MATHEMATICS OF COMPUTATION
VOLUME 61, NUMBER 204
OCTOBER 1993, PAGES 799–820

# LATTICE RULES BY COMPONENT SCALING

J. N. LYNESS AND T. SØREVIK

ABSTRACT. We introduce a theory of rectangular scaling of integer lattices. This may be used to construct families of lattices. We determine the relation between the Zaremba index $\rho(\Lambda)$ of various members of the same family. It appears that if one member of a family has a high index, some of the other family members of higher order may have extraordinarily high indices.

We have applied a technique based on this theory to lists of good lattices available to us. This has enabled us to construct lists of excellent previously unknown lattices of high order in three and four dimensions and of moderate order in five dimensions.

## 1. BACKGROUND

The purpose of this paper is to find $s$-dimensional integer lattices $\Lambda$ that have relatively high Zaremba indices or figures of merit. This index may be defined in terms of absolute values of the nonzero components of a lattice point $\mathbf{x} = (x_1, x_2, \dots, x_s)$.

**Definition 1.** The product coordinate distance function of $\mathbf{x}$ is

$$\rho(\mathbf{x}) = \overline{x}_1 \overline{x}_2 \cdots \overline{x}_s, \tag{1.1}$$

where

$$\overline{x}_i = \max(|x_i|, 1). \tag{1.2}$$

In terms of this, we have the following definition.

**Definition 2.** The Zaremba index or figure of merit, $\rho(\Lambda)$, of an $s$-dimensional integer lattice is

$$\rho(\Lambda) = \min_{\mathbf{x}\in\Lambda;\, \mathbf{x}\neq 0} \rho(\mathbf{x}). \tag{1.3}$$

Note that all lattice points of an integer lattice have integer components. Thus $\rho(\mathbf{x})$ and $\rho(\Lambda)$ are positive integers. In §3 we shall generalize this definition to other point sets.

Received by the editor February 18, 1992 and, in revised form, August 26, 1992.

1991 *Mathematics Subject Classification*. Primary 65D32, 06B99.

*Key words and phrases.* Lattice rules, number-theoretic rules, Zaremba index, figure of merit, scaled lattice.

This work was supported by the Applied Mathematical Sciences subprogram of the Office of Energy Research, U.S. Department of Energy, under Contract W-31-109-Eng-38, and by the Norwegian Council for Humanities and Science.

The Zaremba index, $\rho(\Lambda)$, is a recognized measure of efficiency of the multidimensional quadrature rule based on $\Lambda_Q$, the lattice reciprocal to $\Lambda$. This lattice rule employs all $N$ points of $\Lambda_Q$ lying in $[0, 1)^s$, $N$ being the order of $\Lambda$. Thus, attention has been devoted to searching in $\mathcal{L}_s(N)$, the set of $s$-dimensional integer lattices of order $N$, with a view to finding optimal lattices of this order, that is, lattices $\Lambda'$ for which $\rho(\Lambda')$ coincides with

$$\rho_s(N) = \max_{\Lambda \in \mathcal{L}_s(N)} \rho(\Lambda). \tag{1.4}$$

It is convenient to introduce a "measure of goodness" of a lattice by means of which one can compare lattices having different values of $N$. Our measure is based on the Zaremba [9] conjecture that suggests that there exists a constant $z_s$ such that

$$\rho_s(N) \geq z_s \frac{N}{(\log N)^{s-2}}, \qquad s \geq 2. \tag{1.5}$$

Following Kedem and Zaremba [2], we assign to every lattice a value defined by

$$z(\Lambda) = \frac{\rho(\Lambda)(\log N)^{s-2}}{N}. \tag{1.6}$$

This is of course nothing more than a scaled version of $\rho(\Lambda)$. However, this value is useful when examining a list that contains lattices of different orders $N$ to recognize quickly those lattices which have an outstanding value of $\rho$.

This paper is not directly concerned with the conjectures on which such estimates are based. We note, however, that there exists a bound on $\rho_s(N)$ of order $O(N/(\log N)^{s-1})$ [10] and that both the conjecture and bound are in the context of number-theoretic rules; that is, they are restricted to lattice rules of rank 1.

For an account of the general theory, of which this conjecture forms part, we refer the reader to recent papers by Niederreiter [6, 7], who has extended the theory to cover lattice rules of general rank. This developing theory is mainly devoted to existence proofs and asymptotic bounds. The present paper, on the other hand, is devoted to providing concrete examples of lattices whose reciprocal may be used to construct cost-effective lattice rules. These examples seem to confirm the theory and are in compliance with the truth of the conjecture.

In our searches [3] and [4], each integer lattice $\Lambda$ is represented by an $s \times s$ generator matrix $B$. All elements of $\Lambda$ are integer weighted sums of the rows of $B$, and $\Lambda$ is of order $N = |\det B|$. The lattice $\Lambda_Q$ on which the corresponding lattice rule is based has a generator matrix $A = (B^{-1})^T$.

## 2. Component scaled lattices

**Theorem 3.** *Given $s$ nonzero and real numbers $k_1, k_2, \dots, k_s$ and a lattice $\Lambda$, there exists a lattice $\Lambda'$ such that*

$$\mathbf{p} = (p_1, p_2, \dots, p_s) \in \Lambda \Leftrightarrow \mathbf{p}' = (k_1 p_1, k_2 p_2, \dots, k_s p_s) \in \Lambda'. \tag{2.1}$$

The proof is almost trivial, whatever definition of a lattice is invoked.

**Definition 4.** The lattice $\Lambda'$ in the theorem is termed a *rectangularly scaled version* of $\Lambda$, obtained by using an $s$-dimensional *scaling factor* $(k_1, k_2, \dots, k_s)$

or a *scaling matrix* $K = \operatorname{diag}(k_1, k_2, \dots, k_s)$. $N_K = |\det(K)| = |\prod_{i=1}^{s} k_i|$ is called the *order* of this scaling.

A special case of rectangular scaling occurs when all components of the scaling factor are equal. In this case the scaling matrix $mI$ is a multiple of the unit matrix $I$; the subsequent theory can then be applied in the context of the $m^s$-copy rules discussed in [8].

Rectangular scaling of a lattice has several trivial and obvious properties. In particular, a set of successive scaling operations is itself a scaling operation, and the scaling operation is commutative. If $B$ is a generator matrix for $\Lambda$, then $BK$ is one for $\Lambda'$. When $\Lambda$ and $\Lambda'$ are scaled versions of one another, so are their reciprocal lattices, $\Lambda^{\perp}$ and $\Lambda^{\perp'}$; the scaling matrices involved are inverses of one another.

It appears that, when one confines oneself to the set of integer lattices, one may construct distinct families of lattices, in which each member is a rectangularly scaled version of every other member. Each family is specified by a unique family root lattice $\Lambda$.

**Definition 5.** A *family root* lattice is one whose generator matrix, $B$, has columns each of whose greatest common divisor is $1$.

Other members of the family are precisely those whose generator matrices are $B' = B\operatorname{diag}(k_1, k_2, \dots, k_s)$ with $k_i$ integer. To determine to which family some integer lattice $\Lambda'$ belongs, one takes its generator matrix $B'$ and calculates the greatest common divisors, $h_1, h_2, \dots, h_s$ of its columns. Then the matrix $B = B'\operatorname{diag}(h_1^{-1}, h_2^{-1}, \dots, h_s^{-1})$ is a generator matrix of the family root lattice that generates the family to which $\Lambda'$ belongs.

We are interested in the relation between $\rho(\Lambda)$ and $\rho(\Lambda')$.

We consider first the scaling of only one coordinate using a scaling factor $(k_1, 1, \dots, 1)$ with $k_1 > 1$. As mentioned before, corresponding to every point $\mathbf{x} = (x_1, x_2, \dots, x_s)$ of $\Lambda$ is a point $\mathbf{x}' = (x_1', x_2', \dots, x_s')$. Applying Definition 1, we find

$$\rho(\mathbf{x}') = \begin{cases} k_1\rho(\mathbf{x}) & \text{when } x_1 \neq 0, \\ \rho(\mathbf{x}) & \text{when } x_1 = 0. \end{cases}$$

It follows from this and Definition 2 that

$$\rho(\Lambda) \leq \rho(\Lambda') \leq k_1\rho(\Lambda). \tag{2.2}$$

The possibility of successive scaling in each coordinate in turn, and the commutative property of the scaling operation, allows us to state the following theorem.

**Theorem 6.** *Let $\Lambda'$ be a rectangularly scaled version of $\Lambda$ obtained by using a scaling factor $\mathbf{k} = (k_1, k_2, \dots, k_s)$ with each $k_i \geq 1$. Then*

$$N' = k_1k_2\cdots k_sN \tag{2.3}$$

*and*

$$\rho(\Lambda) \leq \rho(\Lambda') \leq k_1k_2\cdots k_s\rho(\Lambda). \tag{2.4}$$

We note that, under the hypotheses of this theorem, $\rho(\Lambda')/N'$ cannot exceed $\rho(\Lambda)/N$. However, since

$$z(\Lambda) = \frac{\rho(\Lambda)(\log N)^{s-2}}{N}, \tag{2.5}$$

we see that if, in fact, $\rho(\Lambda')/N' = \rho(\Lambda)/N$, then the $z$-value of $\Lambda'$ is greater than the $z$-value of $\Lambda$; in this case, if $\Lambda$ is a "good" lattice, then $\Lambda'$ is better. Because of this, the present authors decided to carry out scaling of lattices already known to be good lattices, to see whether we could discover some better lattices or "good" lattices of higher order.

In its simplest terms, the idea developed in this paper is to take a set of lattices that are known to be good, scale them in various ways, and inspect the scaled lattices (which are generally of higher order) to see whether any of them are good. In some cases, if we are lucky, we may find that the value of $z(\Lambda')$ is close to or even exceeds the upper limit of those already known. In this very fortunate case the new lattice is relatively as good or even better than the original lattice and has a higher order.

The underlying philosophy of this approach is that a calculation of this sort, while nontrivial, is orders of magnitude shorter than, for example, a direct search to find $\rho_s(N')$. If in a minor proportion of the calculations, say one in a thousand, we find a good lattice, the whole calculation can be termed a success.

The organization of this search requires some care. One can find lattices with arbitrarily high indices $\rho(\Lambda)$ by making $N$ sufficiently large. To see this, simply consider the scaled versions of the unit lattice $\Lambda_0$. The lattice $\Lambda' = k\Lambda_0$ (which can be obtained from $\Lambda_0$ using $\mathbf{k} = (k, k, \dots, k)$) has $\rho(\Lambda) = k$ and $N(\Lambda) = k^s$.

In providing guidelines for the scope of the search, the following theorem is helpful.

**Theorem 7.** *Under the hypothesis of the previous theorem,*

$$\rho(\Lambda') \le N(N'/N)^{1/s}. \tag{2.6}$$

*Proof.* All integer lattices of order $N$ contain the sublattice $N\Lambda_0$ (where $\Lambda_0$ is the unit lattice). Thus, $\Lambda'$ contains each of the points $(k_1 N, 0, 0, \dots, 0)$, $(0, k_2 N, 0, \dots, 0), \dots, (0, 0, \dots, k_s N)$, and it follows that

$$\rho(\Lambda') \le N k_i, \qquad i = 1, 2, \dots, s. \tag{2.7}$$

Since all $k_i$ are positive integers, we may take the geometric mean of the $s$ equations in (2.7) and, using (2.3), obtain (2.6). □

In order to make the search finite, we choose a lower bound $\bar{\bar{z}}$ specified in (5.1) below and limit the search to lattices $\Lambda'$ for which $z(\Lambda') > \bar{\bar{z}}$. From the theorem we see that

$$z(\Lambda') \le (N/N')^{1-1/s} \log^{s-2} N', \tag{2.8}$$

and so it is bounded by a quantity that approaches zero with large $N'$. Thus, since $N'$ is restricted to integer multiples of $N$, the number of family members to be treated is finite. In fact, elementary manipulation yields the following lemma.

**Lemma 8.** *We have* $z(\Lambda') < \bar{\bar{z}}$ *when*

$$N'/\log^{s-1} N' > N/\bar{\bar{z}} \tag{2.9}$$

*and*

$$\bar{\bar{z}} \log N' > 1. \tag{2.10}$$

This is a direct consequence of (2.8) and is readily established by eliminating $N$ between inequalities (2.8) and (2.9). In practice, (2.10) is satisfied trivially, and so we may restrict our search to values of $N'$ violating (2.9).

## 3. Scaling an individual lattice

In §§4–6 we shall describe and analyze results obtained by scaling lists of lattices, all of which are reasonably good lattices. In this section we present a more detailed theory about rectangular scaling. The thrust of this section is to provide information by means of which $\rho(\Lambda')$ for a family of lattices may be readily calculated. Readers interested principally in the results of our numerical experiments may omit this section in a first reading. Without loss in generality we shall assume as before that the lattice $\Lambda$ is a family root lattice (see Definition 5) and that $\Lambda'$ is a scaled version obtained using a scaling factor $\mathbf{k}$, all of whose components are positive integers.

The behavior of $\rho(\Lambda')$ as a function of $\mathbf{k}$ is given by the function in (3.1) below.

**Theorem 9.** *Under the hypotheses of the preceding theorems, there exist $2^s - 1$ positive integers $A, A_1, \dots, A_{23\cdots s}$, which depend only on $\Lambda$, such that*

$$(3.1)\quad \frac{\rho(\Lambda')}{N'} = \frac{1}{N}\min\left(A, \frac{A_1}{k_1}, \frac{A_2}{k_2}, \frac{A_3}{k_3}, \dots, \frac{A_{12}}{k_1k_2}, \frac{A_{13}}{k_1k_3}, \dots, \frac{A_{23}}{k_2k_3}, \dots\right).$$

Note that these denominators comprise all distinct products of up to $s-1$ distinct components of $\mathbf{k}$. There is no term in $(k_1k_2\cdots k_s)^{-1}$.

Note also that (3.1), while implicitly containing many inequalities, actually defines a function of $\mathbf{k}$. The rest of this section is devoted to establishing Theorem 9 and to showing how to calculate a set of constants $A_{i,j,\dots}$ from a generator matrix $B$ of $\Lambda$. It will appear that each coefficient $A_{i,j,\dots}$ can be conveniently defined in terms of functions of the form $\rho(S)$, where $S$ is a specified set of points belonging to $\Lambda_0$ and

$$(3.2)\qquad \rho(S) = \min_{\mathbf{x}\in S;\,\mathbf{x}\neq 0} \rho(\mathbf{x}),$$

where $\rho(\mathbf{x})$ as defined in §1 is the absolute value of the product of the nonzero components of $\mathbf{x}$. This notation is a natural generalization of that introduced in Definition 2.

We now partition the elements of $\Lambda$ into $2^s$ distinct sets. We distinguish these using an $s$-component binary index $\mathbf{u} = (u_1, u_2, \dots, u_s)$, that is, one in which each component is either 1 or 0.

**Definition 10.** Let $\Lambda$ be an $s$-dimensional integer lattice and $\mathbf{t}$ an $s$-component binary index. Then

$$(3.3)\qquad \Gamma^{(\mathbf{t})} = \{\mathbf{x} | \mathbf{x} \in \Lambda \text{ and } x_i = 0 \text{ when } t_i = 0 \text{ and } x_i \neq 0 \text{ when } t_i \neq 0\}.$$

Note that $\Gamma^{(\mathbf{t})}$ is not a lattice and $\Gamma^{(0,0,\dots,0)}$ is the single point $(0, 0, \dots, 0)$. These $2^s$ distinct sets form a partition of $\Lambda$; that is,

$$(3.4)\qquad \Lambda = \bigcup_{\substack{u_i=0,1\\ 1\le i\le s}} \Gamma^{(\mathbf{u})}.$$

This partition has been constructed with the following property in view.

**Lemma 11.** *When* $(x_1, x_2, \dots, x_s) \in \Gamma^{(\mathbf{u})}$,

$$\rho(k_1x_1, k_2x_2, \dots, k_sx_s) = k^{(\mathbf{u})}\rho(x_1, x_2, \dots, x_s), \tag{3.5}$$

*where*

$$k^{(\mathbf{u})} = k_1^{u_1}k_2^{u_2}\cdots k_s^{u_s} \tag{3.6}$$

*is the product of those components* $k_i$ *of* $\mathbf{k}$ *for which* $u_i = 1$.

**Theorem 12.** *Under the hypotheses of Theorem* 6,

$$\rho(\Lambda') = \min_{\substack{u_i=0,1\\1\le i\le s\\\mathbf{u}\ne 0}} (k^{(\mathbf{u})}\rho(\Gamma^{(\mathbf{u})})), \tag{3.7}$$

*where these quantities are defined in* (3.6), (3.2), *and* (3.3).

*Proof*. The theorem follows because

$$\begin{aligned}\rho(\Lambda') &= \min_{\substack{\mathbf{x}\in\Lambda\\\mathbf{x}\ne 0}} \rho(k_1x_1, k_2x_2, \dots, k_sx_s)\\ &= \min_{\substack{u_i=0,1\\1\le i\le s\\\mathbf{u}\ne 0}} \min_{\mathbf{x}\in\Gamma^{(\mathbf{u})}} \rho(k_1x_1, k_2x_2, \dots, k_sx_s).\end{aligned} \tag{3.8}$$

The first equality above follows from the definition of $\rho$ and of the scaled lattice. The second follows from the partition (3.4) above. When we apply successively Lemma 11 and (3.2), we find that the expression on the right in (3.8) reduces to the right-hand side of (3.7). □

Theorem 9 may be obtained from this theorem by simply dividing by $N' = k_1k_2\cdots k_sN$ and recognizing that, when $\mathbf{u} \ne 0$, the set $\Gamma^{(\mathbf{u})}$ is not empty and $\rho(\Gamma^{(\mathbf{u})})$ is a positive integer.

One readily identifies

$$A_{i,j,\dots} = \rho(\Gamma^{(\mathbf{u})}), \tag{3.9}$$

where $\mathbf{u}$ is the binary index that has zeros in positions corresponding to $i, j, \dots$, the subscripts of $A$, and units elsewhere.

Since the point set $\Gamma^{(\mathbf{u})}$ contains the point $(u_1N, u_2N, \dots, u_sN)$, it follows that when $\mathbf{u} \ne 0$,

$$1 \le \rho(\Gamma^{(\mathbf{u})}) \le N^{u_1+u_2+\cdots+u_s}, \tag{3.10}$$

and (3.7) supports the $2^s - 1$ inequalities

$$\begin{aligned}&\rho(\Lambda') \le (k_1N)^{u_1}(k_2N)^{u_2}\cdots(k_sN)^{u_s},\\ &\qquad\qquad \mathbf{u}\ne 0,\ u_i = 0, 1,\ i = 1, 2, \dots, s.\end{aligned} \tag{3.11}$$

There is a somewhat unexpected reformulation of Theorem 12. We recall that the points of $\Gamma^{(\mathbf{u})}$ of Definition 10 do not form a lattice. We may, however, form a lattice $\Lambda^{(\mathbf{u})}$ from the points of $\Gamma^{(\mathbf{u})}$ by adding all points of the form $\mathbf{x}\pm\mathbf{y}$, where $\mathbf{x}, \mathbf{y} \in \Gamma^{(\mathbf{u})}$, and iterating. This turns out to be a $(u_1+u_2+\cdots+u_s)$-dimensional projection of $\Lambda$, defined by the following

**Definition 13.** Let $\Lambda$ be an $s$-dimensional lattice and $\mathbf{t}$ an $s$-component binary index. Then

$$\Lambda^{(\mathbf{t})} = \{\mathbf{x}|\mathbf{x} \in \Lambda \text{ and } x_i = 0 \text{ when } t_i = 0\}. \tag{3.12}$$

It follows quite simply that partition (3.4) of $\Lambda$ induces a similar partition of $\Lambda^{(\mathbf{t})}$, namely,

$$\Lambda^{(\mathbf{t})} = \bigcup_{\substack{0\le u_i\le t_i \\ 1\le i\le s}} \Gamma^{(\mathbf{u})}, \tag{3.13}$$

from which it follows that

$$\rho(\Lambda^{(t)}) = \min_{\substack{0\le u_i\le t_i \\ 1\le i\le s \\ \mathbf{u}\neq 0}} \rho(\Gamma^{(\mathbf{u})}). \tag{3.14}$$

**Lemma 14.** *For a given $s$-dimensional lattice $\Lambda$ and $s$-dimensional binary index $\mathbf{t}$,*

$$\min_{\substack{0\le u_i\le t_i \\ 1\le i\le s \\ \mathbf{u}\neq 0}} k^{(\mathbf{u})}\rho(\Gamma^{(\mathbf{u})}) = \min\left( k^{(\mathbf{t})}\rho(\Lambda^{(\mathbf{t})}),\ \min_{\substack{0\le u_i\le t_i \\ 1\le i\le s \\ \mathbf{u}\neq \mathbf{t} \\ \mathbf{u}\neq 0}} k^{(\mathbf{u})}\rho(\Gamma^{(\mathbf{u})}) \right), \tag{3.15}$$

*where $\Lambda^{(\mathbf{t})}$ and $\Gamma^{(\mathbf{u})}$ are defined in terms of $\Lambda$ in Definitions* 13 *and* 10, *and $\rho$ is defined in* (3.2).

The reader will recognize that the two sides of equation (3.15) differ only in that a single term has been changed.

*Proof.* To establish the lemma, we take the right-hand side of (3.15) and replace the cofactor of $k^{(\mathbf{t})}$ by the expression given in (3.14). This procedure leaves us with an expression involving two somewhat similar sets of terms. By inspection we see that, except for the principal term in which $\mathbf{u} = \mathbf{t}$, there are a pair of terms corresponding to each $\mathbf{u}$, one of which has a factor $k^{(\mathbf{t})}$ and the other $k^{(\mathbf{u})}$. In all cases $k^{(\mathbf{u})} \le k^{(\mathbf{t})}$, and the first term can be discarded. Doing this leaves the expression on the left-hand side of (3.15) and so establishes the lemma. □

**Theorem 15.** *There holds*

$$\rho(\Lambda') = \min_{\substack{u_i=0,1 \\ 1\le i\le s \\ \mathbf{u}\neq 0}} (k^{(\mathbf{u})}\rho(S^{(\mathbf{u})})),$$

*where $S$ stands for $\Gamma$ or $\Lambda$ and may be chosen variously in each of $2^s - 1$ terms.*

*Proof.* One may successively apply the lemma to the right-hand side of (3.7). Each application alters one $\Gamma$ to $S$. The lemma must be applied in a proper order. Any ordering in which all terms having $\sum t_i = d$ are treated before any having $\sum t_i > d$ with $d = 1, 2, \ldots, s$ is suitable. □

Theorem 15 sets the stage for the calculation of $\rho(\Lambda')$ in the situation in which $\Lambda$ is defined by a generator matrix $B$ in *utlf* (Hermite normal form) and in which software is available to calculate $\rho(\Lambda)$ for up to $s$-dimensional lattices from its $B$ matrix. The problem is to identify a generator matrix of $\Lambda^{(\mathbf{t})}$.

Let $B$ be in $utlf$ and the binary index vector $\mathbf{t} = (0, 0, \dots, 0, 1, 1, \dots, 1)$ be a string of $s - \sigma$ zeros followed by a string of $\sigma$ ones with, of course, $\sigma = \sum_{i=1}^{s} t_i$. In this case it is almost self-evident that a generator matrix of $\Lambda^{(\mathbf{t})}$ is obtained by replacing the first $s - \sigma$ rows of $B$ by zeros. Thus, $\rho(\Lambda^{(\mathbf{t})})$ may be obtained by applying the software to the $\sigma$-dimensional lattice whose generator is the $\sigma \times \sigma$ lower right-hand minor of $B$.

When $\mathbf{t} = (t_1, t_2, \dots, t_s)$ is not of that form, we exploit the circumstance that $\rho$ is invariant under permutations of the coordinate system. Thus, let $P$ be an $s \times s$ permutation matrix, set $\mathbf{t}' = P\mathbf{t}$, $\overline{B} = BP$, and let $\overline{\Lambda}$ be the lattice whose generator matrix is $\overline{B}$. Then $\rho(\Lambda^{(\mathbf{t})}) = \rho(\overline{\Lambda}^{(\mathbf{t}')})$. Thus, one finds the permutation $P$ which takes $\mathbf{t}$ into $\mathbf{t}'$ of form $(0, 0, 0, 1, \dots, 1)$, applies it to the columns of $B$ to obtain $\overline{B}$, and then puts $\overline{B}$ in $utlf$. This problem is now reduced to the one described in the preceding paragraph.

In our numerical calculations, in pilot schemes we calculated each $\rho(\Lambda')$ individually using our own software. However, applying the results of the previous two paragraphs led to a much faster code. For each root lattice $\Lambda$ we calculated $2^s - 1$ constants required in (3.1). This involved calculating only one $s$-dimensional figure of merit $A = \rho(\Lambda)$, the other constants $A_{i,j,\dots}$ being lower-dimensional figures of merit. Then we relied on (3.1) to calculate $z(\Lambda')$ for all lattices $\Lambda'$ in which we were interested. These included at most those with $N'$ violating (2.9).

## 4. The highlight lists

Applying the technique of §3, we have found apparently endless lists of lattices, hundreds of which are excellent or interesting by previously acceptable standards. In order not to overwhelm the reader, we are presenting our results in two parts. In this section we present two "highlight" lists. These include three- and four-dimensional lattices with exceptionally high $z$-values and also lattices with moderate $z$-values but exceptionally high values of $N$.

In §5 we shall give in more detail some of the actual results and explain precisely how they were obtained; then in §6 we shall comment on some aspects of these results.

To provide criteria for our lists, we have defined an $s$-dimensional benchmark lattice as follows:

**Definition 16.** The $s$-dimensional lattice $\overline{\Lambda}_s$ of order $2^{s+1}$ whose generator matrix in $utlf$ is

$$B(\overline{\Lambda}_s) = \begin{pmatrix} 2 & 0 & \cdots & 0 & 2 \\ 0 & 2 & \cdots & 0 & 2 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 2 & 2 \\ 0 & 0 & \cdots & 0 & 4 \end{pmatrix} \tag{4.1}$$

is termed the $s$-dimensional benchmark lattice.

Clearly, $\rho(\overline{\Lambda}_s) = 4$ and

$$\overline{z}_s = z(\overline{\Lambda}_s) = \frac{1}{2}\left(\frac{(s+1)\log 2}{2}\right)^{(s-2)}. \tag{4.2}$$

The authors have introduced this benchmark simply because it is convenient in the context of discussing our lists of lattice rules. No intrinsic mathematical property is implied or conjectured.

The highlight lists include:

1. all $s$-dimensional lattices $\Lambda$ known to us having $z(\Lambda) \geq \overline{z}_s$, and
2. all $s$-dimensional lattices $\Lambda$ known to us satisfying both
   - $z(\Lambda) > \frac{2}{3}\overline{z}_s$ and
   - $z(\Lambda) > z(\widetilde{\Lambda})$ for all $\widetilde{\Lambda}$ known to us whose order, $\widetilde{N}$, exceeds $N$, the order of $\Lambda$.

Tables 1–8, A1, and A2 contain lists of lattices. Each line corresponds to a single lattice $\Lambda$. The $s(s+1)/2$ entries which follow $N$ and $\rho$ are elements of an upper triangular matrix $B$. This is the upper triangular lattice form or Hermite normal form of any generator matrix of $\Lambda$ (see the remarks at the end of §1). Then comes the rank of the corresponding lattice rule. In this column an entry 0 indicates rank 1 simple (see [3]), and an entry 1 indicates rank 1 not simple. An $s$-dimensional copy rule can be recognized as one having rank $s$. (See [8] for full discussions of rank and of copy rules.)

In Tables 1 and 2, we identify the list from which this lattice was taken. These lists are specified in §5; the abbreviations are B = Blue, G = Green, SG = Scaled Green, and SB = Scaled Blue.

The authors must emphasize that these are lists of lattices that *happen to be known to us at this time.* In §6 we shall discuss the question of how many other lattices there may be that belong on such a list but have not been encountered yet. Only for $N < \overline{N}$ (= 4000 for $s = 3$ and 600 for $s = 4$) are these lists complete.

It is of interest to note the extremely disparate values of $N$ involved. From complete lists of optimal lattices of order up to $\overline{N}$, we obtain excellent lattices of order up to say $5\overline{N}$, a few of these being better than any found previously. After this, the list degrades in quality only slowly, containing lattices of good (but not top) quality up to order $50\overline{N}$.

The tail of the list is unlikely to include any optimal lattices at all. However, for these extraordinarily high orders, an example of a lattice of moderate quality is of some interest.

Undoubtedly, the most outstanding lattices on these lists are

(i) a three-dimensional lattice having $N = 9760$, $\rho = 864$, and $z = 0.81319$, and
(ii) a four-dimensional lattice having $N = 8992$, $\rho = 212$, and $z = 1.95413$.

The results of scaling the short list [1] of five-dimensional rank-1 simple lattices were relatively unexciting. Possible reasons for this are mentioned in §5. We found no lattices whose $z$-values exceeded $\overline{z}_5$, and only 25 whose $z$-values exceeded $\frac{2}{3}\overline{z}_5$. We have listed in Table 8 all the lattices known to us whose $z$-values exceed $\frac{2}{3}\overline{z}_5$.

TABLE 1. A highlight list of three-dimensional lattices

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{22}$ | $b_{23}$ | $b_{33}$ | rank | $z$ | source |
|---|---|---|---|---|---|---|---|---|---|---|
| 16 | 4 | 2 | 0 | 2 | 2 | 2 | 4 | 3 | 0.69315 | G |
| 1672 | 160 | 2 | 0 | 90 | 2 | 130 | 418 | 3 | 0.71022 | G |
| 2352 | 216 | 2 | 0 | 111 | 2 | 171 | 588 | 2 | 0.71293 | G |
| 3069 | 270 | 1 | 1 | 464 | 3 | 168 | 1023 | 2 | 0.70637 | G |
| 4704 | 390 | 2 | 0 | 228 | 3 | 148 | 784 | 2 | 0.70109 | SG,SB |
| 4880 | 432 | 1 | 0 | 638 | 2 | 1002 | 2440 | 2 | 0.75183 | SG,SB |
| 4900 | 400 | 1 | 0 | 452 | 2 | 748 | 2450 | 2 | 0.69363 | SG |
| 8922 | 700 | 1 | 0 | 823 | 2 | 1362 | 4461 | 1 | 0.71367 | SB |
| 9760 | 864 | 2 | 0 | 638 | 2 | 1002 | 2440 | 3 | 0.81319 | SG,SB |
| 9800 | 800 | 2 | 0 | 452 | 2 | 748 | 2450 | 3 | 0.75022 | SG |
| 17844 | 1356 | 1 | 0 | 2656 | 2 | 4062 | 8922 | 2 | 0.74392 | SB |
| 19416 | 1404 | 1 | 0 | 1431 | 2 | 3540 | 9708 | 2 | 0.71399 | SG |
| 20008 | 1440 | 2 | 0 | 1314 | 2 | 2048 | 5002 | 3 | 0.71279 | SG |
| 45576 | 2968 | 2 | 0 | 1658 | 2 | 4192 | 11394 | 3 | 0.69857 | SB |
| 48264 | 2864 | 2 | 0 | 1820 | 3 | 1184 | 8044 | 2 | 0.63995 | SG |
| 67410 | 3822 | 2 | 0 | 2469 | 3 | 1548 | 11235 | 2 | 0.63040 | SG |
| 67527 | 3762 | 3 | 0 | 1971 | 3 | 3072 | 7503 | 3 | 0.61952 | SG |
| 68238 | 3678 | 2 | 0 | 3441 | 3 | 5328 | 11373 | 2 | 0.59994 | SG |
| 90984 | 4680 | 2 | 0 | 5368 | 3 | 3680 | 15164 | 2 | 0.58734 | SG |
| 109050 | 5310 | 2 | 0 | 7940 | 3 | 3080 | 18175 | 1 | 0.56482 | SG |
| 130860 | 5808 | 2 | 0 | 9528 | 3 | 3696 | 21810 | 2 | 0.52292 | SG |
| 153819 | 6678 | 3 | 0 | 2487 | 3 | 6288 | 17091 | 3 | 0.51852 | SB |
| 160064 | 6688 | 4 | 0 | 2628 | 4 | 4096 | 10004 | 3 | 0.50070 | SG |
| 179760 | 7424 | 3 | 0 | 3292 | 4 | 2064 | 14980 | 2 | 0.49970 | SG |
| 227460 | 8880 | 3 | 0 | 6710 | 4 | 4600 | 18955 | 1 | 0.48155 | SG |

TABLE 2. A highlight list of four-dimensional lattices

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{14}$ | $b_{22}$ | $b_{23}$ | $b_{24}$ | $b_{33}$ | $b_{34}$ | $b_{44}$ | rank | $z$ | source |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 32 | 4 | 2 | 0 | 0 | 2 | 2 | 0 | 2 | 2 | 2 | 4 | 4 | 1.50142 | G |
| 928 | 32 | 2 | 0 | 0 | 34 | 2 | 0 | 44 | 2 | 52 | 116 | 4 | 1.61001 | SG |
| 992 | 32 | 2 | 0 | 0 | 20 | 2 | 0 | 46 | 2 | 54 | 124 | 4 | 1.53568 | SG |
| 1008 | 32 | 2 | 0 | 0 | 16 | 2 | 2 | 8 | 6 | 12 | 42 | 4 | 1.51832 | SG |
| 1008 | 32 | 2 | 0 | 0 | 16 | 2 | 2 | 8 | 6 | 30 | 42 | 4 | 1.51832 | SG |
| 1354 | 40 | 1 | 0 | 0 | 492 | 1 | 0 | 550 | 1 | 658 | 1354 | 0 | 1.53607 | B |
| 1748 | 48 | 1 | 0 | 0 | 286 | 1 | 0 | 360 | 1 | 472 | 1748 | 0 | 1.53074 | B |
| 2097 | 54 | 1 | 0 | 0 | 435 | 1 | 0 | 936 | 1 | 1035 | 2097 | 0 | 1.50633 | B |
| 2112 | 55 | 1 | 0 | 0 | 100 | 1 | 0 | 162 | 1 | 830 | 2112 | 0 | 1.52617 | B |
| 2215 | 60 | 1 | 0 | 0 | 257 | 1 | 0 | 448 | 1 | 558 | 2215 | 0 | 1.60730 | B |
| 2248 | 60 | 1 | 0 | 0 | 106 | 1 | 0 | 178 | 2 | 442 | 1124 | 2 | 1.58980 | SG |
| 2320 | 64 | 2 | 0 | 0 | 34 | 2 | 0 | 56 | 2 | 82 | 290 | 4 | 1.65661 | SG |
| 2477 | 63 | 1 | 0 | 0 | 128 | 1 | 0 | 701 | 1 | 915 | 2477 | 0 | 1.55328 | B |
| 2570 | 65 | 1 | 0 | 0 | 787 | 1 | 0 | 1138 | 1 | 1246 | 2570 | 0 | 1.55921 | B |
| 2686 | 66 | 1 | 0 | 0 | 852 | 1 | 0 | 1142 | 1 | 1218 | 2686 | 0 | 1.53190 | B |
| 2730 | 68 | 1 | 0 | 0 | 170 | 1 | 0 | 452 | 1 | 1328 | 2730 | 0 | 1.55928 | B |
| 2836 | 72 | 1 | 0 | 0 | 418 | 1 | 0 | 1010 | 1 | 1290 | 2836 | 0 | 1.60464 | B |
| 3298 | 84 | 1 | 0 | 0 | 535 | 1 | 0 | 701 | 1 | 937 | 3298 | 0 | 1.67153 | B |
| 4496 | 106 | 1 | 0 | 0 | 106 | 2 | 0 | 178 | 2 | 442 | 1124 | 3 | 1.66790 | SG,SB |
| 8992 | 212 | 2 | 0 | 0 | 106 | 2 | 0 | 178 | 2 | 442 | 1124 | 4 | 1.95413 | SG,SB |
| 9908 | 180 | 1 | 0 | 0 | 256 | 1 | 0 | 1402 | 2 | 1830 | 4954 | 2 | 1.53803 | SB |
| 20232 | 318 | 2 | 0 | 0 | 159 | 2 | 0 | 267 | 3 | 663 | 1686 | 2 | 1.54517 | SG,SB |
| 52768 | 672 | 2 | 0 | 0 | 1070 | 2 | 0 | 1402 | 2 | 1874 | 6596 | 4 | 1.50574 | SB |
| 267138 | 2268 | 3 | 0 | 0 | 1605 | 3 | 0 | 2103 | 3 | 2811 | 9894 | 4 | 1.32561 | SB |
| 474912 | 3024 | 3 | 0 | 0 | 2140 | 3 | 0 | 3748 | 4 | 2804 | 13192 | 2 | 1.08787 | SB |

## 5. LISTS OF SCALED LATTICES

In the preceding section we presented two short lists that included the best lattices we have found so far. These were extracted from results that we outline in detail in this section. This is in order that subsequent workers can relate their investigation to ours for purposes that may include confirming or extending our work.

All our work involves taking a list of lattices and treating each member of the list in the way described in §3. We now describe the seven different lists that were used as input. The three blue lists (containing only rank-1 simple lattices) have been available in the literature for several years. The two red lists have appeared in the literature only very recently. The two green lists have not been published. Each list is in a format corresponding to that used in Table 1.

1. Three dimensions:
   - *Green list.* $N \in [16, 4000]$; contains 6557 entries. These are all the lattices in this range for which $\rho(\Lambda) = \rho_3(N)$. This list is unpublished.
   - *Red list.* $N \in [16, 3916]$; contains 68 entries. This is a subset of the green list above, containing entries for $N$ if and only if $\rho_3(N) > \rho_3(M)$ for all $M < N$. This list is published in [3].
   - *Blue list.* $N \in [21, 6066]$; contains 101 entries. This is a concatenation of lists published by Maisonneuve [5] and Kedem and Zaremba [2]. It contains only rank-1 simple lattices, assembled from this subclass, using the standard red list convention described above.
2. Four dimensions:
   - *Green list.* As above; $N \in [20, 600]$; contains 16127 entries, but is not complete. It contains most lattices for which $\rho(\Lambda) = \rho_4(N)$. If there exist more than ten lattices $\Lambda$ for a single value of $N$, some may be missing, but at least ten are included. This list is unpublished.
   - *Red list.* As above; $N \in [32, 562]$; contains 23 entries. This list is published in [4].
   - *Blue list.* As above; $N \in [52, 3298]$; contains 47 entries. This is a concatenation of lists published by Maisonneuve [5] and Bourdeau and Pitre [1].
3. Five dimensions:
   - *Blue list.* $N \in [112, 772]$; contains nine entries. This list is published in [1]. Seven of these are repeated in the first part of Table 8.

Each of these seven lists were processed in the same way. This process produced three more lists from each input list. These are specified below in the case that the input list is the three-dimensional *green list.*

1. We form first a three-dimensional raw *scaled green list.* For this, we require $\bar{\bar{z}}$, a cutoff value specified in (5.1) below. This list contains each lattice with $z(\Lambda) \geq \bar{\bar{z}}$ obtained by scaling every member of the green list. This huge list includes duplicate entries, and for some values of $N$, entries with different $\rho$-values.

2. From this, by cutting out all duplicate entries and any entries for which there is another lattice of the same order with a higher value of $\rho$, we produce a *green scaled green list.* We have retained this list in our files.

3. Next we use the standard procedure to produce a *red scaled green list.* This, as usual, retains only lattices on the green scaled green list for which $\rho(\Lambda) > \rho(\widetilde{\Lambda})$ for all $\widetilde{\Lambda}$ on that list having $\widetilde{N} < N$. This list is given in Table 5.

TABLE 3. Length and scope of lists involved

| Kind of list | | Input lists | | | Green scaled lists | | | Red scaled lists | |
|---|---|---|---|---|---|---|---|---|---|
| Dim. | Type | Length | $N_{min}$ | $N_{max}$ | Length | $N_{min}$ | $N_{max}$ | Length | Table |
| | Blue | 101 | 21 | 6066 | 177 | 4044 | 153819 | 29 | 4 |
| 3-d | Green | 6557 | 16 | 4000 | 4910 | 4002 | 227460 | 80 | 5 |
| | Red | 68 | 16 | 3916 | 42 | 4032 | 31328 | 17 | A1 |
| | Blue | 46 | 52 | 3298 | 162 | 624 | 474912 | 47 | 6 |
| 4-d | Green | 16127 | 20 | 600 | 4750 | 602 | 365625 | 55 | 7 |
| | Red | 23 | 32 | 562 | 51 | 640 | 80928 | 25 | A2 |
| 5-d | Blue | 7 | 112 | 772 | 117 | 112 | 15768 | 17 | |

TABLE 4. Red scaled blue list in three dimensions

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{22}$ | $b_{23}$ | $b_{33}$ | rank | $z$ |
|---|---|---|---|---|---|---|---|---|---|
| 4108 | 270 | 1 | 0 | 556 | 1 | 1408 | 4108 | 0 | 0.546880 |
| 4142 | 312 | 1 | 0 | 966 | 1 | 1422 | 4142 | 0 | 0.627380 |
| 4358 | 336 | 1 | 0 | 1398 | 1 | 1998 | 4358 | 0 | 0.646080 |
| 4704 | 390 | 2 | 0 | 228 | 3 | 148 | 784 | 2 | 0.701090 |
| 4880 | 432 | 1 | 0 | 638 | 2 | 1002 | 2440 | 2 | 0.751830 |
| 5862 | 450 | 1 | 0 | 538 | 1 | 1902 | 5862 | 0 | 0.666040 |
| 6066 | 460 | 1 | 0 | 600 | 1 | 1581 | 6066 | 0 | 0.660540 |
| 7430 | 544 | 1 | 0 | 1039 | 2 | 1425 | 3715 | 1 | 0.652600 |
| 7664 | 552 | 1 | 0 | 1194 | 2 | 1600 | 3832 | 2 | 0.644210 |
| 7698 | 576 | 1 | 0 | 603 | 2 | 1701 | 3849 | 1 | 0.669580 |
| 7734 | 588 | 1 | 0 | 600 | 1 | 2019 | 7734 | 0 | 0.680710 |
| 8922 | 700 | 1 | 0 | 823 | 2 | 1362 | 4461 | 1 | 0.713670 |
| 9760 | 864 | 2 | 0 | 638 | 2 | 1002 | 2440 | 3 | 0.813190 |
| 14112 | 908 | 2 | 0 | 454 | 2 | 746 | 3528 | 3 | 0.614780 |
| 15328 | 958 | 2 | 0 | 728 | 2 | 1778 | 3832 | 3 | 0.602340 |
| 15396 | 1032 | 2 | 0 | 603 | 2 | 1701 | 3849 | 2 | 0.646300 |
| 17436 | 1040 | 1 | 0 | 2094 | 2 | 1299 | 8718 | 1 | 0.582530 |
| | | 1 | 0 | 1299 | 2 | 2094 | 8718 | 2 | 0.582530 |
| 17844 | 1356 | 1 | 0 | 2656 | 2 | 4062 | 8922 | 2 | 0.743920 |
| 22788 | 1484 | 1 | 0 | 1658 | 2 | 4192 | 11394 | 2 | 0.653430 |
| 26766 | 1646 | 1 | 0 | 2656 | 3 | 4062 | 8922 | 2 | 0.626940 |
| 31008 | 1680 | 2 | 0 | 2190 | 2 | 3664 | 7752 | 3 | 0.560330 |
| 32940 | 2016 | 3 | 0 | 957 | 3 | 1503 | 3660 | 3 | 0.636650 |
| 34872 | 2080 | 2 | 0 | 1299 | 2 | 2094 | 8718 | 2 | 0.623870 |
| 45576 | 2968 | 2 | 0 | 1658 | 2 | 4192 | 11394 | 3 | 0.698570 |
| 69282 | 3042 | 3 | 0 | 1206 | 3 | 3402 | 7698 | 3 | 0.489390 |
| 69744 | 3120 | 2 | 0 | 1732 | 3 | 2792 | 11624 | 2 | 0.498910 |
| 78080 | 3584 | 4 | 0 | 1276 | 4 | 2004 | 4880 | 3 | 0.517100 |
| 102546 | 4452 | 2 | 0 | 2487 | 3 | 6288 | 17091 | 2 | 0.500920 |
| 153819 | 6678 | 3 | 0 | 2487 | 3 | 6288 | 17091 | 3 | 0.518520 |

The cutoff values we used were

$$\bar{\bar{z}} = \tfrac{2}{3}\bar{z}_3 \simeq 0.46, \quad \bar{\bar{z}} = \tfrac{2}{3}\bar{z}_4 \simeq 1.00, \quad \bar{\bar{z}} = \tfrac{5}{9}\bar{z}_5 \simeq 2.50, \tag{5.1}$$

in 3, 4, and 5 dimensions, respectively. Table 3 gives some information on the length and the scope of the lists in this section.

We note that, when the input is a green or a red list of lattices with $N \in [N_{\min}, \overline{N}]$, there is no need to retain scaled lattices having $N \leq \overline{N}$ because these lattices, or better ones having the same $N'$, are available by definition on the input list. This is not the case when the input is a blue list. The input blue list comprises excellent lattices, all of which are rank-1 simple. One may well find an interesting lattice of higher rank having $N' \geq N$ but $N' < \overline{N}$. Tables 4 and 6 (in which $\overline{N} = 6066$ and $3298$, respectively) contain a handful of such lattices. These are generally of technical interest only. By including them we specify precisely the effect of scaling a blue list.

TABLE 5. Red scaled green list in three dimensions

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{22}$ | $b_{23}$ | $b_{33}$ | rank | $z$ |
|---|---|---|---|---|---|---|---|---|---|
| 4002 | 280 | 1 | 0 | 958 | 1 | 1258 | 4002 | 0 | 0.580330 |
| 4008 | 288 | 2 | 0 | 219 | 2 | 294 | 1002 | 2 | 0.596120 |
| 4044 | 308 | 1 | 0 | 400 | 1 | 1054 | 4044 | 0 | 0.632530 |
| 4050 | 312 | 1 | 3 | 178 | 9 | 410 | 450 | 1 | 0.639910 |
| 4185 | 324 | 3 | 0 | 108 | 3 | 168 | 465 | 3 | 0.645620 |
| 4358 | 336 | 1 | 0 | 1398 | 1 | 1998 | 4358 | 0 | 0.646080 |
| 4528 | 344 | 2 | 0 | 218 | 2 | 316 | 1132 | 3 | 0.639530 |
| 4580 | 348 | 1 | 0 | 348 | 1 | 2002 | 4580 | 0 | 0.640490 |
| 4588 | 360 | 1 | 0 | 808 | 2 | 588 | 2294 | 2 | 0.661560 |
| 4704 | 390 | 2 | 0 | 228 | 3 | 148 | 784 | 2 | 0.701090 |
| 4880 | 432 | 1 | 0 | 638 | 2 | 1002 | 2440 | 2 | 0.751830 |
| 5862 | 450 | 1 | 0 | 538 | 1 | 1902 | 5862 | 0 | 0.666040 |
| 6066 | 460 | 1 | 0 | 600 | 1 | 1581 | 6066 | 0 | 0.660540 |
| 6198 | 468 | 1 | 0 | 1203 | 2 | 1470 | 3099 | 1 | 0.659340 |
| | | 1 | 0 | 864 | 2 | 234 | 3099 | 1 | 0.659340 |
| 6322 | 480 | 1 | 0 | 800 | 1 | 2998 | 6322 | 0 | 0.664480 |
| 6682 | 504 | 1 | 0 | 1808 | 1 | 2624 | 6682 | 0 | 0.664290 |
| 6976 | 506 | 1 | 0 | 1644 | 1 | 3034 | 6976 | 0 | 0.641950 |
| 7116 | 510 | 1 | 0 | 1606 | 1 | 2120 | 7116 | 0 | 0.635720 |
| 7184 | 560 | 1 | 2 | 586 | 4 | 1544 | 1796 | 2 | 0.692170 |
| 7544 | 572 | 2 | 0 | 336 | 2 | 582 | 1886 | 3 | 0.676980 |
| 7698 | 576 | 1 | 0 | 603 | 2 | 1701 | 3849 | 1 | 0.669580 |
| 7734 | 588 | 1 | 0 | 600 | 1 | 2019 | 7734 | 0 | 0.680710 |
| 8391 | 598 | 1 | 0 | 1635 | 1 | 3849 | 8391 | 0 | 0.643890 |
| 8628 | 630 | 1 | 0 | 792 | 1 | 3363 | 8628 | 0 | 0.661750 |
| 8836 | 660 | 1 | 0 | 942 | 2 | 2126 | 4418 | 2 | 0.678720 |
| 9297 | 702 | 1 | 0 | 864 | 3 | 234 | 3099 | 2 | 0.689950 |
| 9760 | 864 | 2 | 0 | 638 | 2 | 1002 | 2440 | 3 | 0.813190 |
| 12944 | 936 | 1 | 0 | 954 | 2 | 2360 | 6472 | 2 | 0.684670 |
| 13524 | 940 | 1 | 0 | 2488 | 2 | 984 | 6762 | 2 | 0.661160 |
| 14068 | 948 | 1 | 0 | 1880 | 2 | 800 | 7034 | 2 | 0.643660 |
| 14260 | 980 | 1 | 0 | 996 | 2 | 2440 | 7130 | 2 | 0.657360 |
| 14820 | 1032 | 1 | 0 | 1702 | 2 | 2650 | 7410 | 2 | 0.668760 |
| 15420 | 1080 | 1 | 0 | 2002 | 2 | 3050 | 7710 | 2 | 0.675410 |
| 16914 | 1120 | 1 | 0 | 2973 | 2 | 1755 | 8457 | 1 | 0.644690 |
| 16926 | 1152 | 1 | 0 | 1712 | 3 | 644 | 5642 | 1 | 0.662680 |
| 18372 | 1160 | 2 | 0 | 1713 | 2 | 2049 | 4593 | 2 | 0.619940 |
| 18882 | 1224 | 2 | 0 | 669 | 3 | 1449 | 3147 | 2 | 0.638250 |
| 19194 | 1260 | 1 | 0 | 1839 | 2 | 1161 | 9597 | 1 | 0.647420 |
| 19416 | 1404 | 1 | 0 | 1431 | 2 | 3540 | 9708 | 2 | 0.713990 |

TABLE 5 (*continued*)

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{22}$ | $b_{23}$ | $b_{33}$ | rank | $z$ |
|---|---|---|---|---|---|---|---|---|---|
| 20008 | 1440 | 2 | 0 | 1314 | 2 | 2048 | 5002 | 3 | 0.712790 |
| 21810 | 1470 | 1 | 0 | 4764 | 2 | 1848 | 10905 | 1 | 0.673340 |
| 22980 | 1512 | 1 | 0 | 4266 | 2 | 771 | 11490 | 1 | 0.660750 |
| 24132 | 1584 | 2 | 0 | 888 | 2 | 1365 | 6033 | 2 | 0.662380 |
| 25888 | 1618 | 1 | 0 | 1908 | 2 | 4720 | 12944 | 2 | 0.635100 |
| 27048 | 1764 | 2 | 0 | 984 | 2 | 2488 | 6762 | 3 | 0.665570 |
| 29080 | 1936 | 1 | 0 | 6352 | 2 | 2464 | 14540 | 2 | 0.684240 |
| 32176 | 1992 | 2 | 0 | 1184 | 2 | 1820 | 8044 | 3 | 0.642560 |
| 32940 | 2016 | 3 | 0 | 957 | 3 | 1503 | 3660 | 3 | 0.636650 |
| 33075 | 2034 | 3 | 0 | 678 | 3 | 1122 | 3675 | 3 | 0.639970 |
| 34758 | 2052 | 2 | 0 | 1101 | 3 | 2688 | 5793 | 2 | 0.617300 |
| 35868 | 2136 | 2 | 0 | 1167 | 2 | 4131 | 8967 | 2 | 0.624550 |
| 36198 | 2148 | 2 | 0 | 1365 | 3 | 888 | 6033 | 2 | 0.622880 |
| 38388 | 2232 | 2 | 0 | 1161 | 2 | 1839 | 9597 | 2 | 0.613730 |
| 39348 | 2418 | 2 | 0 | 2220 | 2 | 2931 | 9837 | 2 | 0.650170 |
| 42064 | 2448 | 2 | 0 | 2164 | 2 | 4876 | 10516 | 3 | 0.619620 |
| 44940 | 2548 | 2 | 0 | 1548 | 2 | 2469 | 11235 | 2 | 0.607410 |
| | | 2 | 0 | 2112 | 2 | 3249 | 11235 | 2 | 0.607410 |
| 45492 | 2664 | 2 | 0 | 2760 | 2 | 4026 | 11373 | 2 | 0.628070 |
| 48264 | 2864 | 2 | 0 | 1820 | 3 | 1184 | 8044 | 2 | 0.639950 |
| 54525 | 2904 | 1 | 0 | 7940 | 3 | 3080 | 18175 | 1 | 0.580880 |
| 57582 | 3006 | 2 | 0 | 1839 | 3 | 1161 | 9597 | 2 | 0.572200 |
| 60656 | 3120 | 2 | 0 | 3680 | 2 | 5368 | 15164 | 3 | 0.566480 |
| 60858 | 3180 | 2 | 0 | 3732 | 3 | 1476 | 10143 | 2 | 0.575630 |
| 65430 | 3186 | 2 | 0 | 4044 | 3 | 1788 | 10905 | 2 | 0.539950 |
| 67410 | 3822 | 2 | 0 | 2469 | 3 | 1548 | 11235 | 2 | 0.630400 |
| 76776 | 4008 | 2 | 0 | 2452 | 3 | 1548 | 12796 | 2 | 0.587220 |
| 87240 | 4248 | 2 | 0 | 6352 | 3 | 2464 | 14540 | 2 | 0.553950 |
| 90495 | 4440 | 3 | 0 | 1480 | 3 | 2275 | 10055 | 2 | 0.559960 |
| 90984 | 4680 | 2 | 0 | 5368 | 3 | 3680 | 15164 | 2 | 0.587340 |
| 109050 | 5310 | 2 | 0 | 7940 | 3 | 3080 | 18175 | 1 | 0.564820 |
| 120660 | 5370 | 3 | 0 | 2275 | 4 | 1480 | 10055 | 1 | 0.520740 |
| 130860 | 5808 | 2 | 0 | 9528 | 3 | 3696 | 21810 | 2 | 0.522920 |
| 144792 | 5976 | 3 | 0 | 2730 | 4 | 1776 | 12066 | 2 | 0.490450 |
| 151640 | 6130 | 2 | 0 | 6710 | 4 | 4600 | 18955 | 2 | 0.482240 |
| 160064 | 6688 | 4 | 0 | 2628 | 4 | 4096 | 10004 | 3 | 0.500700 |
| 174480 | 7104 | 2 | 0 | 9528 | 4 | 3696 | 21810 | 3 | 0.491420 |
| 179760 | 7424 | 3 | 0 | 3292 | 4 | 2064 | 14980 | 2 | 0.499700 |
| 191940 | 7440 | 3 | 0 | 3065 | 4 | 1935 | 15995 | 1 | 0.471540 |
| 227460 | 8880 | 3 | 0 | 6710 | 4 | 4600 | 18955 | 1 | 0.481550 |

TABLE 6. Red scaled blue list in four dimensions

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{14}$ | $b_{22}$ | $b_{23}$ | $b_{24}$ | $b_{33}$ | $b_{34}$ | $b_{44}$ | rank | $z$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 624 | 16 | 2 | 0 | 0 | 12 | 2 | 0 | 21 | 2 | 27 | 78 | 3 | 1.06215 |
| | | 2 | 0 | 0 | 9 | 2 | 0 | 15 | 2 | 36 | 78 | 3 | 1.06215 |
| 708 | 18 | 1 | 0 | 0 | 27 | 2 | 0 | 60 | 2 | 78 | 177 | 2 | 1.09489 |
| 718 | 22 | 1 | 0 | 0 | 158 | 1 | 0 | 210 | 1 | 234 | 718 | 0 | 1.32521 |
| 732 | 24 | 1 | 0 | 0 | 248 | 1 | 0 | 294 | 1 | 324 | 732 | 0 | 1.42637 |
| 932 | 26 | 1 | 0 | 0 | 116 | 1 | 0 | 288 | 1 | 314 | 932 | 0 | 1.30416 |
| 1124 | 30 | 1 | 0 | 0 | 106 | 1 | 0 | 178 | 1 | 442 | 1124 | 0 | 1.31706 |
| 1234 | 36 | 1 | 0 | 0 | 170 | 1 | 0 | 306 | 1 | 404 | 1234 | 0 | 1.47811 |
| 1354 | 40 | 1 | 0 | 0 | 492 | 1 | 0 | 550 | 1 | 658 | 1354 | 0 | 1.53607 |
| 1748 | 48 | 1 | 0 | 0 | 286 | 1 | 0 | 360 | 1 | 472 | 1748 | 0 | 1.53074 |
| 1990 | 50 | 1 | 0 | 0 | 256 | 1 | 0 | 584 | 1 | 684 | 1990 | 0 | 1.44969 |
| 2052 | 51 | 1 | 0 | 0 | 184 | 1 | 0 | 282 | 1 | 598 | 2052 | 0 | 1.44561 |
| 2097 | 54 | 1 | 0 | 0 | 435 | 1 | 0 | 936 | 1 | 1035 | 2097 | 0 | 1.50633 |
| 2112 | 55 | 1 | 0 | 0 | 100 | 1 | 0 | 162 | 1 | 830 | 2112 | 0 | 1.52617 |
| 2248 | 60 | 1 | 0 | 0 | 106 | 1 | 0 | 178 | 2 | 442 | 1124 | 2 | 1.58980 |
| 2686 | 66 | 1 | 0 | 0 | 852 | 1 | 0 | 1142 | 1 | 1218 | 2686 | 0 | 1.53190 |
| 2730 | 68 | 1 | 0 | 0 | 170 | 1 | 0 | 452 | 1 | 1328 | 2730 | 0 | 1.55928 |
| 2836 | 72 | 1 | 0 | 0 | 418 | 1 | 0 | 1010 | 1 | 1290 | 2836 | 0 | 1.60464 |
| 4496 | 106 | 1 | 0 | 0 | 106 | 2 | 0 | 178 | 2 | 442 | 1124 | 3 | 1.66790 |
| 5672 | 108 | 1 | 0 | 0 | 194 | 1 | 0 | 718 | 2 | 994 | 2836 | 2 | 1.42248 |
| 6744 | 120 | 1 | 0 | 0 | 159 | 2 | 0 | 267 | 2 | 663 | 1686 | 2 | 1.38308 |
| 8448 | 126 | 1 | 0 | 0 | 830 | 2 | 0 | 100 | 2 | 162 | 2112 | 3 | 1.21931 |
| 8508 | 144 | 1 | 0 | 0 | 627 | 1 | 0 | 1935 | 2 | 1515 | 4254 | 1 | 1.38584 |
| 8992 | 212 | 2 | 0 | 0 | 106 | 2 | 0 | 178 | 2 | 442 | 1124 | 4 | 1.95413 |
| 16116 | 218 | 1 | 0 | 0 | 1278 | 2 | 0 | 1713 | 2 | 1827 | 4029 | 2 | 1.26949 |
| 20232 | 318 | 2 | 0 | 0 | 159 | 2 | 0 | 267 | 3 | 663 | 1686 | 2 | 1.54517 |
| 26384 | 336 | 1 | 0 | 0 | 614 | 2 | 0 | 1402 | 2 | 1874 | 6596 | 3 | 1.31989 |
| | | 1 | 0 | 0 | 1070 | 2 | 0 | 1402 | 2 | 1874 | 6596 | 3 | 1.31989 |
| 32232 | 432 | 2 | 0 | 0 | 1278 | 2 | 0 | 1713 | 2 | 1827 | 4029 | 3 | 1.44428 |
| 42976 | 436 | 2 | 0 | 0 | 1704 | 2 | 0 | 2284 | 2 | 2436 | 5372 | 4 | 1.15467 |
| 43821 | 468 | 3 | 0 | 0 | 156 | 3 | 0 | 228 | 3 | 495 | 1623 | 4 | 1.21996 |
| 45522 | 477 | 3 | 0 | 0 | 159 | 3 | 0 | 267 | 3 | 663 | 1686 | 4 | 1.20550 |
| 48348 | 480 | 2 | 0 | 0 | 1035 | 2 | 0 | 1281 | 3 | 882 | 4029 | 2 | 1.15504 |
| | | 2 | 0 | 0 | 375 | 2 | 0 | 1257 | 3 | 585 | 4029 | 2 | 1.15504 |
| 52768 | 672 | 2 | 0 | 0 | 614 | 2 | 0 | 1402 | 2 | 1874 | 6596 | 4 | 1.50574 |
| 89046 | 756 | 1 | 0 | 0 | 1605 | 3 | 0 | 2103 | 3 | 2811 | 9894 | 3 | 1.10276 |
| | | 1 | 0 | 0 | 79 | 3 | 0 | 2103 | 3 | 2811 | 9894 | 3 | 1.10276 |
| 108783 | 972 | 3 | 0 | 0 | 327 | 3 | 0 | 558 | 3 | 1386 | 4029 | 4 | 1.20172 |
| 118728 | 1008 | 2 | 0 | 0 | 79 | 2 | 0 | 2103 | 3 | 2811 | 9894 | 2 | 1.15914 |
| | | 2 | 0 | 0 | 1605 | 2 | 0 | 2103 | 3 | 2811 | 9894 | 2 | 1.15914 |
| | | 2 | 0 | 0 | 1605 | 2 | 0 | 2811 | 3 | 2103 | 9894 | 2 | 1.15914 |
| 145044 | 1080 | 3 | 0 | 0 | 1176 | 3 | 0 | 1380 | 3 | 1708 | 5372 | 3 | 1.05174 |
| 178092 | 1512 | 2 | 0 | 0 | 1605 | 3 | 0 | 2103 | 3 | 2811 | 9894 | 3 | 1.24098 |
| | | 2 | 0 | 0 | 79 | 3 | 0 | 2103 | 3 | 2811 | 9894 | 3 | 1.24098 |
| 257856 | 1728 | 3 | 0 | 0 | 744 | 4 | 0 | 436 | 4 | 1848 | 5372 | 3 | 1.04043 |
| 267138 | 2268 | 3 | 0 | 0 | 79 | 3 | 0 | 2103 | 3 | 2811 | 9894 | 3 | 1.32561 |
| 474912 | 3024 | 3 | 0 | 0 | 2140 | 3 | 0 | 3748 | 4 | 2804 | 13192 | 2 | 1.08787 |

## 6. COMMENTS ABOUT LISTS

6.1. **Evaluation of scaled lists.** An immediate question that comes to mind is to what extent any list obtained here compares with the corresponding complete list. The authors believe that, at best, one retains about 70% of a complete list, and that this percentage diminishes to zero as the order $N$ significantly exceeds the order $\overline{N}$ of the input list. The rest of this subsection is devoted to this question.

We carried out some numerical experiments in an environment in which the answer, in the form of a complete red list, is available. We applied our scaling technique to only part of our three-dimensional green list, the first part having $N \leq \overline{N} = 250$. This produced first a long repetitive raw scaled green list and after massaging, as described in §5, a green scaled green list containing 450 lattices sharing 326 distinct values of $N$ lying in $[251, 13376]$. Since we have

TABLE 7. Red scaled green list in four dimensions

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{14}$ | $b_{22}$ | $b_{23}$ | $b_{24}$ | $b_{33}$ | $b_{34}$ | $b_{44}$ | rank | $z$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 676 | 22 | 1 | 0 | 0 | 50 | 1 | 0 | 158 | 2 | 34 | 338 | 2 | 1.38186 |
| | | 1 | 0 | 0 | 54 | 1 | 0 | 128 | 2 | 12 | 338 | 2 | 1.38186 |
| 688 | 24 | 1 | 0 | 0 | 106 | 1 | 0 | 144 | 2 | 22 | 344 | 2 | 1.48920 |
| 900 | 25 | 1 | 0 | 4 | 9 | 1 | 6 | 25 | 30 | 0 | 30 | 2 | 1.28535 |
| 928 | 32 | 2 | 0 | 0 | 34 | 2 | 0 | 44 | 2 | 52 | 116 | 4 | 1.61001 |
| 1281 | 34 | 1 | 0 | 0 | 54 | 1 | 0 | 129 | 3 | 98 | 427 | 1 | 1.35893 |
| 1344 | 36 | 1 | 0 | 0 | 45 | 2 | 0 | 99 | 2 | 162 | 336 | 2 | 1.38989 |
| 1556 | 40 | 1 | 0 | 0 | 84 | 1 | 0 | 218 | 2 | 244 | 778 | 2 | 1.38871 |
| 1692 | 42 | 1 | 0 | 0 | 96 | 1 | 0 | 412 | 2 | 134 | 846 | 2 | 1.37169 |
| 1952 | 48 | 2 | 0 | 0 | 24 | 2 | 0 | 42 | 2 | 56 | 244 | 4 | 1.41160 |
| | | 2 | 0 | 0 | 24 | 2 | 0 | 40 | 2 | 92 | 244 | 4 | 1.41160 |
| 2200 | 50 | 1 | 0 | 2 | 28 | 2 | 2 | 94 | 10 | 20 | 110 | 3 | 1.34617 |
| 2248 | 60 | 1 | 0 | 0 | 106 | 1 | 0 | 178 | 2 | 442 | 1124 | 2 | 1.58980 |
| 2320 | 64 | 2 | 0 | 0 | 34 | 2 | 0 | 56 | 2 | 82 | 290 | 4 | 1.65661 |
| 3132 | 72 | 2 | 0 | 0 | 51 | 3 | 0 | 66 | 3 | 78 | 174 | 3 | 1.48950 |
| 4080 | 80 | 2 | 0 | 0 | 40 | 2 | 0 | 62 | 2 | 154 | 510 | 4 | 1.35530 |
| | | 2 | 0 | 0 | 40 | 2 | 0 | 62 | 2 | 134 | 510 | 4 | 1.35530 |
| 4496 | 106 | 1 | 0 | 0 | 106 | 2 | 0 | 178 | 2 | 442 | 1124 | 3 | 1.66790 |
| 6192 | 108 | 2 | 0 | 0 | 196 | 2 | 0 | 234 | 3 | 168 | 516 | 3 | 1.32960 |
| 6736 | 120 | 2 | 0 | 0 | 62 | 2 | 0 | 206 | 2 | 294 | 842 | 4 | 1.38435 |
| 7312 | 128 | 2 | 0 | 0 | 218 | 2 | 0 | 340 | 2 | 414 | 914 | 4 | 1.38576 |
| 7888 | 144 | 2 | 0 | 0 | 172 | 2 | 0 | 314 | 2 | 382 | 986 | 4 | 1.46987 |
| 8992 | 212 | 2 | 0 | 0 | 106 | 2 | 0 | 178 | 2 | 442 | 1124 | 4 | 1.95413 |
| 14112 | 216 | 2 | 0 | 0 | 174 | 2 | 0 | 284 | 3 | 510 | 1176 | 3 | 1.39735 |
| 18816 | 224 | 2 | 0 | 0 | 174 | 2 | 0 | 284 | 4 | 510 | 1176 | 4 | 1.15326 |
| 19632 | 252 | 2 | 0 | 0 | 616 | 2 | 0 | 684 | 3 | 180 | 1636 | 3 | 1.25424 |
| 20232 | 318 | 2 | 0 | 0 | 159 | 2 | 0 | 267 | 3 | 663 | 1686 | 2 | 1.54517 |
| 26622 | 324 | 2 | 0 | 0 | 258 | 3 | 0 | 471 | 3 | 573 | 1479 | 3 | 1.26360 |
| 31590 | 360 | 2 | 0 | 0 | 327 | 3 | 0 | 483 | 3 | 543 | 1755 | 3 | 1.22327 |
| 37017 | 432 | 3 | 0 | 0 | 327 | 3 | 0 | 510 | 3 | 621 | 1371 | 4 | 1.29135 |
| 39933 | 486 | 3 | 0 | 0 | 258 | 3 | 0 | 471 | 3 | 573 | 1479 | 4 | 1.36616 |
| 47385 | 540 | 3 | 0 | 0 | 327 | 3 | 0 | 483 | 3 | 543 | 1755 | 4 | 1.32089 |
| 65808 | 576 | 3 | 0 | 0 | 680 | 3 | 0 | 828 | 4 | 436 | 1828 | 2 | 1.07736 |
| | | 3 | 0 | 0 | 436 | 3 | 0 | 680 | 4 | 828 | 1828 | 2 | 1.07736 |
| 70992 | 648 | 3 | 0 | 0 | 344 | 3 | 0 | 628 | 4 | 764 | 1972 | 2 | 1.13893 |
| | | 3 | 0 | 0 | 628 | 3 | 0 | 764 | 4 | 344 | 1972 | 2 | 1.13893 |
| 81360 | 720 | 3 | 0 | 0 | 632 | 3 | 0 | 916 | 4 | 392 | 2260 | 2 | 1.13133 |
| 87744 | 768 | 3 | 0 | 0 | 436 | 4 | 0 | 680 | 4 | 828 | 1828 | 3 | 1.13395 |
| 94656 | 792 | 3 | 0 | 0 | 344 | 4 | 0 | 628 | 4 | 764 | 1972 | 3 | 1.09849 |
| 108480 | 864 | 3 | 0 | 0 | 392 | 4 | 0 | 472 | 4 | 1056 | 2260 | 3 | 1.07067 |
| | | 3 | 0 | 0 | 392 | 4 | 0 | 632 | 4 | 916 | 2260 | 3 | 1.07067 |
| 112320 | 960 | 3 | 0 | 0 | 436 | 4 | 0 | 644 | 4 | 724 | 2340 | 3 | 1.15586 |
| 116992 | 1024 | 4 | 0 | 0 | 436 | 4 | 0 | 680 | 4 | 828 | 1828 | 4 | 1.19200 |
| 126208 | 1056 | 4 | 0 | 0 | 344 | 4 | 0 | 628 | 4 | 764 | 1972 | 4 | 1.15434 |
| 144640 | 1152 | 4 | 0 | 0 | 392 | 4 | 0 | 472 | 4 | 1056 | 2260 | 4 | 1.12446 |
| 149760 | 1280 | 4 | 0 | 0 | 436 | 4 | 0 | 644 | 4 | 724 | 2340 | 4 | 1.21376 |
| 219375 | 1500 | 3 | 0 | 0 | 545 | 5 | 0 | 805 | 5 | 905 | 2925 | 3 | 1.03422 |
| 228500 | 1600 | 4 | 0 | 0 | 545 | 5 | 0 | 850 | 5 | 1035 | 2285 | 3 | 1.06614 |
| 285625 | 2000 | 5 | 0 | 0 | 545 | 5 | 0 | 850 | 5 | 1035 | 2285 | 4 | 1.10505 |
| 365625 | 2500 | 5 | 0 | 0 | 545 | 5 | 0 | 805 | 5 | 905 | 2925 | 4 | 1.12191 |

available a complete green list for $N \in [1, 4000]$, we were able to observe the quality of this particular green scaled green list.

Table 9 gives a breakdown of the distribution of lattices in this list and their quality. Here, $\rho_L(N)$ is the lower bound on $\rho_3(N)$ based only on the lattices in this list. Examination of this table shows that for values of $N$ near to $\overline{N}$, we seem to be obtaining lattices for about half the values of $N$. Of these, 80% are optimal, the rest being generally of reasonably high quality. On the other hand, for values of $N$ exceeding $8\overline{N}$ in a range containing 2000 values of $N$, we have found lattices for only 30 of these values, and only four of these are optimal. Fourteen of these 450 lattices may also be found on the three-dimensional red list which has 45 entries for $N \in [251, 4000]$.

A second numerical experiment concerns a scaled three-dimensional red list.

TABLE 8. Five-dimensional lattices having $z(\Lambda) > 3.0$. Rank-1 simple lattices in this list having $N \in [112, 772]$ have been taken from [1]. The others are scaled versions of these or of the benchmark lattice

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{14}$ | $b_{15}$ | $b_{22}$ | $b_{23}$ | $b_{24}$ | $b_{25}$ | $b_{33}$ | $b_{34}$ | $b_{35}$ | $b_{44}$ | $b_{45}$ | $b_{55}$ | rank | $z$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 64 | 4 | 2 | 0 | 0 | 0 | 2 | 2 | 0 | 0 | 2 | 2 | 0 | 2 | 2 | 2 | 4 | 5 | 4.49583 |
| 96 | 4 | 2 | 0 | 0 | 0 | 3 | 2 | 0 | 0 | 3 | 2 | 0 | 3 | 2 | 3 | 6 | 4 | 3.96210 |
| 128 | 4 | 2 | 0 | 0 | 0 | 4 | 2 | 0 | 0 | 4 | 2 | 0 | 4 | 2 | 4 | 8 | 5 | 3.56961 |
| | | 1 | 0 | 0 | 0 | 12 | 1 | 0 | 0 | 22 | 1 | 0 | 48 | 1 | 52 | 128 | 0 | 3.56961 |
| 144 | 4 | 2 | 0 | 0 | 0 | 3 | 2 | 0 | 0 | 3 | 2 | 0 | 3 | 3 | 3 | 96 | 3 | 3.40971 |
| 160 | 4 | 2 | 0 | 0 | 0 | 5 | 2 | 0 | 0 | 5 | 2 | 0 | 5 | 2 | 5 | 10 | 4 | 3.26808 |
| 192 | 4 | 2 | 0 | 0 | 0 | 4 | 2 | 0 | 0 | 4 | 2 | 0 | 4 | 3 | 4 | 8 | 4 | 3.02758 |
| | | 2 | 0 | 0 | 0 | 6 | 2 | 0 | 0 | 6 | 2 | 0 | 6 | 2 | 6 | 12 | 5 | 3.02758 |
| | | 1 | 0 | 0 | 0 | 18 | 1 | 0 | 0 | 33 | 1 | 0 | 72 | 1 | 78 | 192 | 0 | 3.02758 |
| 275 | 5 | 1 | 0 | 0 | 0 | 31 | 1 | 0 | 0 | 71 | 1 | 0 | 91 | 1 | 136 | 275 | 0 | 3.22179 |
| 308 | 6 | 1 | 0 | 0 | 0 | 18 | 1 | 0 | 0 | 62 | 1 | 0 | 70 | 1 | 102 | 308 | 0 | 3.66511 |
| 438 | 8 | 1 | 0 | 0 | 0 | 38 | 1 | 0 | 0 | 50 | 1 | 0 | 96 | 1 | 168 | 438 | 0 | 4.10962 |
| 448 | 6 | 1 | 0 | 0 | 0 | 12 | 1 | 0 | 0 | 54 | 2 | 0 | 22 | 2 | 28 | 112 | 3 | 3.04710 |
| | | 1 | 0 | 0 | 0 | 22 | 1 | 0 | 0 | 54 | 2 | 0 | 12 | 2 | 28 | 112 | 3 | 3.04710 |
| 512 | 8 | 1 | 0 | 0 | 0 | 22 | 1 | 0 | 0 | 48 | 2 | 0 | 12 | 2 | 52 | 128 | 3 | 3.79336 |
| 657 | 8 | 1 | 0 | 0 | 0 | 57 | 1 | 0 | 0 | 75 | 1 | 0 | 144 | 1 | 252 | 657 | 0 | 3.32502 |
| 666 | 9 | 1 | 0 | 0 | 0 | 15 | 1 | 0 | 0 | 42 | 1 | 0 | 175 | 1 | 269 | 666 | 0 | 3.71336 |
| | | 1 | 0 | 0 | 0 | 57 | 1 | 0 | 0 | 137 | 1 | 0 | 223 | 1 | 240 | 666 | 0 | 3.71336 |
| | | 1 | 0 | 0 | 0 | 57 | 1 | 0 | 0 | 221 | 1 | 0 | 240 | 1 | 307 | 666 | 0 | 3.71336 |
| 768 | 8 | 1 | 0 | 0 | 0 | 22 | 1 | 0 | 0 | 48 | 2 | 0 | 12 | 3 | 52 | 128 | 2 | 3.05476 |
| | | 1 | 0 | 0 | 0 | 22 | 1 | 0 | 0 | 48 | 2 | 0 | 52 | 3 | 12 | 128 | 2 | 3.05476 |
| | | 1 | 0 | 0 | 0 | 33 | 1 | 0 | 0 | 72 | 2 | 0 | 18 | 2 | 78 | 192 | 3 | 3.05476 |
| 772 | 10 | 1 | 0 | 0 | 0 | 154 | 1 | 0 | 0 | 170 | 1 | 0 | 230 | 1 | 256 | 772 | 0 | 3.80758 |
| 924 | 9 | 1 | 0 | 0 | 0 | 93 | 1 | 0 | 0 | 105 | 1 | 0 | 153 | 2 | 27 | 462 | 0 | 3.10161 |
| 1158 | 10 | 1 | 0 | 0 | 0 | 231 | 1 | 0 | 0 | 255 | 1 | 0 | 345 | 1 | 384 | 1158 | 0 | 3.03166 |
| 1536 | 12 | 1 | 0 | 0 | 0 | 33 | 2 | 0 | 0 | 18 | 2 | 0 | 72 | 2 | 78 | 192 | 4 | 3.08556 |
| 1544 | 12 | 1 | 0 | 0 | 0 | 78 | 1 | 0 | 0 | 264 | 1 | 0 | 378 | 2 | 10 | 772 | 2 | 3.07610 |
| 2048 | 16 | 2 | 0 | 0 | 0 | 12 | 2 | 0 | 0 | 22 | 2 | 0 | 48 | 2 | 52 | 128 | 5 | 3.46294 |
| | | 1 | 0 | 0 | 0 | 44 | 2 | 0 | 0 | 24 | 2 | 0 | 96 | 2 | 104 | 256 | 4 | 3.46294 |
| 2560 | 16 | 1 | 0 | 0 | 0 | 55 | 2 | 0 | 0 | 30 | 2 | 0 | 120 | 2 | 130 | 320 | 4 | 3.02077 |
| 2664 | 18 | 1 | 0 | 0 | 0 | 442 | 1 | 0 | 0 | 480 | 1 | 0 | 614 | 2 | 114 | 1332 | 2 | 3.31566 |
| 3088 | 20 | 1 | 0 | 0 | 0 | 78 | 1 | 0 | 0 | 378 | 2 | 0 | 10 | 2 | 264 | 772 | 3 | 3.36013 |
| 4632 | 24 | 1 | 0 | 0 | 0 | 117 | 1 | 0 | 0 | 567 | 2 | 0 | 15 | 2 | 396 | 1158 | 2 | 3.11591 |
| 7008 | 32 | 2 | 0 | 0 | 0 | 16 | 2 | 0 | 0 | 70 | 2 | 0 | 144 | 2 | 186 | 438 | 5 | 3.17025 |

TABLE 9. The three-dimensional green scaled green list with $\overline{N} = 250$

| Interval | Scaled Lattices | Distinct $N$-values | $N$-values for which $\rho_L(N)/\rho_3(N)$ | | |
|---|---|---|---|---|---|
| | | | $=1$ | $\epsilon(1, 0.9)$ | $\leq 0.9$ |
| $N \in [251, 500]$ | 209 | 1112 | 94 | 15 | 3 |
| $N \in [501, 1000]$ | 120 | 100 | 56 | 32 | 12 |
| $N \in [1001, 2000]$ | 75 | 69 | 19 | 27 | 23 |
| $N \in [2001, 4000]$ | 31 | 30 | 4 | 2 | 24 |
| $N > 4000$ | 15 | 15 | ? | ? | ? |
| Total | 450 | 326 | 173 | 76 | 62 |

We compare a plot of $\rho_3(N)$ based on a scaled red list with a similar plot based on the complete red list. We consider five such plots. For the complete red list we define a piecewise constant function

$$\tilde{\rho}(N) = \max_{\substack{\Lambda \in \mathscr{L}_3(M) \\ M \leq N}} \rho(\Lambda) . \tag{6.1}$$

The discontinuities of $\tilde{\rho}(N)$ occur at values of $N$ appearing on the red list. At such values, $\tilde{\rho}(N-1) < \tilde{\rho}(N)$. One can define the analogous function for a red

TABLE 10. Information about red scaled green lists

| % of $N$ satisfying | $(N_1, N_2]$ | $(N_2, N_3]$ | $(N_3, N_4]$ | $(N_4, N_5]$ |
|---|---|---|---|---|
| $\tilde{\rho}_1(N) = \tilde{\rho}(N)$ | 51.6 | 62.4 | 46.6 | 0.0 |
| $\tilde{\rho}_2(N) = \tilde{\rho}(N)$ | | 75.0 | 63.4 | 38.15 |
| $\tilde{\rho}_3(N) = \tilde{\rho}(N)$ | | | 95.2 | 74.4 |
| $\tilde{\rho}_4(N) = \tilde{\rho}(N)$ | | | | 85.65 |

scaled list based on an input green list for $N \in [1, N_i]$, where $N_i = 125 \cdot 2^i$.

We have constructed four functions $\tilde{\rho}_i(N)$, $i = 1, 2, 3, 4$. In general, $\tilde{\rho}_i(N) \leq \tilde{\rho}(N)$, but for some values of $N$ this is an equality. We list in Table 10 the percentage of values of $N$ in an interval $(N_j, N_{j+1}]$, $j = 1, 2, 3, 4$, for which $\tilde{\rho}_i(N) = \rho_3(N)$.

Naturally, when $j < i$, this is 100%, and when $j >> i$, this reduces to zero.

Examination of the complete red lists in three and four dimensions which appeared in [3] and [4], respectively, shows that a proportion that varies unsteadily between 15% and 40% are root lattices, the majority of lattices on these lists being scaled lattices. If this state of affairs were to prevail for higher values of $N$, then any red scaled list would omit between 15% and 40% of the optimal lattices since it cannot by definition include root lattices.

Finally, we state one further reason why we believe these lists to be incomplete. This one is based on the actual lists, rather than on extrapolation. We have presented separately in Tables 5 and 4 a red scaled green list and the corresponding red scaled blue list. The first contains 80 entries and appears to be an excellent list in many ways, having as far as one can see the same characteristics as the actual red list for $N < 4000$. However, there are some known rules missing. We know this because they appear on the clearly inferior and shorter red scaled blue list. These two lists contain six entries in common. There are eight entries on the red scaled blue list that merit inclusion on the red scaled green list, but are not there. If included, they would in total displace eight entries already there.

One sees that a few missing entries do not alter the overall nature of the list very much. The missing entries are simply replaced by entries representing marginally inferior lattices; the effect on the list as a whole is local. Also, it is not particularly the entries with the highest $z$-values that seem to be missing.

6.2. **Suitable input for a scaled list.** We have listed the three-dimensional red scaled red list (Table A1) and the red scaled green list (Table 5). Only two elements $N = 4185$ and $N = 4704$ occur on both. Thereafter, the red scaled red list deteriorates significantly when compared with the red scaled green list. However, the input red list contains all the really good elements of the input green list. The heuristic conclusion in this case is that, for scaling purposes, one does not want to start with optimal lattices having optimal $N$ values. It appears that one will discover more if one inputs a list of good but not excellent lattices.

All our results appear to support to some extent this conclusion. We have found the red scaled blue list to be intermediate. The blue list being restricted to rank-1 simple lattices is not as good as the red list but seems to provide better scaled lattices. Any conclusion based on our four-dimensional lists must take into account that the blue list includes much higher values of $N$ than the green list.

Theoretical support for this state of affairs can be found in §3. There it is noted that, starting with a family root lattice, the effect of scaling is in general to improve the $z$-value at first, but then there is a steady decay in $z$-value. It is consistent with this situation that, for optimal values of $N$, the best lattices are not root lattices but are already scaled versions—but not very highly scaled. As mentioned above, the majority of the lattices on our red list are like this, and scaling them is unlikely to provide better ones.

6.3. **Comments on red lists.** It has been traditional to report results of the type treated here using red lists (i.e., lists of optimal lattices). One reason is that it is feasible to publish such a list. A red list contains in one page an excellent selection from a green list of fifty pages. Another reason is linked to the numerical quadrature application in which the cost is taken to be proportional to $N$, the number of function values, and the quality of the result to $\rho$. However, the present authors believe that, for the values of $N$ now reached in three or four dimensions, the red list has become an anachronism. For many purposes a highlight list is adequate. For deeper investigation, the green list is probably needed. And, in applications, questions such as embedding of one rule in another and convenience in locating points using the relevant machine architecture may be much more significant than a small margin in the plot of $N$ versus $\rho$.

While the red list contains an excellent selection, occasionally good lattices are excluded because they are "in the shadow of" marginally better lattices. An example of this occurs in three dimensions with $N = 9760$ and $N = 9800$. The first has $\rho = 864$ and the second $\rho = 800$, so the second does not occur on a red list. In fact, we know only three lattices with $z(\Lambda) > 0.75$; these are the two mentioned above and one with $N = 4880$. Thus, our red list has omitted what might be considered the third best lattice available. In investigations relating to the distribution of good lattices, one may prefer to know about all good lattices, even if in applications some are not going to be used.

6.4. **The tail of the list.** We mentioned towards the end of §2 that it is trivial to find infinite sequences of lattices having monotonic increasing $\rho(\Lambda')$ and $N'$. Thus, an incomplete red list can be extended indefinitely. The lists we have presented have the additional requirement that $z(\Lambda)$ should exceed a specified amount $\overline{\overline{z}}$. The reader should note that this by itself need not render a list finite. In fact, numerical and theoretical evidence suggests the opposite. Our list deteriorates and so is finite simply because it can contain only a subset of lattices, namely, those which are scaled versions of root lattices having $N \leq \overline{N}$. Inequalities (2.8) and (2.9) apply to the scaled versions of each of this finite collection of root lattices, and so to the concatenation from which our lists are formed. It is important to realize that this deterioration is a property of our selection process and has nothing to do with the asymptotic behavior of a complete red list of optimal lattices.

## 7. Concluding remarks

The basic contribution of this paper is the introduction of a very simple theory of rectangular scaling of lattices and a description of the behavior of

$\rho(\Lambda)$ under such scaling. This theory, described in §§2 and 3, remains to be fully exploited. In the rest of this paper we have used it only to provide lists of good lattices from existing lists. Applications of a more detailed and innovative nature may exist.

The rest of this paper is concerned with carrying out this scaling process on lists of lattices. By any measure, this has been very successful, producing a cornucopia of new good lattices. Indeed, so many and varied are the outputs of this process that organization and selection of results for publication has become a problem in itself. This aspect of the work is described and discussed in §§5 and 6.

We have uncovered many high-order lattice rules in dimensions 3, 4, and, to some extent, 5. The best are listed in Tables 1, 2, and 8, respectively. These turn out as might be expected in view of the current advanced theory (see, for example, Niederreiter [7]). It is our hope that these concrete examples will provide a spur to the recognition and practical application of lattice rules in actual scientific projects involving multidimensional quadrature.

## Appendix. Red scaled red lists

The two lists in this appendix are included to illustrate the discussion in §6.2. At first glance both lists appear reasonable. However, in fact, these lists as a whole are significantly *inferior* to those given in Tables 4 and 5, and 6 and 7, respectively, though they do contain some very good lattices.

Table A1. Red scaled red list in three dimension

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{22}$ | $b_{23}$ | $b_{33}$ | rank | $z$ |
|---|---|---|---|---|---|---|---|---|---|
| 4185 | 324 | 3 | 0 | 108 | 3 | 168 | 465 | 3 | 0.645620 |
| 4704 | 390 | 2 | 0 | 228 | 3 | 148 | 784 | 2 | 0.701090 |
| 5874 | 414 | 1 | 0 | 1044 | 2 | 303 | 2937 | 1 | 0.611650 |
| 7056 | 444 | 3 | 0 | 148 | 3 | 228 | 784 | 2 | 0.557620 |
| 7248 | 448 | 2 | 0 | 260 | 3 | 376 | 1208 | 2 | 0.549400 |
| 7696 | 480 | 2 | 0 | 466 | 2 | 834 | 1924 | 3 | 0.558120 |
| 8811 | 540 | 1 | 0 | 621 | 3 | 378 | 2937 | 2 | 0.556720 |
| 8820 | 555 | 3 | 0 | 185 | 3 | 285 | 980 | 2 | 0.571660 |
| 9408 | 588 | 3 | 0 | 228 | 4 | 148 | 784 | 2 | 0.571830 |
| 11748 | 621 | 1 | 0 | 1392 | 3 | 404 | 3916 | 1 | 0.495370 |
| 13212 | 648 | 2 | 0 | 1137 | 2 | 1428 | 3303 | 2 | 0.465390 |
| 13376 | 672 | 4 | 0 | 180 | 4 | 260 | 836 | 3 | 0.477330 |
| 15664 | 808 | 2 | 0 | 404 | 2 | 1392 | 3916 | 3 | 0.498250 |
| 17622 | 909 | 2 | 0 | 621 | 3 | 378 | 2937 | 2 | 0.504320 |
| 23352 | 1080 | 2 | 0 | 1304 | 2 | 2140 | 5838 | 3 | 0.465190 |
| 23496 | 1212 | 2 | 0 | 1392 | 3 | 404 | 3916 | 2 | 0.519160 |
| 31328 | 1440 | 2 | 0 | 828 | 4 | 504 | 3916 | 3 | 0.475840 |

TABLE A2. Red scaled red list in four dimensions

| $N$ | $\rho$ | $b_{11}$ | $b_{12}$ | $b_{13}$ | $b_{14}$ | $b_{22}$ | $b_{23}$ | $b_{24}$ | $b_{33}$ | $b_{34}$ | $b_{44}$ | rank | $z$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 640 | 16 | 2 | 0 | 2 | 6 | 4 | 0 | 8 | 4 | 16 | 20 | 4 | 1.04376 |
| 864 | 24 | 1 | 0 | 0 | 42 | 2 | 0 | 69 | 3 | 54 | 144 | 2 | 1.26997 |
| | | 2 | 0 | 0 | 15 | 2 | 0 | 33 | 3 | 21 | 72 | 2 | 1.26997 |
| | | 1 | 0 | 0 | 42 | 2 | 0 | 69 | 3 | 12 | 144 | 2 | 1.26997 |
| 1124 | 30 | 1 | 0 | 0 | 106 | 1 | 0 | 178 | 1 | 442 | 1124 | 0 | 1.31706 |
| 1944 | 36 | 3 | 0 | 0 | 15 | 3 | 3 | 12 | 6 | 30 | 36 | 4 | 1.06190 |
| | | 3 | 0 | 0 | 15 | 3 | 0 | 21 | 3 | 33 | 72 | 4 | 1.06190 |
| 2164 | 40 | 1 | 0 | 0 | 152 | 1 | 0 | 330 | 2 | 104 | 1082 | 2 | 1.09017 |
| 2248 | 60 | 1 | 0 | 0 | 106 | 1 | 0 | 178 | 2 | 442 | 1124 | 2 | 1.58980 |
| 4328 | 72 | 1 | 0 | 0 | 104 | 2 | 0 | 152 | 2 | 330 | 1082 | 3 | 1.16625 |
| | | 1 | 0 | 0 | 104 | 2 | 0 | 242 | 2 | 400 | 1082 | 3 | 1.16625 |
| 4496 | 106 | 1 | 0 | 0 | 106 | 2 | 0 | 178 | 2 | 442 | 1124 | 3 | 1.66790 |
| 6744 | 120 | 1 | 0 | 0 | 159 | 2 | 0 | 267 | 2 | 663 | 1686 | 2 | 1.38308 |
| 8656 | 144 | 2 | 0 | 0 | 104 | 2 | 0 | 152 | 2 | 330 | 1082 | 4 | 1.36734 |
| 8992 | 212 | 2 | 0 | 0 | 106 | 2 | 0 | 178 | 2 | 442 | 1124 | 4 | 1.95413 |
| 19476 | 216 | 2 | 0 | 0 | 156 | 2 | 0 | 228 | 3 | 495 | 1623 | 2 | 1.08193 |
| | | 2 | 0 | 0 | 156 | 2 | 0 | 495 | 3 | 228 | 1623 | 2 | 1.08193 |
| 20232 | 318 | 2 | 0 | 0 | 159 | 2 | 0 | 267 | 3 | 663 | 1686 | 2 | 1.54517 |
| 35968 | 356 | 2 | 0 | 0 | 212 | 2 | 0 | 356 | 4 | 884 | 2248 | 4 | 1.08922 |
| 38952 | 360 | 2 | 0 | 0 | 208 | 3 | 0 | 304 | 3 | 660 | 2164 | 2 | 1.03259 |
| 40464 | 424 | 2 | 0 | 0 | 212 | 3 | 0 | 356 | 3 | 884 | 2248 | 2 | 1.17917 |
| 43821 | 468 | 3 | 0 | 0 | 156 | 3 | 0 | 228 | 3 | 495 | 1623 | 4 | 1.21996 |
| 45522 | 477 | 3 | 0 | 0 | 159 | 3 | 0 | 267 | 3 | 663 | 1686 | 4 | 1.20550 |
| 77904 | 624 | 3 | 0 | 0 | 208 | 3 | 0 | 304 | 4 | 660 | 2164 | 2 | 1.01613 |
| 80928 | 636 | 3 | 0 | 0 | 212 | 3 | 0 | 356 | 4 | 884 | 2248 | 2 | 1.00373 |

## BIBLIOGRAPHY


1. M. Bourdeau and A. Pitre, *Tables of good lattices in four and five dimensions*, Numer. Math. **47** (1985), 39–43.

2. G. Kedem and S. K. Zaremba, *A table of good lattice points in three dimensions*, Numer. Math. **23** (1974), 175–180.

3. J. N. Lyness and T. Sørevik, *A search program for finding optimal integration lattices*, Computing **47** (1991), 103–120.

4. ——, *An algorithm for finding optimal integration lattices of composite order*, BIT **32** (1992), 665–675.

5. D. Maisonneuve, *Recherche et utilisation des bons treillis, programmation et résultats numériques*, Applications of Number Theory to Numerical Analysis (S. K. Zaremba, ed.), Academic Press, London, 1972, pp. 121–201.

6. H. Niederreiter, *Quasi-Monte Carlo methods for multidimensional numerical integration*, Numerical Integration III (G. Hämmerlin and H. Brass, eds.), Birkhäuser Verlag, Boston, 1988, pp. 157–171.

7. ——, *The existence of efficient lattice rules for multidimensional numerical integration*, Math. Comp. **58** (1992), 305–314, S7–S16.

8. I. H. Sloan and J. N. Lyness, *The representation of lattice quadrature rules as multiple sums*, Math. Comp. **52** (1989), 81–94.

9. S. K. Zaremba, *Good lattice points, discrepancy and numerical integration*, Ann. Mat. Pura Appl. **73** (1966), 293–317.

10. ——, *Good lattice points modulo composite numbers*, Monatsh. Math. **78** (1974), 446–460.

Mathematics and Computer Science Division, Argonne National Laboratory, Argonne, Illinois 60439-4844

*E-mail address*: lyness@mcs.anl.gov

Department of Informatics, University of Bergen, Thormohlensgate 55, 5020 Bergen, Norway

*E-mail address*: tors@ii.uib.no