	18.52665	1.41161	0.02675	2.58028
40	6	2586.457	44.257	0.78176	0.006153	11.37038	1.32871	0.05905	2.44113
41	39	2505.993	289.095	0.08543	0.000290	27.46621	1.61801	0.01642	2.84475
43	41	2505.223	306.837	0.13837	0.000026	27.88252	1.59767	0.00472	2.83346
44	9	2535.143	67.753	0.12372	0.000830	12.58658	1.36247	0.01587	2.49687
45	12	2517.289	90.410	0.14131	0.000297	15.32316	1.43865	0.01316	2.49948
47	45	2506.052	342.626	0.09038	0.000517	29.20429	1.64085	0.01762	2.88328
48	4	2521.707	30.553	0.60932	0.006073	9.27006	1.29900	0.04740	2.36446
49	36	2508.798	275.753	0.09468	0.000230	26.84519	1.62201	0.01366	2.74264
51	15	2526.701	115.205	0.56066	0.001197	18.69700	1.53718	0.02985	2.56331
52	11	2553.836	84.952	0.48615	0.000310	13.38493	1.41161	0.01423	2.46340
53	51	2548.480	398.252	0.00643	0.000399	29.53259	1.67729	0.02156	2.96062
55	27	2509.998	210.645	0.13205	0.002016	22.27891	1.56915	0.03851	2.63395
56	10	2516.789	78.323	0.52630	0.000730	12.70161	1.36402	0.02221	2.44271
57	17	2509.169	133.213	0.31033	0.000356	19.79805	1.55137	0.01533	2.75323
59	57	2505.503	452.691	0.05165	0.000126	29.42993	1.74724	0.00986	2.94160
60	3	2513.720	23.761	0.93760	0.007494	7.45733	1.27038	0.05993	2.24913
61	59	2512.627	471.598	0.02825	0.000109	29.98239	1.68757	0.01400	3.03949
63	20	2503.432	160.015	0.21101	0.000553	17.34584	1.43459	0.01819	2.51445
64	16	2508.451	128.467	0.37696	0.000095	15.06637	1.51673	0.00851	2.74952
65	33	2516.155	265.591	0.18416	0.000867	23.68456	1.59941	0.02218	2.72152
67	65	2506.077	328.090	0.11303	0.000263	30.46775	1.75660	0.01495	2.96521
68	15	2506.422	121.598	0.05346	0.001380	14.93672	1.47440	0.03542	2.55268
69	21	2512.006	170.883	0.22273	0.003124	19.76238	1.56791	0.04035	2.65577
71	69	2515.050	566.761	0.09859	0.000353	31.24616	1.68277	0.01463	2.97862
72	8	2614.933	66.205	0.27916	0.002330	9.81096	1.39838	0.03341	2.44542

Table 5.1 (Cont.): STATISTICS ABOUT L-SERIES, BY MODULUS

Q	L-SERIES	ERR HEIGHT	SUM 1/R	LEAST ROOT	LEAST MAX	GREATEST MAX	GREATEST GAP	LEAST GAP	S(T)	ERR HEIGHT	SUM 1/R	LEAST ROOT	LEAST MAX	GREATEST MAX	GREATEST GAP
75	16	2529.460	132.413	0.000000	16.78104	1.42782	0.02480	2.57318							
76	17	2505.056	140.502	0.000016	15.20326	1.71011	0.00152	2.62377							
77	44	2505.664	373.562	0.000014	25.86384	1.50013	0.00131	2.76008							
80	12	2511.333	101.597	0.000000	12.66968	1.42782	0.04196	2.42879							
81	36	2511.658	301.428	0.000017	21.89038	1.50013	0.01412	2.58920							
84	5	2512.153	63.005	0.000000	6.64278	1.42782	0.03611	2.28934							
85	45	2512.244	239.499	0.000014	26.55321	1.50013	0.01849	2.64839							
87	27	2512.539	122.635	0.000018	12.19910	1.42782	0.01078	2.65824							
88	18	2512.539	149.854	0.000018	15.91243	1.42782	0.01052	2.62959							
91	55	2512.429	469.865	0.000022	15.91243	1.42782	0.01052	2.62959							
92	21	2512.203	248.075	0.000025	21.25503	1.50013	0.01318	2.76921							
93	29	2512.712	438.298	0.000034	26.90276	1.50013	0.01318	2.76921							
95	51	2512.712	688.584	0.000049	26.90276	1.50013	0.01318	2.76921							
96	8	2513.747	68.584	0.000013	10.68359	1.42782	0.04663	2.74894							
99	36	2513.335	319.027	0.000049	21.39496	1.50013	0.01663	2.69791							
100	16	2513.853	192.814	0.000049	13.25361	1.50013	0.02345	2.70441							
104	22	2513.713	191.942	0.000219	15.30344	1.42779	0.02345	2.70441							
105	15	2513.713	191.942	0.000161	15.71932	1.42012	0.02345	2.74152							
108	12	2513.200	105.636	0.00012	20.21365	1.51110	0.02345	2.57007							
111	35	2513.740	177.399	0.00053	14.07386	1.45388	0.03451	2.71117							
112	20	2513.642	240.304	0.00060	16.60488	1.54730	0.03451	2.72823							
119	46	2513.900	392.705	0.00013	22.02380	1.51906	0.03451	2.45700							
120	10	2513.000	897.144	0.000072	8.31769	1.27648	0.03451	2.45700							
121	100	2513.360	260.710	0.000072	31.26244	1.72166	0.03451	2.96501							
124	80	2510.799	721.304	0.000072	16.78331	1.44015	0.03451	2.73119							
125	32	2505.179	289.578	0.000014	16.75118	1.52485	0.03451	2.73119							
132	9	2513.920	81.745	0.000008	3.86952	1.30689	0.03451	2.73119							
140	15	2513.469	138.254	0.000008	11.33315	1.67142	0.03451	2.73119							
143	99	2513.355	911.957	0.000008	11.33315	1.67142	0.03451	2.73119							
144	16	2513.320	147.599	0.000008	11.33315	1.67142	0.03451	2.73119							
156	11	2513.002	102.925	0.000008	10.03451	1.42782	0.03451	2.73119							
163	161	2513.024	1515.771	0.000008	10.03451	1.42782	0.03451	2.73119							
168	10	2513.277	94.186	0.000008	10.03451	1.42782	0.03451	2.73119							
189	144	2513.534	1362.403	0.000008	10.03451	1.42782	0.03451	2.73119							
180	12	2513.959	114.467	0.000008	10.03451	1.42782	0.03451	2.73119							
216	24	2513.764	1078.863	0.000008	10.03451	1.42782	0.03451	2.73119							
246	186	2513.776	1078.863	0.000008	10.03451	1.42782	0.03451	2.73119							
256	168	2513.776	1078.863	0.000008	10.03451	1.42782	0.03451	2.73119							
320	24	2513.628	955.117	0.000008	10.03451	1.42782	0.03451	2.73119							
430	15	2504.628	631.117	0.000008	10.03451	1.42782	0.03451	2.73119							
432	48	2509.744	519.392	0.02054	12.89282	1.46812	0.03451	2.32022							

Notes

- The values given are extrema over all L-series with conductor Q. ERR HEIGHT and SUM 1/R: BOUND were computed using interval arithmetic, and are mathematically rigorous, though not sharp. Other entries in the table are believed accurate to the last digit but are not rigorous.
- ERR HEIGHT is a lower bound for the height to which the ERH has been proved uniformly for all L-series with conductor Q.
- SUM 1/R: BOUND is an upper bound for the sum over all L-series with conductor Q of the sums of the reciprocal moduli of the roots  $0.5+it$  with  $0 < t \leq$  ERR HEIGHT.
- LEAST ROOT is the least  $T_0$  for which  $0.5+it$  is a root.
- ERR MAX and GREATST MAX are the smallest and largest maxima between roots.
- GREATST S(T) is the largest value of  $S(T)$  encountered.
- LEAST GAP and GREATST GAP are gaps between consecutive roots, on a Gram point scale

Table 5.2: CHAMPIONS

SMALL GAPS:										LARGE GAPS:									
Q	K	(GRAM) GAP LENGTH	LOW	HIGH	T	Q	K	(GRAM) GAP LENGTH	LOW	HIGH	T	Q	K	(GRAM) GAP LENGTH	LOW	HIGH	T		
95	8	0.001831	357.3404	1.754738		163	71	3.164799	2376.7616	4.771501	50142								
77	4	0.004341	815.1073	6.41037		163	56	3.940178	1525.2514	1.9713672	36782								
169	104	0.004689	810.1140	6.41037		61	56	3.939193	1961.6720	1.9616748	67487								
43	19	0.004719	1843.8981	13.118218		61	13	3.167752	1148.3243	1.7503655	3515								
111	25	0.005684	1429.2120	11.9123673		71	68	2.975667	1218.1938	2.6013930	8750								
163	118	0.007289	272.1413	2.46947		163	46	2.966101	2203.3813	2.6013930	08500								
64	11	0.008754	2348.3232	17.622572		121	16	2.966101	1307.1111	1.9114700	47001								
13	10	0.008990	1516.4579	11.43186		67	16	2.964932	832.4015	1.9114700	85461								
35	0	0.008990	7958.6949	63.68512		169	90	2.964932	1856.2718	1.9114700	84624								
55	0	0.008955	467.6576	6.97972		163	116	2.964932	680.0717	1.9114700	88126								
88	8	0.010521	967.6576	6.97972		53	36	2.964932	1490.0717	1.9114700	35150								
127	46	0.012283	1069.1796	17.6114572		169	120	2.964932	2109.1462	2.1112308	12308								
243	74	0.012283	322.0671	3.253666		71	57	2.964932	2495.7616	2.1112308	37495								
169	12	0.014398	981.31417	9.231417		39	49	2.964932	2385.4816	2.1112308	04617								

SMALL CONSECUTIVE PAIRS OF GAPS:										LARGE S(T) (given with sign):									
Q	K	(GRAM) GAP LENGTH	LOW	MIDDLE	HIGH	T	Q	K	S(T)	T	Q	K	S(T)	T					
121	99	0.773990	1766.6061	1766.6061	1766.6061	1766.6061	163	105	-1.79084	1513.6016									
93	4	0.812536	2414.3961	2444.3961	2444.3961	2444.3961	163	9	1.76726	2120.4576									
111	35	0.913736	216.7137	2116.6565	2116.6565	2116.6565	67	42	-1.75660	2016.0576									
163	105	0.963511	1513.9965	1513.73471	1513.73471	1513.73471	163	101	-1.74983	2467.0116									
163	23	0.973272	374.3524	601.2739	601.2739	601.2739	59	32	1.74724	1723.0310									
163	94	0.980768	394.9473	1074.2593	1074.2593	1074.2593	121	17	-1.72166	1170.7147									
123	94	0.980768	394.9473	1074.2593	1074.2593	1074.2593	59	48	-1.71527	1448.2463									
100	13	0.979191	647.9637	1994.1857	1994.1857	1994.1857	121	24	-1.71139	2008.4817									
163	126	0.979191	647.9637	1994.1857	1994.1857	1994.1857	169	49	-1.70714	2654.1917									
29	17	0.984350	104.9637	2384.1919	2384.1919	2384.1919	169	53	-1.70314	2652.7793									
243	92	0.984350	104.9637	2384.1919	2384.1919	2384.1919	163	44	1.69647	2151.7283									
256	63	0.984350	104.9637	2384.1919	2384.1919	2384.1919	163	36	1.69138	1800.7471									
169	117	0.984350	104.9637	2384.1919	2384.1919	2384.1919	71	56	-1.68477	1194.3641									
							163	28	1.68133	774.8827									

SMALL MAXIMA: T: [Z(T,X)]

LARGE MAXIMA: T: [Z(T,X)]

Table 5.4: STATISTICS ABOUT INDIVIDUAL L-SERIES, THROUGH MODELS 13

CHAR ORDER	ERH HEIGHT	# ROOTS	SUM 1 R ROUND	LEAST ROOT	GREATEST MAX	GREATEST S(T)	LEAST GAP
1	10029.021	10177	4.322	14.13473	16.90615	1.34847	0.04149
1	10133.702	12071	5.672	8.03974	15.04223	1.34760	0.05364
1	10248.864	12696	6.092	6.02095	11.86842	1.37010	0.05353
2	10072.973	12909	6.349	6.18358	19.12447	1.36199	0.06692
2	10073.585	12809	6.439	4.13290	19.26218	1.38385	0.03512
3	10073.585	12809	6.312	6.64845	19.62612	1.30896	0.04192
3	10019.301	13268	6.826	4.35640	23.50929	1.49081	0.01357
3	10017.838	13266	6.793	5.19812	22.40548	1.52313	0.02132
3	10017.500	13265	6.752	6.20123	21.69485	1.46549	0.05511
4	10017.838	13266	7.044	2.50937	22.26818	1.45655	0.06420
5	10019.033	13267	6.863	4.47574	22.68375	1.50453	0.04195
7	10021.581	13484	7.081	3.57615	12.37598	1.37152	0.04986
2	10021.581	13484	6.975	4.89997	13.59221	1.37205	0.04961
6	10157.746	13893	7.361	2.90199	16.91375	1.47636	0.02545
1	10158.709	13881	7.189	3.44409	18.18760	1.42000	0.05004
2	10158.709	13881	7.189	4.57574	17.55217	1.42780	0.03586
4	10160.035	13883	7.120	5.31958	18.22939	1.41943	0.02855
1	10022.682	13994	7.514	3.54704	27.66859	1.55705	0.07743
2	10020.541	13991	7.497	3.61004	26.92461	1.52217	0.04342
3	10021.848	13992	7.383	5.13370	24.52623	1.59421	0.05023
4	10021.848	13993	7.383	4.62935	24.89686	1.53289	0.05824
5	10019.810	13990	8.081	3.41492	25.11366	1.56344	0.05265
6	10019.810	13990	8.081	1.23119	26.31236	1.59970	0.05007
7	10021.457	13992	7.380	5.07032	26.08453	1.59036	0.02760
8	10020.905	13991	7.575	2.69600	25.06758	1.61149	0.05042
9	10019.810	13990	7.627	2.47724	24.10710	1.61542	0.02522
1	10026.176	14138	7.973	3.80463	9.30459	1.28586	0.03403
2	10021.747	14259	7.692	4.24461	25.84377	1.49491	0.04801
3	10022.752	14260	7.592	4.93859	26.86664	1.64596	0.04900
4	10020.006	14257	7.984	4.45485	27.30308	1.54824	0.03671
5	10019.592	14256	8.499	2.19555	28.54738	1.51429	0.03346
6	10021.378	14259	7.933	0.88396	25.46452	1.60102	0.05848
7	10020.006	14256	7.692	3.74382	27.52673	1.54761	0.02850
8	10020.551	14257	7.888	2.27313	26.40368	1.65258	0.05447
9	10020.006	14256	7.726	3.66097	26.20502	1.59870	0.03252
10	10019.119	14255	7.712	3.32883	25.34196	1.58678	0.00899
11	10019.582	14256	7.727	3.11834	26.00151	1.50557	0.05511

Figure 5.3: STATISTICS FOR L-SERIES WITH  $Q \ll 132$ . PLOTTED BY MODULUS

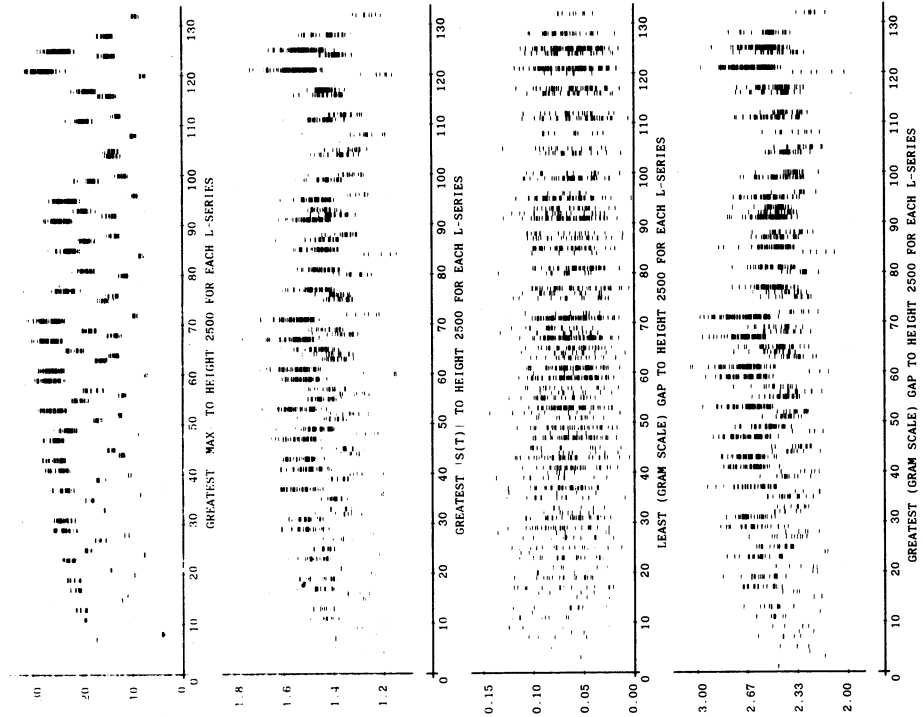
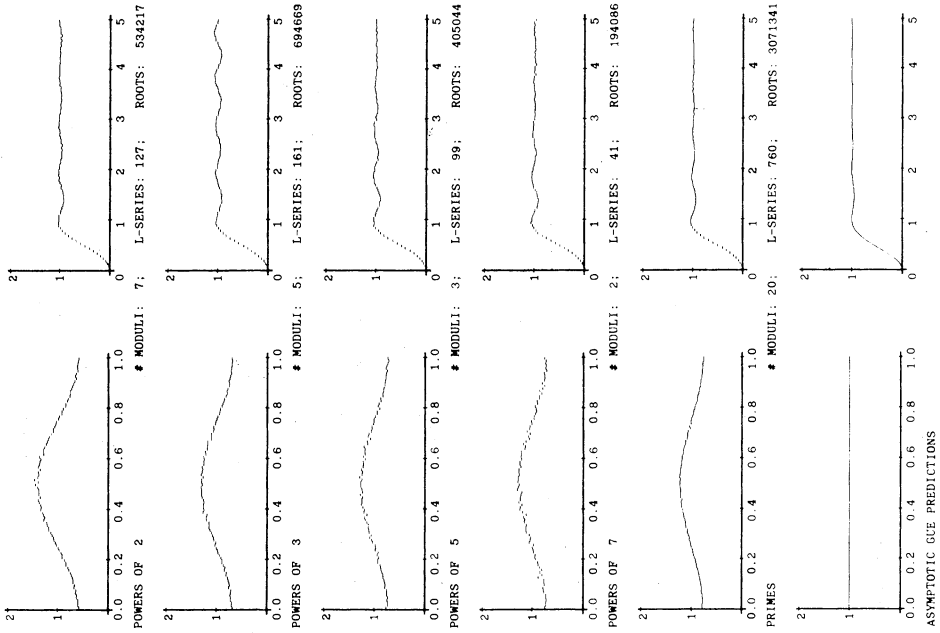


Table 5.5: ROOT COUNTS AND SUM  $1/|R|$  BOUNDS FOR INDIVIDUAL L-SERIES, THROUGH MODULUS 13

K	K̄	HEIGHT:			
		100	500	2500	10000
1	1	25 0.598	269 1.515	1985 2.845	4520 3.543
1	1	45 1.131	356 2.328	2421 3.938	5393 4.758
1	1	50 1.317	379 2.594	2536 4.278	5623 5.129
1	2	54 1.441	397 2.770	2624 4.510	5799 5.386
2	1	53 1.518	397 2.859	2624 4.599	5801 5.476
3	3	54 1.405	397 2.733	2624 4.473	5800 5.349
1	3	59 1.675	423 3.092	2758 4.920	6068 5.833
2	4	59 1.643	423 3.060	2758 4.897	6068 5.801
3	1	59 1.603	423 3.021	2758 4.846	6068 5.760
4	2	59 1.890	424 3.312	2758 5.138	6069 6.052
5	5	59 1.710	424 3.131	2759 4.957	6068 5.870
1	1	61 1.830	435 3.266	2811 5.145	6174 6.073
2	2	61 1.725	434 3.179	2811 5.039	6174 5.968
1	3	63 2.003	444 3.488	2858 5.379	6268 6.320
2	4	63 1.889	443 3.372	2858 5.264	6268 6.205
3	1	63 1.832	443 3.314	2859 5.206	6268 6.147
4	2	63 1.765	444 3.248	2858 5.137	6267 6.078
1	5	67 2.038	460 3.566	2973 5.547	6496 6.471
2	3	66 2.012	459 3.517	2973 5.547	6496 6.471
3	2	67 1.910	459 3.416	2973 5.440	6496 6.380
4	8	66 1.900	459 3.411	2973 5.439	6496 6.379
5	1	66 2.075	459 3.599	2973 5.619	6496 6.519
6	6	66 1.897	460 3.418	2973 5.444	6496 6.382
7	4	66 1.897	460 3.418	2973 5.444	6496 6.382
8	4	66 2.090	460 3.621	2973 5.621	6496 6.521
9	9	66 2.141	460 3.671	2973 5.671	6496 6.571
1	1	67 2.021	466 3.581	2973 5.547	6496 6.519
1	6	69 2.088	472 3.662	3005 5.649	6561 6.631
2	8	69 1.991	472 3.563	3005 5.549	6561 6.531
3	9	69 2.032	473 3.607	3005 5.591	6560 6.573
4	7	69 2.376	473 3.956	3005 5.941	6561 6.923
5	10	70 2.901	473 4.471	3005 6.457	6561 7.438
6	1	69 2.325	473 3.904	3004 5.889	6561 6.871
7	4	69 2.089	473 3.665	3004 5.649	6560 6.631
8	2	69 2.283	473 3.860	3004 5.845	6561 6.827
9	3	66 2.112	473 3.698	3003 5.663	6561 6.665
10	7	69 2.199	473 3.805	3004 5.689	6560 6.651
11	11	69 2.124	473 3.700	3004 5.684	6560 6.666

FIGURE 5.6: DISTRIBUTION OF ROOTS MODULO 1 AND PAIR CORRELATION FUNCTIONS



ASYMPTOTIC GUE PREDICTIONS

FIGURE 5.7: PAIR CORRELATION FUNCTIONS OF INDIVIDUAL L-SERIES

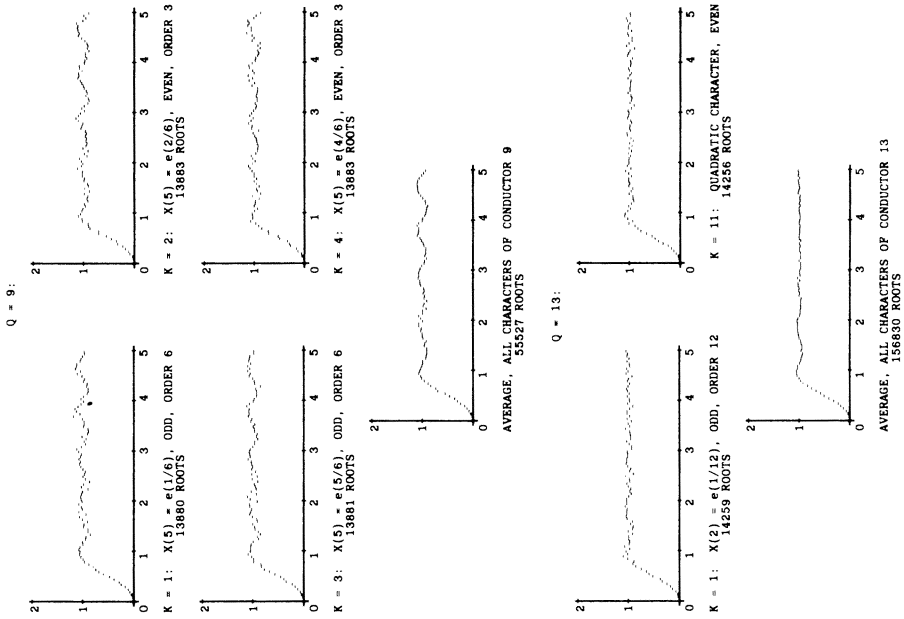


FIGURE 5.6 (Cont.): ROOTS MODULO 1 AND PAIR CORRELATION FUNCTIONS

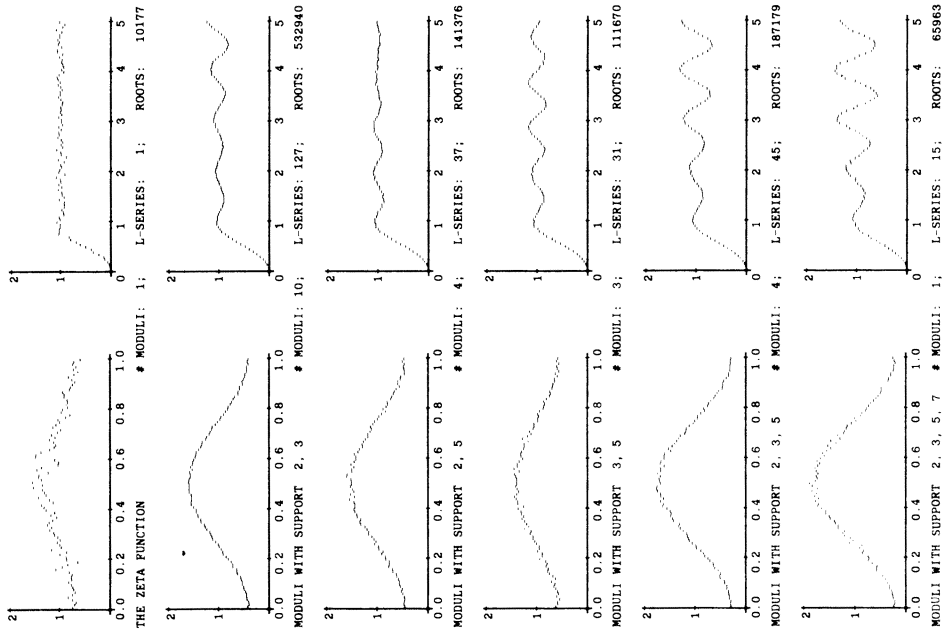


FIGURE 5.8: EXTENDED PLOTS OF PAIR CORRELATION FUNCTIONS

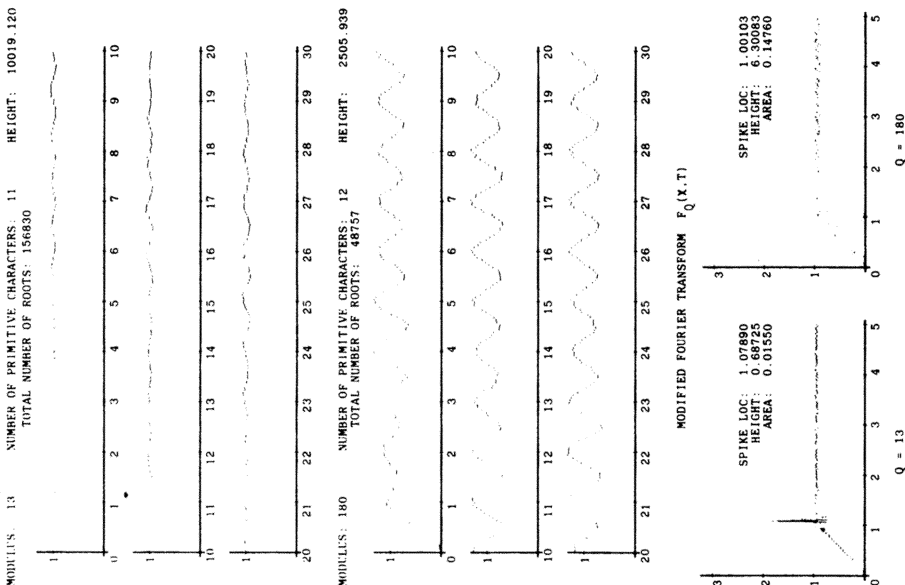


FIGURE 5.9: NEAREST NEIGHBOR AND SECOND NEAREST NEIGHBOR GAP DISTRIBUTIONS

NEAREST NEIGHBOR MOMENTS:

MOMENT TYPE	MULTIPLES OF 12	ALL MODULI	PRIME MODULI	ON GUE
2	0.138	0.149	0.158	0.180
3	0.011	0.022	0.030	0.038
4	0.052	0.067	0.079	0.101
5	0.014	0.031	0.045	0.066
6	0.031	0.055	0.074	0.111
7	0.016	0.046	0.072	0.124
8	0.026	0.068	0.105	0.197
9	0.019	0.078	0.132	0.290
10	0.027	0.112	0.194	0.488
LOG	-0.082	-0.086	-0.090	-0.104
1/7	1.728	1.926	2.254	3.544
1/7***2	2.185	2.247	2.254	2.563

SECOND NEAREST NEIGHBOR MOMENTS:

MOMENT TYPE	MULTIPLES OF 12	ALL MODULI	PRIME MODULI	ON GUE
2	0.140	0.181	0.206	0.249
3	-0.002	0.019	0.029	0.030
4	0.055	0.097	0.124	0.185
5	-0.003	0.031	0.050	0.073
6	0.034	0.087	0.123	0.237
7	-0.003	0.050	0.085	0.185
8	0.028	0.108	0.167	0.451
9	-0.004	0.090	0.158	0.544
10	0.028	0.169	0.280	1.178

SECOND NEAREST NEIGHBOR DENSITY PLOTS:

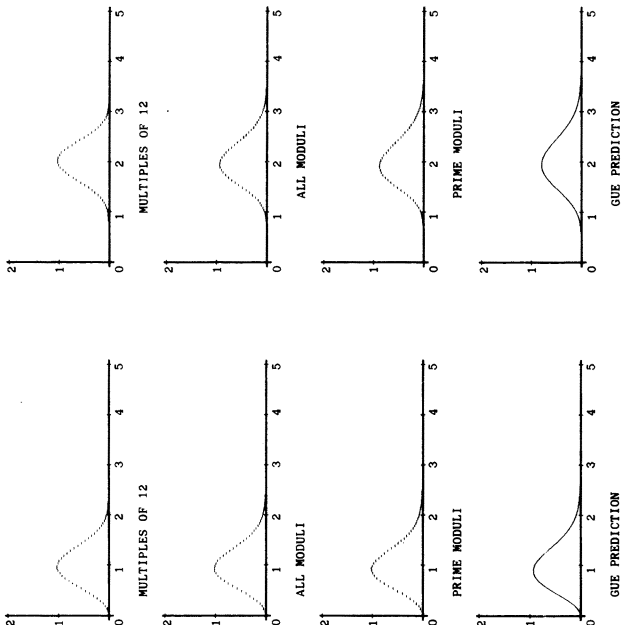


FIGURE 5.11: NEAREST NEIGHBOR GAP DISTRIBUTIONS FOR SEVERAL L-SERIES

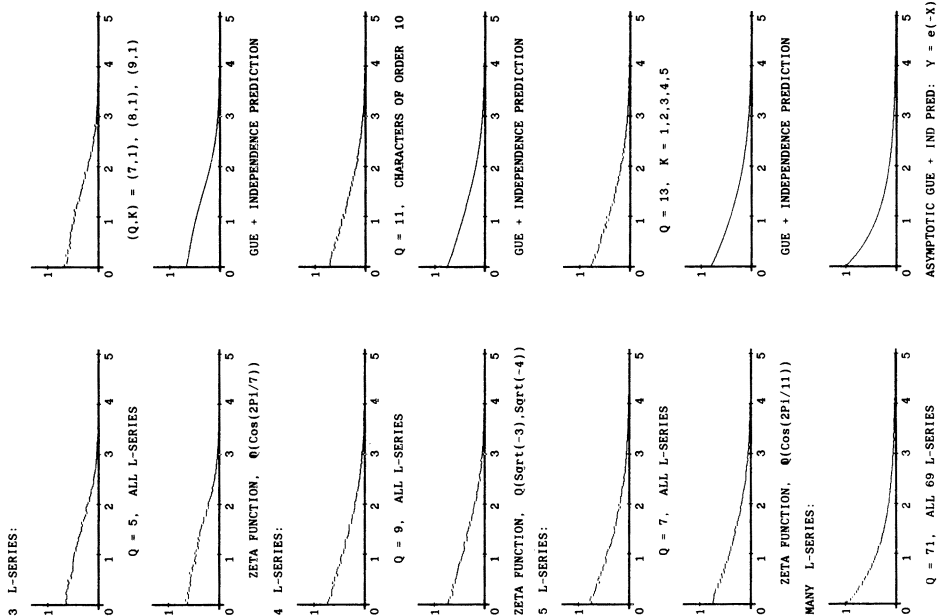


FIGURE 5.10: NEAREST NEIGHBOR GAP DISTRIBUTIONS FOR PAIRS OF L-SERIES

