MATHEMATICS OF COMPUTATION
VOLUME 61, NUMBER 204
OCTOBER 1993, PAGES 931–934

CHECKING THE GOLDBACH CONJECTURE UP TO $4 \cdot 10^{11}$

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ABSTRACT. One of the most studied problems in additive number theory, Goldbach's conjecture, states that every even integer greater than or equal to 4 can be expressed as a sum of two primes. In this paper checking of this conjecture up to $4 \cdot 10^{11}$ by the IBM 3083 mainframe with vector processor is reported.

1. INTRODUCTION

The Goldbach conjecture states that every even integer greater than or equal to 4 can be expressed as a sum of two primes. This problem appeared for the first time in a letter from Goldbach to Euler in the year 1742.

A direct consequence of the Goldbach conjecture would be that every odd integer greater than or equal to 7 can be expressed as a sum of three primes.

It should be noted, that Goldbach treated the number 1 as a prime. Here we consider the number 2 as the first prime.

Mok Kong Shen [1] reported in 1964 about checking the Goldbach conjecture up to 33,000,000. Stein and Stein [2] checked the conjecture up to 10^8 in 1965 and Light, Forrest, Hammond, and Roe [4] in 1980 up to the same bound, independently. To the best of our knowledge the latest published result is due to Granville, van de Lune, and te Riele [5], who checked the conjecture up to $2 \cdot 10^{10}$ in 1989.

As in [5], we will use the following terminology. By *minimal Goldbach partition* for an even integer n we mean the representation $n = p + q$, where p and q are primes and p is such that $n - p'$ is composite for every prime $p' < p$. The smallest prime in the minimal Goldbach partition of n is denoted by $p(n)$. For every prime q we denote by $S(q)$ the least even number n such that $p(n) = q$.

2. ON THE COMPUTATIONAL PROCESS

The basic method used in our computations was Eratosthenes' sieve method. No primality test was needed in the actual computational process.

The details of the method have been presented in [5]. This paper also includes some valuable statistics concerning the subject. The published values in [5] agree with our results.

At the first step a bit matrix representing the prime numbers up to 1048576 ($= 2^{20}$) was made by using Eratosthenes' sieve method. A 32-bit integer

Received by the editor September 10, 1991 and, in revised form, August 5, 1992.
1991 *Mathematics Subject Classification*. Primary 11P32.
Key words and phrases. Goldbach conjecture, Eratosthenes sieve method.

array of 32768 elements ($= 1048576/32$) was needed for this. Every integer $[1, 1048576]$ was represented by one bit in this table. These primes were stored into another integer table. This table can be used to make prime number tables on intervals up to 2^{40} ($\approx 1.0995 \cdot 10^{12}$), using Eratosthenes' sieve method.

The basic step size was chosen as $2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 = 1021020$. Hence, an interval of more than one million numbers was checked at a time. This selection made the program run faster since we did not need to use the small primes from 2 to 17 in the sieving process. The divisibility by the primes from 19 up to the square root of the last integer of the interval had to be checked by sieving.

The addition of small primes was replaced by shifting of the bit matrix as a whole by the number of bits corresponding to these prime values. The shifted bit matrices were joined together using the logical OR operation. The remaining zeros were handled separately.

The programs were written in VS FORTRAN and interpreted by the IBM FORTVS2 vectorizing interpreter. The logical (IAND, IOR) and shifting operations (ISHFT) available in FORTRAN were used whenever it was possible. The main parts of the program vectorized quite effectively.

TABLE 1

n	$p(n)$	q_1	q_2	n	$p(n)$	q_1	q_2
6	3	1.485	1.602	3,526,958	727	0.347	1.179
12	5	0.959	0.890	3,807,404	751	0.346	1.203
30	7	0.898	0.494	10,759,922	829	0.359	1.136
98	19	0.529	0.594	24,106,882	929	0.364	1.135
220	23	0.549	0.469	27,789,878	997	0.360	1.194
308	31	0.486	0.541	37,998,938	1039	0.362	1.193
556	47	0.426	0.638	60,119,912	1093	0.366	1.181
992	73	0.375	0.794	113,632,822	1163	0.372	1.158
2,642	103	0.367	0.804	187,852,862	1321	0.369	1.235
5,372	139	0.353	0.876	335,070,838	1427	0.372	1.244
7,426	173	0.336	0.996	419,911,924	1583	0.366	1.344
43,532	211	0.373	0.781	721,013,438	1789	0.364	1.426
54,244	233	0.367	0.821	1,847,133,842	1861	0.376	1.336
63,274	293	0.343	0.998	7,473,202,036	1877	0.400	1.163
113,672	313	0.353	0.941	11,001,080,372	1879	0.407	1.119
128,168	331	0.349	0.971	12,703,943,222	2029	0.401	1.191
194,428	359	0.352	0.968	21,248,558,888	2089	0.407	1.166
194,470	383	0.344	1.033	35,884,080,836	2803	0.386	1.487
413,572	389	0.364	0.909	105,963,812,462	3061	0.394	1.469
503,222	523	0.335	1.178	244,885,595,672	3163	0.404	1.408
1,077,422	601	0.339	1.184				

3. The results

In Table 1 we list the champions for the $p(n)$ function: values of n such that $p(m) < p(n)$ for all even integers $m < n$. This table is an extension of Table 3 in [3] (up to $n = 40 \cdot 10^6$) and of Table 3 in [5] (by four new entries). As in [5], we list the quotient $q_1 = \log(n)/(\log p(n))^2$ and also the quotient $q_2 = p(n)/((\log n)^2 \log\log n)$. It was conjectured on probabilistic grounds in [5] that the latter quotient would be bounded above and for all $n \geq 10$, that is, we should have $p(n) << (\log n)^2 \log\log n$.

Table 1 implies that for all even $n \leq 4 \cdot 10^{11}$ we have $p(n) \leq 3163$.

Table 2 presents a list of champions for the function $S(p)$, that is, primes p such that $S(q) < S(p)$ for all primes $q < p$.

Table 2

p	$S(p)$	p	$S(p)$	p	$S(p)$
3	6	347	1,042,078	1,091	678,546,502
5	12	379	1,172,918	1,097	1,168,888,534
7	30	401	2,041,402	1,283	1,673,268,292
11	124	419	2,406,448	1,301	1,927,528,888
17	418	463	4,288,574	1,327	2,331,465,314
37	1,274	487	4,938,848	1,429	2,538,833,642
53	2,512	509	9,292,156	1,439	2,816,593,312
59	3,526	521	14,341,888	1,451	4,407,165,118
71	4,618	569	17,726,098	1,493	5,801,828,806
83	7,432	593	20,757,292	1,559	8,946,630,856
89	12,778	659	32,507,242	1,571	21,439,965,412
101	26,098	739	34,362,758	1,787	26,070,202,114
131	34,192	743	37,890,844	1,811	30,325,742,068
149	37,768	761	49,358,128	1,867	30,834,371,756
167	59,914	773	68,788,066	1,873	32,652,627,542
179	88,786	839	129,796,642	1,889	44,460,316,708
191	97,768	853	144,516,902	1,907	64,243,962,808
197	112,558	911	150,386,932	1,997	65,334,725,368
223	221,942	941	206,892,484	2,027	113,843,130,358
257	237,544	977	247,013,164	2,153	244,808,993,116
263	485,326	1,031	299,434,108	2,351	384,619,217,512
281	642,358	1,049	379,410,652	2,441	$> 400,000,000,000$
317	686,638	1,061	554,463,808		

4. Remarks

Vinogradov showed in 1937 that every odd integer which is large enough can be expressed as a sum of three primes. Several authors have since then improved the lower bound in this statement and it has been shown to be true for every odd integer n greater than $\exp(\exp(11.503)) \approx 10^{43000}$ [6].

On the other hand, let n be an odd integer. If there is a prime p_1 on the interval $(n - 4 \cdot 10^{11}, n-2)$, then $n - p_1$ is even and it can be expressed as a sum of two primes, say $n - p_1 = p_2 + p_3$. Thus $n = p_1 + p_2 + p_3$. Hence, the Vinogradov statement has been checked up to the first prime number gap (the difference between two consecutive primes) of $4 \cdot 10^{11}$ integers.

Large prime number gaps seem to be quite rare. Young and Potler [7] have investigated prime number gaps up to $7.263 \cdot 10^{13}$. The largest gap found by them was 778. It may be that there is no prime number gap of length $4 \cdot 10^{11}$ or larger below $\exp(\exp(11.503))$.

5. CPU-time

Verification of the Goldbach conjecture required about 130 hours of cpu-time on the IBM 3083 mainframe. In total, about 170 hours of computing time was used on this project.

6. Acknowledgments

I would like to express my gratitude to Dr. Tapani Matala-aho and Professor Keijo Väänänen for valuable advice during this work, to Professor te Riele for informing me about the earlier work [5], and to the referee of this paper for proposing some improvements to the manuscript.

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