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A NEW FORMULA FOR BOX SPLINES ON THREE-DIRECTIONAL MESHES

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ABSTRACT. A new formula for s-variate box splines on three-directional meshes is derived. An application to evaluation of a certain multiple integral is included.

1. INTRODUCTION

Box splines on three-directional meshes and their translates provide a useful tool in the finite element method (see, e.g., [4–5]). This class of multivariate splines is contained in a broader class of splines introduced by de Boor and DeVore [1]. Some basic recurrence relations obeyed by the latter splines have been derived in [2]. Computations with these recursions could be cumbersome and time-consuming. In this note we offer a closed formula for multivariate box splines on three-directional meshes (see (3)).

In §2 we give notation and definitions which will be used in subsequent sections. The main result is derived in §3. An application is given in §4.

2. NOTATION AND DEFINITIONS

By e_j $(1 \le j \le s)$ we will denote the jth coordinate vector in $\mathbb{R}^s$. Also, let $e_{s+1} = e_1 + \cdots + e_s$. For $n_1, \ldots, n_{s+1} \in \{1, 2, \ldots\}$ let $m = n_1 + \cdots + n_{s+1}$. The three-directional mesh in $\mathbb{R}^s$ is the matrix

$$(1) \qquad X = (\underbrace{e_1, \ldots, e_1}_{n_1 \text{ times}}, \ldots, \underbrace{e_{s+1}, \ldots, e_{s+1}}_{n_{s+1} \text{ times}}).$$

The centered box spline $B_n(\cdot|X)$ can be realized as the piecewise polynomial kernel of the distribution

$$f \to \int_{I^m} f\left(\sum_{j=1}^{m} t_j x^j\right) dt, \qquad f \in C_0^\infty(\mathbb{R}^s).$$

Here, $n = (n_1, \ldots, n_{s+1})$, $I = [-1/2, 1/2]$, and $x^1, \ldots, x^m$ denote the columns of X. It is well known that $B_n(\cdot|X)$ is an s-variate piecewise polynomial of total degree $m - s$, $B_n \in C^{\mu-2}(\mathbb{R}^s)$, where μ is the minimal number of columns of X to be deleted to end up with a matrix X' so that $\operatorname{rank}(X') < \operatorname{rank}(X)$, and

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$$\operatorname{supp} B_n(\cdot|X) = \left\{ x = \sum_{j=1}^{m} t_j x^j : t_j \in I ,\ \ x^j \in X \right\}.$$

The Fourier transform of B_n is [3, (2.3)]

$$\widehat{B}_n(y|X) := \int_{\mathbb{R}^s} e^{iy\cdot x} B_n(x|X)\, dx = \prod_{j=1}^{s+1} \left(\operatorname{Sinc} \frac{y \cdot e_j}{2} \right)^{n_j}, \tag{2}$$

where $y \cdot x$ is the dot product, and $\operatorname{Sinc} t = (\sin t)/t$ is the sinc function.

3. A closed formula for B_n

We shall now establish the following formula:

$$B_n(x|X) = \int_\alpha^\beta \prod_{j=1}^{s+1} M_{n_j}(x_j + t) dt , \qquad x_{s+1} \equiv 0. \tag{3}$$

Here, M_{n_j} stands for the univariate central B-spline of degree $n_j - 1$ with knots at $-n_j/2$, $-n_j/2 + 1$, $\ldots$, $n_j/2$ (see, e.g., [6, Chapter 2]), $x^T = (x_1, x_2, \ldots, x_s)$, and

$$\begin{aligned} \alpha &= \max\{-x_j - n_j/2 : 1 \le j \le s+1\}, \\ \beta &= \min\{-x_j + n_j/2 : 1 \le j \le s+1\}. \end{aligned} \tag{4}$$

For the proof of (3) we apply the inversion formula to the first and third members of (2) to obtain

$$B_n(x|X) = (2\pi)^{-s} \int_{\mathbb{R}^s} e^{ix\cdot y} \prod_{j=1}^{s+1} \left(\operatorname{Sinc} \frac{y \cdot e_j}{2} \right)^{n_j} dy. \tag{5}$$

Use of [6, (1.7)]

$$\left(\operatorname{Sinc} \frac{y \cdot e_{s+1}}{2} \right)^{n_{s+1}} = \int_{\mathbb{R}} e^{i(y\cdot e_{s+1})t} M_{n_{s+1}}(t)\, dt$$

on (5) and the fact that $e_{s+1} = e_1 + \cdots + e_s$ gives

$$B_n(x|X) = (2\pi)^{-s} \int_{\mathbb{R}} M_{n_{s+1}}(t) \left[\int_{\mathbb{R}^s} \prod_{j=1}^{s} e^{i(x_j+t)y_j} \left(\operatorname{Sinc} \frac{y_j}{2} \right)^{n_j} dy_j \right] dt.$$

Combining this with

$$\int_{\mathbb{R}} e^{i(x_j+t)y_j} \left(\operatorname{Sinc} \frac{y_j}{2} \right)^{n_j} dy_j = 2\pi M_{n_j}(x_j + t),$$

we obtain

$$B_n(x|X) = \int_{\mathbb{R}} \prod_{j=1}^{s+1} M_{n_j}(x_j + t)\, dt , \qquad x_{s+1} \equiv 0. \tag{6}$$

To complete the proof, let us note that

$$\operatorname{supp} M_{n_j}(x_j + \cdot) = [-x_j - n_j/2, -x_j + n_j/2],$$

$j = 1, 2, \ldots, s+1$. This, in conjunction with (6), yields the assertion.

4. An application

We will now deal with a multiple integral which is defined by

$$J_{s,n} := \int_{\mathbb{R}^s} (\operatorname{Sinc} t_1)^{n_1} \cdots (\operatorname{Sinc} t_s)^{n_s} (\operatorname{Sinc}(t_1 + \cdots + t_s))^{n_{s+1}}\, dt.$$

We show that

$$J_{s,n} = \pi^s B_n(0_s|X) = 2\pi^s \int_0^\beta \prod_{j=1}^{s+1} M_{n_j}(t)\, dt, \tag{7}$$

where now $\beta = \min\{n_j/2 : 1 \le j \le s+1\}$ and 0_s stands for the zero vector in $\mathbb{R}^s$. For the proof of (7) we make a change of variables in (5) putting $y = 2v$. It is readily seen that the Jacobian of this transformation equals 2^s. Next, letting $x = 0_s$ we obtain the first and second members of (7). To obtain the third member of (7), we use the formula (3) with $x = 0_s$. Note that in this case the integrand of (3) is an even function.

If $n = (k, \ldots, k) \in \mathbb{R}^{s+1}$, k a positive integer, then (7) simplifies to

$$J_{s,n} = 2\pi^s \int_0^{k/2} [M_k(t)]^{s+1}\, dt.$$

In particular, $J_{s,n} = \pi^s$ if $k = 1$, and $J_{s,n} = 2\pi^s/(s+2)$ for $k = 2$.

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