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ARITHMETICAL STUDY OF A CERTAIN TERNARY RECURRENCE SEQUENCE AND RELATED QUESTIONS

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ABSTRACT. The complete solution in $(n, y_1, y_2) \in \mathbf{Z}^3$ of the Diophantine equation

$$b_n = \pm 2^{y_1} 3^{y_2}$$

is given, where $(b_n)_{n\in\mathbf{Z}}$ is Berstel's recurrence sequence defined by

$$b_0 = b_1 = 0, \quad b_2 = 1, \quad b_{n+3} = 2b_{n+2} - 4b_{n+1} + 4b_n.$$

1. INTRODUCTION

Let $(u_n)_{n\in\mathbf{Z}}$ be a linear recurrence sequence in $\mathbf{Q}$ whose characteristic polynomial has at least two distinct roots and suppose that this sequence is nondegenerate, i.e., the ratio of any distinct roots of the characteristic polynomial is not a root of unity. Let c be an integer which is either a constant or an S-integer (i.e., an integer whose prime divisors belong to a finite fixed set of primes). Under these assumptions, the equation $u_n = c$ (in the unknown n) has at most finitely many solutions; see, for example, Corollary 3 of J. H. Evertse [3]. The problem of the *explicit* computation of these solutions is a difficult one, and in a previous paper of ours [6] we propose a general practical method for the explicit solution of equations as above. The purpose of our present paper is to give an interesting application of our method [6] to the equation

$$(1) \qquad b_n = \pm 2^r 3^s,$$

where $(b_n)_{n\in\mathbf{Z}}$ is Berstel's ternary recurrence sequence defined by

$$b_0 = b_1 = 0, \quad b_2 = 1, \quad b_{n+3} = 2b_{n+2} - 4b_{n+1} + 4b_n.$$

We quote from the introduction of our paper [6]: "Among ternary linear recurrence sequences, it seems that Berstel's sequence...plays a very special role. Firstly, it is the only known example of a nondegenerate ternary linear recurrence sequence which has six zeros (by definition, a nondegenerate linear recurrence sequence has only finitely many zeros). It was proved in [4] that it contains exactly six zeros. F. Beukers has just proved [in the meantime, this has appeared in [1]] that six is the right upper bound for the number of zeros of nondegenerate ternary recurrence sequences of integers. Secondly, Berstel's sequence contains

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many repetitions; indeed it was proved in [5] that the equation $b_m = \pm b_n$ for rational integers $m, n \in \mathbf{Z}$ has exactly 21 solutions (m, n) with $m < n$, and these solutions were explicitly computed. For the problem studied here, i.e., the equation $u_n = \pm 2^r 3^s$, it seems again that Berstel's sequence has remarkable properties: we can prove that there are exactly 44 solutions (n, r, s)." In our aforementioned paper we announce without proof the complete solution of (1) (see the theorem in §IV of [6]). Here we will give all the details of the solution. In particular, we hope to make clear, by means of the concrete example which we study, the part of our method described only in general terms in the remark of §III of [6].

2. Preliminaries

We work in the field $\mathbf{Q}(\theta)$, where $\theta^3 - 2\theta^2 + 4\theta - 4 = 0$. In this field, $\pi = \theta^2/2$ is a prime element and $(2) = \pi^3$. More precisely, $2 = \pi^3\varepsilon$, where $\varepsilon = 3 - \theta + \theta^2$ is a unit and $\theta = \pi^2\mu$, where $\mu = 1 + \theta^2/2$ is a unit ($1, \theta, \theta^2/2$ is an integral basis in the field $\mathbf{Q}(\theta)$). If $\theta^{(1)}, \theta^{(2)}, \theta^{(3)}$ are the conjugates of θ in $\mathbf{C}$ (exactly one is real), then it is easy to see that

$$b_n = \sum_{i=1}^{3} \alpha_i \theta^{(i)^n}, \quad \text{where } \alpha_1 = \frac{\theta^{(3)} - \theta^{(2)}}{4\sqrt{-11}}, \alpha_2 = \frac{\theta^{(3)} - \theta^{(1)}}{4\sqrt{-11}}, \alpha_3 = \frac{\theta^{(2)} - \theta^{(1)}}{4\sqrt{-11}}.$$

We write $n = 3m + j$, with $j \in \{0, 1, 2\}$. Then

$$\begin{aligned} b_n &= \sum_{i=1}^{3} \alpha_i \pi^{(i)^{6m+2j}} \mu^{(i)^{3m+j}} = \sum_{i=1}^{3} \alpha_i (\pi^{(i)^2} \mu^{(i)})^j (\pi^{(i)^3})^{2m} \mu^{(i)^{3m}} \\ &= \sum_{i=1}^{3} \alpha_i (\pi^{(i)^2} \mu^{(i)})^j (2\varepsilon^{(i)^{-1}})^{2m} \mu^{(i)^{3m}} \\ &= 2^{2m} \sum_{i=1}^{3} \alpha_i (\pi^{(i)^2} \mu^{(i)})^j (\mu^{(i)^3} \varepsilon^{(i)^{-2}})^m = 2^{2m} \sum_{i=1}^{3} \alpha_i \theta^{(i)^j} \omega^{(i)^m}, \end{aligned}$$

where $\omega = \mu^3 \varepsilon^{-2}$ and $\omega^3 + \omega^2 + \omega - 1 = 0$.

Thus, for $j = 1, 2$, $b_{3m+j} = 2^{2m} u_{jm}$, where u_{jm} is given by the formula

$$u_{jm} = \sum_{i=1}^{3} \alpha_i \theta^{(i)^j} \omega^{(i)^m} \tag{2}$$

and

$$b_{3m} = 2^{2m-1} u_{0m}, \quad \text{where } u_{0m} = \sum_{i=1}^{3} 2\alpha_i \omega^{(i)^m},$$

and in all the three cases, $u_{j,m+3} = -u_{j,m+2} - u_{j,m+1} + u_{j,m}$.

It is easy to see that $(u_{j0}, u_{j1}, u_{j2}) = (0, 1, 0), (0, 0, 1), (1, -1, 1)$ according as $j = 0, 1, 2$, respectively. In the following sections we shall solve the equations

$$u_{jm} = \pm 2^r 3^s \tag{3}$$

for each value $j \in \{0, 1, 2\}$ separately. The advantage of working with the sequence (u_{jm}) instead of (b_n) is that the first one assumes only integral values, even for negative index n. For simplicity in our notations we will omit the index j, but in the beginning of each section it will be clear which sequence we study.

In the present paper we will often apply Theorem 1 of §II of [6]. We describe its use in our situation: Let p be a prime $\neq 2, 11$ (these are the primes dividing the discriminant of the minimal polynomial of ω). Choose a positive integer S such that $\omega^S \equiv A \pmod{p}$ for some $A \in \mathbf{Z}$. Suppose, moreover, that A has been chosen in such a way that the orders of A modulo p and p^2 have the same value R. Then we have the following result (cf. Theorem 1, §II of [6]):

Theorem 1. *Let the rational integer c be such that either $c \not\equiv 0 \pmod{p}$ or $c = 0$. Let $\mathscr{P}$ be a complete system of residues modulo S, and $\mathscr{M}$ a subset of $\mathscr{P}$ satisfying the following conditions:*

(i) *$u_m = c$ for every $m \in \mathscr{M}$,*

(ii) *if $n \in \mathscr{P}$ and $u_n \equiv cA^r \pmod{p}$ for some $r \in \{0, 1, \dots, R-1\}$, then $n \in \mathscr{M}$,*

(iii) *$u_{m+S} \not\equiv Au_m \pmod{p^2}$ for every $m \in \mathscr{M}$.*

Then $u_n = c$ implies $n \in \mathscr{M}$.

In the beginning of §4 we will use another result from [6] (Theorem 2 of [6]):

Theorem 2. *Let p, ω, and A be as in Theorem 1 and $\mathscr{N} = \{n \in \mathbf{Z}: u_n = 0\}$. Let q be a prime $\neq p$ and ν a positive integer such that the following condition is satisfied:*

$$u_m \equiv 0 \pmod{q^\nu} \Rightarrow \exists n \in \mathscr{N} \text{ such that } n \equiv m \pmod{S}.$$

Then $u_m \equiv 0 \pmod{q^\nu}$ implies that p divides u_m.

Remark. More often in this paper we will use, instead of Theorem 2, the following trick (cf. with the remark of §III of [6]): Let p be a prime. Then (u_n) is periodic modulo p, with period P, say. Next, consider a prime $q \neq p$. The sequence (u_n) is periodic modulo q^ν for any positive integer ν, with period length Q, say (depending on ν); therefore, a relation of the form $u_n \equiv 0 \pmod{q^\nu}$ restricts the values of the index n modulo Q, hence, *if* gcd(P, Q) *is not "very small"*, restricts the values of n modulo P to only "a few" possibilities, say $n_1, \dots, n_k \pmod{P}$. With a convenient choice of the prime p, it can happen that p divides u_{n_j} for every $j = 1, \dots, k$, and in this case we get the same conclusion as in Theorem 2, i.e.,

$$u_n \equiv 0 \pmod{q^\nu} \Rightarrow u_n \equiv 0 \pmod{p}.$$

Thus, if it is known a priori that u_n *is not* divisible by p, we conclude that a power of q can divide u_n only if it is lower than q^ν.

3. The case $j = 0$

Here we have

$$u_0 = 0, \quad u_1 = 1, \quad u_2 = 0, \quad u_{m+3} = -u_{m+2} - u_{m+1} + u_m.$$

We apply first the remark at the end of §2, with $q = 2$, $\nu = 3$, $Q = 16$, $p = 7$, and $P = 48$ to conclude that if 8 divides u_m, then 7 also divides u_m. Therefore, we can assume $0 \le r \le 2$.

To find an upper bound for s using Theorem 2 or the above remark seems difficult in this case. Therefore, we work as follows, distinguishing three cases. First note that for $s \geq 2$ one has necessarily $m \equiv 0, 2 \pmod{13}$.

(i) $m \equiv 0 \pmod{13}$ and $s \geq 7$. In this case,

$$u_m \equiv 0 \pmod{3^7} \Rightarrow m \equiv 0 \pmod{13 \cdot 81} \Rightarrow m \equiv 0, 81 \pmod{162}.$$

Since $m \equiv 0 \pmod{162}$ implies that 163 divides u_m, we have $m \equiv 81 \pmod{162}$.

(ii) $m \equiv 2 \pmod{13}$ and $s \geq 6$. In this case an argument similar to the previous one shows that we must have $m \equiv 83 \pmod{162}$.

(iii) $0 \leq s \leq 6$. This case can be treated as §II of [6] suggests (see below).

First we exclude the first two cases. To simplify notations, we shall write

$$(a, b) \equiv (a', b') \quad \mathrm{mod}(m_1, m_2)$$

instead of the two relations

$$a \equiv a' \pmod{m_1} \,\&\, b \equiv b' \pmod{m_2}.$$

Note that in both cases (i) and (ii) the index m is odd, which implies that u_m is also odd, and therefore $r = 0$. Moreover, we have

In case (i),

$$\begin{aligned} m \equiv 81 \pmod{162} &\Rightarrow u_m \equiv 15 \pmod{163} \\ &\Rightarrow 3^s \equiv \pm 15 \pmod{163} \Rightarrow s \equiv 22 \pmod{81}; \end{aligned}$$

therefore

$$(m, s) \equiv (9, 4) \quad \mathrm{mod}(18, 9). \tag{4}$$

We have the table

m	9	27	45	63	81	99	117	135	153	171	189	mod 198
u_m	−3	21	97	−80	−41	91	10	−92	79	−8	−74	mod 199
s	100	143	131	67	68	116	46	33	61	21	18	mod 199

The only pair (m, s) in this table which satisfies (4) is $(63, 67)\,\mathrm{mod}(198, 99)$.

On the other hand, $m \equiv 63 \pmod{198}$ implies $m \equiv 63, 129 \pmod{132}$ and, making use of the auxiliary prime $397 = 3 \cdot 132 + 1$, we see that

$$\begin{aligned} m \equiv 63 \pmod{132} &\Rightarrow u_m \equiv -156 \pmod{397} \Rightarrow 3^s \equiv \pm 156 \pmod{397} \\ &\Rightarrow s \cdot \mathrm{ind}\, 3 = \mathrm{ind}(\pm 156) \pmod{396} \Rightarrow s \equiv 2 \pmod{9}, \end{aligned}$$

which contradicts $s \equiv 67 \pmod{99}$.

If $m \equiv 129 \pmod{132}$, we get analogously $s \equiv 1 \pmod{9}$, which is again a contradiction.

In case (ii) we work as in case (i) to obtain first

$$(m, s) \equiv (11, 8) \quad \mathrm{mod}\,(18, 9), \tag{5}$$

and then we construct a table relative to the auxiliary prime 199, from which we see that the only pair which satisfies (5) is

$$(m, s) \equiv (155, 80) \quad \mathrm{mod}\,(198, 99). \tag{6}$$

In particular, $m \equiv 23, 89 \pmod{132}$ and, as before, we make use of the auxiliary prime 397. If $m \equiv 23 \pmod{132}$, we easily get $s \equiv 0 \pmod 9$, which contradicts (6), and if $m \equiv 89 \pmod{132}$, then $s \equiv 10 \pmod{11}$, which again contradicts (6) (note that 396 is divisible by 99).

Now that we have excluded cases (i) and (ii), we are left with (iii); i.e., we have to solve

$$u_m = \pm 2^r \cdot 3^s, \qquad 0 \le r \le 2, 0 \le s \le 6. \tag{7}$$

We applied Theorem 1 of [6] with $p \in \{47, 53, 103, 163, 199, 397\}$, using a simple computer program. The values of the various parameters and a summary of the application of that theorem to the solution of (7) are given, respectively, in Tables I and II.

TABLE I

p	S	A	R
47	46	1	1
53	52	1	1
103	17	56	3
163	54	−59	3
199	66	−93	3
397	132	1	1

TABLE II. $u_m = \pm 2^r 3^s$, $0 \le r \le 2$, $0 \le s \le 6$

u_m		m		p	
1	−1	1, −1	3, 5	53	103
2	−2	4, −2	6, 12	53	53
3	−3	−3	9	199	103
4	−4	∅	8	47	53
6	−6	−4	∅	53	47
9	−9	15	∅	163	199
12	−12	10	∅	53	47
18	−18	∅	∅	47	47
27	−27	13	∅	103	53
36	−36	∅	∅	47	53
54	−54	∅	∅	47	47
81	−81	∅	∅	53	47
108	−108	∅	∅	47	47
162	−162	∅	∅	103	103
243	−243	∅	∅	47	47
324	−324	∅	∅	103	53
486	−486	∅	∅	47	103
729	−729	∅	∅	47	47
972	−972	∅	∅	47	47
1458	−1458	∅	∅	103	53
2916	−2916	∅	∅	103	103

The symbol $\varnothing$ in the second main column means that the congruence $u_m \equiv c \pmod{p}$, where c and p are the numbers of the first and third main column on the same row, respectively, is impossible.

4. The cases $j = 1$ and $j = 2$

Here we have

$$\begin{aligned} j = 1: \quad & u_0 = 0, \quad u_1 = 0, \quad u_2 = 1, \quad u_{m+3} = -u_{m+2} - u_{m+1} + u_m, \\ j = 2: \quad & u_0 = 1, \quad u_1 = -1, \quad u_2 = 1, \quad u_{m+3} = -u_{m+2} - u_{m+1} + u_m. \end{aligned}$$

This section is mainly devoted to the case $j = 1$. The case $j = 2$ is very easy, and its solution is given at the end of the section.

The case $j = 1$.

First we solve the equation

$$u_m = \pm 3^s. \tag{8}$$

It is easily checked that

$$u_m \equiv 0 \pmod{3^8} \Rightarrow m \equiv 0, 1, 4, 17 \pmod{3^4 \cdot 13}.$$

In particular, if $s \geq 8$, either $m \equiv 0, 1, 4, 17$ or $m \equiv 81, 82, 85, 98 \pmod{162}$. In the first case, the hypotheses of Theorem 2 of [6] are satisfied with $q = 3$, $p = 163$, $S = 162$, $A = 1$, and $\nu = 8$, and we conclude that 163 divides u_m, which contradicts (8). Thus, we are left with the following cases:

(i) $s \geq 8$ and $m \equiv 81, 82, 85, 98 \pmod{162}$,

(ii) $s \leq 7$.

We show that (i) is impossible, as we did in (i) and (ii) when $j = 0$. We have,

$$\begin{aligned} m \equiv 81 \pmod{162} &\Rightarrow u_m \equiv -61 \pmod{163} \\ &\Rightarrow 3^s \equiv \pm 61 \pmod{163} \Rightarrow s \equiv 12 \pmod{81}. \end{aligned}$$

Analogously,

$$\begin{aligned} m \equiv 82 \pmod{162} &\Rightarrow s \equiv 59 \pmod{81}, \\ m \equiv 85 \pmod{162} &\Rightarrow s \equiv 38 \pmod{81}, \\ m \equiv 98 \pmod{162} &\Rightarrow s \equiv 1 \pmod{81}. \end{aligned}$$

Therefore, only the following cases are possible:

$$(m, s) \equiv (9, 3), (10, 5), (13, 2), (8, 1) \quad \bmod (18, 9).$$

Then, we work modulo 199, exactly as we did immediately after (4), to conclude that the fourth case above is impossible and, corresponding to the first three cases, we have respectively

$$(m, s) \equiv (9, 21), (171, 21) \quad \bmod (198, 99), \tag{9}$$

$$(m, s) \equiv (190, 5) \quad \bmod (198, 99), \tag{10}$$

$$(m, s) \equiv (157, 92) \quad \bmod (198, 99). \tag{11}$$

In case (9) we have $m \equiv 9, 171, 207, 369 \pmod{396}$, which implies

$$m \equiv 9, 39, 75, 105 \pmod{132}. \tag{12}$$

Analogously, in cases (10) and (11) we have, respectively,

$$m \equiv 58,\ 124 \pmod{132}, \tag{13}$$

$$m \equiv 25,\ 91 \pmod{132}. \tag{14}$$

Then we work modulo 397 to show that each pair of the relations (9) & (12), (10) & (13), and (11) & (14), is either contradictory or includes an impossible relation. We give some typical examples (note that the order of 3 modulo 397 is 198 and that $-1 \equiv 3^{99} \pmod{397}$):

$$m \equiv 9 \pmod{132} \Rightarrow u_m \equiv -8 \pmod{397} \Rightarrow 3^s \equiv \pm 8 \pmod{397}$$

and the last relation is impossible;

$$m \equiv 75 \pmod{132} \Rightarrow u_m \equiv -136 \pmod{397} \Rightarrow 3^s \equiv \pm 136 \pmod{397}$$
$$\Rightarrow s \equiv 114 \pmod{198} \Rightarrow s \equiv 6 \pmod 9,$$

while (9) implies that $s \equiv 3 \pmod 9$;

$$m \equiv 91 \pmod{132} \Rightarrow u_m \equiv 179 \pmod{397} \Rightarrow 3^s \equiv \pm 179 \pmod{397}$$
$$\Rightarrow s \equiv 47 \pmod{198} \Rightarrow s \equiv 3 \pmod{11},$$

while from (11) we must have $s \equiv 4 \pmod{11}$. All the remaining cases are treated analogously.

Now we are left with case (ii) of (8). We deal with it as we did with equation (7), and we summarize its solution in Table III.

The equation

$$u_m = \pm 2^r \tag{15}$$

is easier to solve than (8). First we observe that

$$u_m \equiv 0 \pmod{2^4} \Rightarrow u_m \equiv 0 \pmod 7, \tag{16}$$

which implies that $r \leq 3$ in (15), and this equation's solution is summarized in Table IV.

TABLE III. $u_m = \pm 3^s$, $0 \leq s \leq 7$

u_m		m		p	
1	−1	−2, −1, 2, 7	3	47	47
3	−3	∅	6	47	397
9	−9	14	∅	163	103
27	−27	∅	∅	103	103
81	−81	30, −9	∅	53	53
243	−243	∅	∅	53	53
729	−729	∅	∅	53	47
2187	−2187	∅	∅	47	47

TABLE IV. $u_m = \pm 2^r$, $1 \leq r \leq 3$

u_m		m		p	
2	−2	5, −3	∅	53	53
4	−4	8, −4	∅	53	53
8	−8	∅	9	53	163

Finally, we are left with the equation

$$u_m = \pm 2^r \cdot 3^s, \qquad 1 \le r \le 3,\, s \ge 1. \tag{17}$$

A main difficulty lies in the problem of finding an upper bound for the exponent s. In §6, we discuss an alternative approach to the solution of (17). We observe the following fact:

$$\begin{aligned} u_m \equiv 0 \pmod 9 &\Rightarrow m \equiv 0,\,1,\,4 \pmod{13} \\ &\Rightarrow m \equiv 0,\,1,\,4,\,13,\,14,\,17,\,26,\,27,\,30,\,39,\,40,\,43 \pmod{52}. \end{aligned}$$

The values $m \equiv 0,\,1,\,4,\,17 \pmod{52}$ are rejected because, for such values, 53 divides u_m. On the other hand, from (17), u_m is even, which implies that $m \equiv 0,\,1 \pmod 4$. Thus, we are left with

$$m \equiv 13,\,40 \pmod{52}. \tag{18}$$

Also, if 3^5 divides u_m, then $m \equiv 0,\,1,\,4,\,17 \pmod{39}$ and the values $m \equiv 4,\,17 \pmod{39}$ are rejected in view of (18). Finally, if 3^6 divides u_m and $m \equiv 0,\,1 \pmod{39}$, then $m \equiv 0,\,1 \pmod{13\cdot 3^4}$. Therefore, in order to solve (16), we distinguish two cases:

(i) $s \ge 6$, and consequently $m \equiv 0,\,1 \pmod{13 \cdot 81}$,

(ii) $s \le 5$.

First we exclude case (i), working modulo 163, 199, and 397, successively. Notice that in case (i) we have, in particular, $m \equiv 81,\,82 \pmod{162}$, as we noticed in the resolution of equation (8).

The number 3 is a primitive root for the modulus 163; therefore,

$$\begin{aligned} m \equiv 81 \pmod{162} &\Rightarrow u_m \equiv -61 \pmod{163} \Rightarrow 2^r 3^s \equiv \pm 61 \pmod{163} \\ &\Rightarrow 3^{77r+s} \equiv \pm 61 \pmod{163} \Rightarrow 77r + s \equiv 12 \pmod{81}. \end{aligned}$$

From this last congruence we see that

$$s \equiv 7,\,2,\,6 \pmod 9 \quad \text{according as } r = 1,\,2,\,3, \text{ respectively.}$$

Analogously, we find that if $m \equiv 82 \pmod{162}$, then

$$s \equiv 0,\,4,\,8 \pmod 9 \quad \text{according as } r = 1,\,2,\,3, \text{ respectively.}$$

Thus, we have the following three cases:

$$r = 1 \quad \text{and} \quad (m,\,s) \equiv (9,\,7),\,(10,\,0) \bmod (18,\,9), \tag{19}$$

$$r = 2 \quad \text{and} \quad (m,\,s) \equiv (9,\,2),\,(10,\,4) \bmod (18,\,9), \tag{20}$$

$$r = 3 \quad \text{and} \quad (m,\,s) \equiv (9,\,6),\,(10,\,8) \bmod (18,\,9). \tag{21}$$

Next we work modulo 199. Equation (17) implies $u_m \equiv \pm 3^{7r+s} \pmod{199}$. We have the following table when $m \equiv 9 \pmod{18}$:

m	9	27	45	63	81	99	117	135	153	171	189	mod 198
u_m	−8	−43	63	13	−7	86	−38	−92	−47	−8	81	mod 199
$7r+s$	21	88	45	73	43	85	62	33	76	21	4	mod 99
$7r+s$	3	7	0	1	7	4	8	6	4	3	4	mod 9

When $m \equiv 10 \pmod{18}$, we have the table

m	10	28	46	64	82	100	118	136	154	172	190	mod 198
u_m	5	64	34	−93	−34	5	48	0	−73	0	44	mod 199
$7r+s$	39	42	31	66	31	39	29	*	52	*	5	mod 99
$7r+s$	3	6	4	3	4	3	2	*	7	*	5	mod 9

(an asterisque means that, modulo 199, the value u_m is not a power of 3). From the above tables it is easy to check the following facts:

(22) relation (19) is possible only if $m \equiv 154 \pmod{198}$,

(23) relation (20) is possible only if $m \equiv 27, 81 \pmod{198}$,

(24) relation (21) is possible only if $m \equiv 45, 118 \pmod{198}$.

The three relations (22), (23), and (24) imply respectively

$$m \equiv 22, 88; 27, 93, 15, 81; 45, 111, 52, 118 \pmod{132}.$$

Since u_m is even (by (17)), we must have $m \equiv 0, 1 \pmod 4$, and consequently the following cases are left:

(25) $r = 1$ and $(m, s) \equiv (88, 0) \bmod (132, 9)$,

(26) $r = 2$ and $(m, s) \equiv (93, 2), (81, 2) \bmod (132, 9)$,

(27) $r = 3$ and $(m, s) \equiv (45, 6), (52, 8) \bmod (132, 9)$.

Finally we work modulo 397. Relation (17) implies

$$t \cdot \mathrm{ind}(-1) + r \cdot \mathrm{ind}(2) + s \cdot \mathrm{ind}(3) \equiv \mathrm{ind}(u_m) \pmod{396},$$

where $t \in \{0, 1\}$. Since $\mathrm{ind}(-1) \equiv \mathrm{ind}(2) \equiv 0 \pmod 9$ and $\mathrm{ind}(3) \equiv 2 \pmod 9$, we must have

$$2s \equiv \mathrm{ind}(u_m) \pmod 9. \tag{28}$$

If (25) is true, then we have the following implications:

$$m \equiv 88 \pmod{132} \Rightarrow u_m \equiv 33 \pmod{397} \Rightarrow \mathrm{ind}(u_m) \equiv 322 \pmod{396}$$
$$\Rightarrow \mathrm{ind}(u_m) \equiv 7 \pmod 9 \Rightarrow s \equiv 8 \pmod 9$$

(in view also of (28)), and this contradicts (25). In an analogous way we prove the impossibility of (26) and (27), and this shows that case (i) (i.e., $s \geq 6$) is impossible.

It remains therefore to solve the equation $u_m = \pm 2^r 3^s$ with $1 \leq r \leq 3$ and $1 \leq s \leq 5$. The usual table corresponding to this equation is found in Table V.

The case $j = 2$. This is the easiest case. It is obvious that u_m is odd for every m; therefore, we have to solve the equation $u_m = \pm 3^s$. It is easily checked that u_m is never divisible by 27. Therefore, $0 \leq s \leq 2$. The usual table of solutions is given in Table VI.

TABLE V. $(j=1)u_m = \pm 2^r 3^s$, $1 \le r \le 3$, $0 \le s \le 3$

u_m		m		p	
6	−6	∅	∅	47	47
12	−12	∅	∅	47	47
18	−18	13	∅	53	53
24	−24	−7	∅	53	47
36	−36	∅	∅	47	53
54	−54	∅	∅	103	47
72	−72	∅	∅	47	47
108	−108	∅	∅	199	47
162	−162	∅	∅	47	47
216	−216	∅	∅	47	47
324	−324	∅	∅	47	103
486	−486	∅	∅	47	47
648	−648	∅	∅	47	47
972	−972	∅	∅	47	47
1944	−1944	∅	∅	47	47

TABLE VI. $(j=1)u_m = \pm 3^s$, $0 \le s \le 2$

u_m		m		p	
1	−1	−2, −1, 0, 2, 3, 6	1, 9	103	47
3	−3	−3, 5	4	47	103
9	−9	−5, 8	∅	47	47

5. THE FINAL RESULT

The results of all the previous sections are summarized as follows:

Theorem 3. *The only solutions* (n, y_1, y_2) *of the equation*

$$b_n = \pm 2^{y_1} \cdot 3^{y_2}, \qquad n \in \mathbf{Z},$$

where $b_0 = b_1 = 0$, $b_2 = 1$, $b_{n+3} = 2b_{n+2} - 4b_{n+1} + 4b_n$, *are the* 44 *ones listed below.*

n	−26	−20	−13	−12	−11	−9	−8	−7	−6	−5	−4
b_n	$2^{-18}3^4$	$2^{-11}3$	$2^{-10}3^2$	$2^{-8}3$	2^{-6}	$2^{-7}3$	2^{-5}	$2^{-6}3$	2^{-4}	2^{-4}	2^{-4}
n	−3	−2	−1	2	3	5	7	8	9	10	11
b^n	2^{-3}	2^{-2}	2^{-2}	1	2	-2^2	2^4	2^4	-2^5	-2^6	2^6
n	12	14	15	16	17	18	19	20	22	24	25
b_n	2^8	$-2^8 3$	-2^9	2^{11}	$2^{10}3$	-2^{12}	$-2^{12}3$	2^{12}	2^{14}	-2^{17}	2^{18}
n	26	27	28	29	30	36	39	40	43	45	91
b_n	$2^{16}3^2$	$-2^{17}3$	-2^{21}	-2^{18}	$2^{21}3$	-2^{24}	$2^{25}3^3$	$2^{27}3^2$	$2^{28}3^2$	$2^{29}3^2$	$2^{60}3^4$

6. An alternative approach to (17)

In this section we indicate how linear forms in logarithms of algebraic numbers, in combination with a recent computational technique, can be applied to the solution of equation (17).

It is not difficult to prove the following (we omit the proof):

Lemma. *If in equation* (17) *we have* $s \geq 5$, *then* $m = 13 \cdot 3^{\nu} \cdot M + j$, *where* $j \in \{0, 1, 4, 17\}$ *and* $\nu \geq s - 4$.

This result will be applied below. In the sequel we will assume that $s \geq 5$. We number the conjugates of θ as follows:

$$\theta^{(1)} \simeq 0.352201129 + i \cdot 1.721433237, \quad \theta^{(2)} = \overline{\theta^{(1)}}, \quad \theta^{(3)} \simeq 1.295597743.$$

Then $|\omega^{(1)}| = |\omega^{(2)}| > 1$ and $|\omega^{(3)}| < 1$. From (1) we have

$$u_m = \sum_{i=1}^{3} \alpha^{(i)} \theta^{(i)} \omega^{(i)^m} = \sum_{i=1}^{3} \beta^{(i)} \omega^{(i)^m},$$

where $\beta^{(i)} = \alpha^{(i)} \theta^{(i)}$, for $i = 1, 2, 3$.

Let $m > 17$. Then $u_m \neq 0$, and consequently $|u_m| \geq 1$. Also, $|\beta^{(3)} \omega^{(3)^m}| < 5.8 \cdot 10^{-6}$, and therefore

$$\begin{aligned} 1 \leq |u_m| &\leq |\beta^{(1)} \omega^{(1)^m} + \beta^{(2)} \omega^{(2)^m}| + |\beta^{(3)} \omega^{(3)^m}| \\ &< |\beta^{(1)} \omega^{(1)^m} + \beta^{(2)} \omega^{(2)^m}| + 5.8 \cdot 10^{-6}. \end{aligned}$$

Then,

$$\begin{aligned} |u_m| &> 0.999994 \cdot |\beta^{(1)} \omega^{(1)^m} + \beta^{(2)} \omega^{(2)^m}| \\ &= 0.999994 \cdot |\beta^{(1)} \omega^{(1)^m}| \cdot \left| \left(-\frac{\beta^{(2)}}{\beta^{(1)}}\right) \cdot \left(-\frac{\omega^{(2)}}{\omega^{(1)}}\right)^m - 1 \right| \\ &> 0.259988 \cdot |\omega^{(1)}|^m \left| \left(-\frac{\beta^{(2)}}{\beta^{(1)}}\right) \cdot \left(-\frac{\omega^{(2)}}{\omega^{(1)}}\right)^m - 1 \right|. \end{aligned}$$

We put

$$\Lambda = \operatorname{Log} \left(-\frac{\beta^{(2)}}{\beta^{(1)}}\right) \left(-\frac{\omega^{(2)}}{\omega^{(1)}}\right)^m = \operatorname{Log} \left(-\frac{\beta^{(2)}}{\beta^{(1)}}\right) + m \cdot \operatorname{Log} \left(\frac{\omega^{(2)}}{\omega^{(1)}}\right) + k \cdot \operatorname{Log}(-1),$$

where Log denotes the principal branch of the logarithmic function and k is some integer with $|k| \leq m + 1$. Then

$$|u_m| > 0.259988 \cdot |\omega^{(1)}|^m \cdot |e^{\Lambda} - 1|. \tag{29}$$

If $|e^{\Lambda} - 1| < 0.5$, then

$$|e^{\Lambda} - 1| > 0.98 \cdot |\Lambda| > 0. \tag{30}$$

We can now apply the theory of linear forms in logarithms of algebraic numbers (see [7]and [2]) to find a lower bound for $|\Lambda|$ as follows: By Waldschmidt's theorem [7] we found

$$|\Lambda| > \exp\{-7.1669 \cdot 10^{25} \cdot (\operatorname{Log} m + 3.991)\}.$$

Therefore,

(31) $$|u_m| > 0.259988 \cdot 1.3562^m \cdot \exp\{-7.1669 \cdot 10^{25} \cdot (\operatorname{Log} m + 3.991)\}.$$

On the other hand, from (17) and the lemma, we have

$$\nu \leq \frac{\operatorname{Log} m - \operatorname{Log} 13}{\operatorname{Log} 3} < 2.335 \cdot \operatorname{Log} m,$$

so that

(32) $$|u_m| \leq 8 \cdot 3^{2.335 \operatorname{Log} m + 4}.$$

Combine the last inequality with (29) to see that

(33) $$m \leq 5.52 \cdot 10^{27},$$

from which we can find an upper bound for s. Indeed, in view of (32),

$$3^s \leq |u_m| \leq 8 \cdot 3^{2.335 \operatorname{Log} m + 4},$$

from which, in combination with (31), we get $s \leq 155$.

If $m \leq 0$, things are much easier: Let us put $m = -n$, where we may suppose that $n \geq 2$. Then, from the equality

$$u_m = \beta^{(1)}(\omega^{(1)^{-1}})^m + \beta^{(2)}(\omega^{(2)^{-1}})^m + \beta^{(3)}(\omega^{(3)^{-1}})^m$$

it is easy to see that

(34) $$|u_m| > 0.75|\beta^{(3)}|\,|\omega^{(3)^{-1}}|^m > 0.252171 \cdot 1.839287^m.$$

On the other hand, from $m - j = 13 \cdot 3^\nu \cdot M$, on putting $M = -N$, $N > 0$, we get the relation $m + j = 13 \cdot 3^\nu \cdot N \geq 13 \cdot 3^\nu$ and $13 \cdot 3^\nu \leq m + 17$, from which

$$\nu \leq \frac{\operatorname{Log} m}{\operatorname{Log} 3} < 0.91024 \cdot \operatorname{Log} m.$$

Then, in view of the lemma,

$$|u_m| \leq 8 \cdot 3^s \leq 8 \cdot 3^{\nu+4} \leq 8 \cdot 3^{0.91024 \operatorname{Log} m + 4},$$

and this relation, combined with (34), gives $m < 18$. Thus, if $m < 0$, we have to check only the values $-17 \leq m \leq -1$, and this is done trivially.

The case $m > 0$ (in fact, we have supposed that $m > 17$) requires much more effort; it is not a realistic task to check u_m for all $m \leq 5.52 \cdot 10^{27}$ (cf. (33)); nor is it realistic to solve all equations $u_m = 2^r \cdot 3^s$ for $0 \leq r \leq 3$ and $0 \leq s \leq 155$. Therefore, we need a practical method for reducing the very large upper bound for m. Note first that in (29) we may suppose $|e^\Lambda - 1| < 0.5$. Indeed, if this is not the case, then (32) and (29) imply

$$8 \cdot 3^{2.335 \operatorname{Log} m + 4} \geq 0.12994 \cdot 1.356204^m,$$

which gives $m \leq 63$. Then, by the lemma, the only possible values for m are 39, 40, 43, and 56, and none of them is a solution of (17). Thus, $|e^\Lambda - 1| < 0.5$ holds, which implies (30). This, in turn, in combination with (32) implies

$$|\Lambda| \leq 2543.291 \cdot 13.00404^{\operatorname{Log} m} \cdot 1.356203^{-m}.$$

If $m \geq 331$, then

$$|\Lambda| < 2543.29 \cdot 1.356203^{-m/2},$$

and we have to solve the last inequality under the restrictions $|\Lambda| > 0$ and $17 < m < 5.52 \cdot 10^{27}$. At this point, we can apply the technique of B. M. M. de Weger (see §§7B and 7C of [8]), which reduces the upper bound of m logarithmically. This requires, however, a considerable amount of computations. After this, only the "small" values of m will remain to be checked, and this can be done easily, provided we have a computer program for doing long-integer arithmetic. Indeed, in the range $m \leq 1000$, say, we have to check (in view of the lemma) only the values which are congruent to $0, 1, 4, 17$ modulo 39, i.e., only about 100 values, to see which of them satisfy (17).

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