

ERROR BOUNDS FOR GAUSS-KRONROD QUADRATURE FORMULAE

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ABSTRACT. The Gauss-Kronrod quadrature formula Q_{2n+1}^{GK} is used for a practical estimate of the error R_n^G of an approximate integration using the Gaussian quadrature formula Q_n^G . Studying an often-used theoretical quality measure, for Q_{2n+1}^{GK} we prove best presently known bounds for the error constants

$$c_s(R_{2n+1}^{GK}) = \sup_{\|f^{(s)}\|_\infty \leq 1} |R_{2n+1}^{GK}[f]|$$

in the case $s = 3n + 2 + \kappa$, $\kappa = \lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor$. A comparison with the Gaussian quadrature formula Q_{2n+1}^G shows that there exist quadrature formulae using the same number of nodes but having considerably better error constants.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

For a given nonnegative and integrable weight function w on $[-1, 1]$, a quadrature formula Q_n and the corresponding remainder R_n of (precise) degree of exactness $\deg(R_n) = s$ are linear functionals defined by

$$Q_n[f] = \sum_{\nu=1}^n a_{\nu,n} f(x_{\nu,n}), \quad R_n[f] = \int_{-1}^1 w(x) f(x) dx - Q_n[f],$$

$$\deg(R_n) = s \Leftrightarrow R_n[p_\nu] \begin{cases} = 0, & \nu = 0, 1, \dots, s, \\ \neq 0, & \nu = s + 1, \end{cases} \quad p_\nu(x) = x^\nu$$

with nodes $-\infty < x_{1,n} < \dots < x_{n,n} < \infty$ and weights $a_{\nu,n} \in \mathbb{R}$. It is well known that the Gaussian quadrature formula $Q_n^G[f] = \sum_{\nu=1}^n a_{\nu,n}^G f(x_{\nu,n}^G)$ having the highest possible degree of exactness $\deg(R_n^G) = 2n - 1$ exists uniquely under these assumptions.

In order to obtain an estimate for $R_n[f]$ in practice, often a second quadrature formula is used whose nodes, for economical reasons, include $x_{1,n}^G, \dots, x_{n,n}^G$. If there exist $n + 1$ further real and distinct nodes $\xi_{1,2n+1}, \dots, \xi_{n+1,2n+1}$ and weights $\beta_{1,2n+1}^{(1)}, \dots, \beta_{n,2n+1}^{(1)}, \beta_{1,2n+1}^{(2)}, \dots, \beta_{n+1,2n+1}^{(2)}$ such that the quadrature formula

$$Q_{2n+1}^{GK}[f] := \sum_{\nu=1}^n \beta_{\nu,2n+1}^{(1)} f(x_{\nu,n}^G) + \sum_{\mu=1}^{n+1} \beta_{\mu,2n+1}^{(2)} f(\xi_{\mu,2n+1})$$

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satisfies $\deg(R_{2n+1}^{GK}) \geq 3n + 1$, then Q_{2n+1}^{GK} is called a Gauss-Kronrod quadrature formula. Considering $Q_{2n+1}^{GK}[f]$ as a much better approximation than $Q_n^G[f]$, their difference serves as an estimate for $R_n^G[f]$.

For surveys on this method, cf. Gautschi [5] and Monegato [8], while for existence results with respect to special weight functions w cf., e.g., Szegő [14] and the recent results of Notaris [9] and Peherstorfer [10].

The Gauss-Kronrod method is basic for several practical integration routines, e.g. in QUADPACK [11], and hence one of the most often used methods for approximate integration with practical error estimate. Yet, there is still a need for a theoretical study of R_{2n+1}^{GK} which could justify the important role Gauss-Kronrod quadrature plays in practical numerical computation (cf. [8, Part II.2]).

As a basis of a systematic study, and as an often-used quality measure, we define the error constants $c_s(R_{2n+1}^{GK})$ by

$$c_s(R_{2n+1}^{GK}) = \sup_{\|f^{(s)}\|_\infty \leq 1} |R_{2n+1}^{GK}[f]|,$$

where $\|g\|_\infty := \sup_{x \in [-1, 1]} |g(x)|$ for $g: [-1, 1] \rightarrow \mathbb{R}$. By definition, the error constants $c_s(R_{2n+1}^{GK})$ are the smallest real values independent of f satisfying the standard error bounds

$$|R_{2n+1}^{GK}[f]| \leq c_s(R_{2n+1}^{GK}) \|f^{(s)}\|_\infty$$

and are well known to exist whenever $\deg(R_{2n+1}^{GK}) \geq s - 1$ (cf. [1]).

Brass and Förster [2] proved

$$\frac{c_{2n}(R_{2n+1}^{GK})}{c_{2n}(R_n^G)} \leq \text{const} \sqrt[4]{n} \left(\frac{1}{3.493\dots} \right)^n$$

and considered this result as a theoretical explanation for the significant superiority of Q_{2n+1}^{GK} over Q_n^G .

Only very little is known about the quality of Q_{2n+1}^{GK} itself. Rabinowitz [12] proved the existence of $c_s(R_{2n+1}^{GK})$ for $s = 1, \dots, 3n + 2 + \kappa$, $\kappa = \lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor$ and nonexistence for $s > 3n + 2 + \kappa$, showing that $\deg(R_{2n+1}^{GK}) = 3n + 1 + \kappa$; cf. also Rabinowitz [13] for a proof of the nondefiniteness of R_{2n+1}^{GK} . Brass and Förster [2], Brass and Schmeisser [3], and Monegato [7] proved upper bounds for $c_{3n+2+\kappa}(R_{2n+1}^{GK})$. In the following theorem, we give lower bounds as well as new upper bounds for $c_{3n+2+\kappa}(R_{2n+1}^{GK})$ in the case $w \equiv 1$ that improve the hitherto best-known bounds.

Theorem. *Let $w \equiv 1$, $n \geq 4$, $\kappa = \lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor$. Then there holds*

$$c_{3n+2+\kappa}(R_{2n+1}^{GK}) \leq \bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK}),$$

where

$$\bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK}) := \frac{1}{(3n + 2 + \kappa)! 2^{3n+\kappa}} \left\{ \frac{17}{10\sqrt{3n - 3(2 + \kappa)}} + \frac{2\sqrt{2}}{\sqrt{(6n + 3 + 2\kappa)(6n + 5 + 2\kappa)}} \right\}.$$

Furthermore, for n even there holds

$$(1.a) \quad c_{3n+2}(R_{2n+1}^{GK}) \geq \frac{1}{3e\sqrt{\pi}} \frac{2^{n+3}}{(3n+2)!} \frac{[n!]^4}{(2n)!(2n+1)!} \cdot \sqrt{\frac{3n-2}{n^2-2n} \frac{(2n+3)(5n-3)^2}{(n-2)^2(n+2)(3n-1)^3(3n+1)^2}},$$

while for n odd there holds

$$(1.b) \quad c_{3n+3}(R_{2n+1}^{GK}) \geq \frac{1}{27e\sqrt{\pi}} \frac{2^{n+3}}{(3n+3)!} \frac{[n!]^4}{(2n)!(2n+1)!} \cdot \sqrt{\frac{3n-1}{n^2-3n} \frac{(2n+3)(5n^2-7n+3)^2}{(n-1)^2n^2(n+2)(n-3)^2(3n-3)(3n-2)^2}}.$$

Remark. While the upper bound in the theorem improves known results, it may still be sharpened. However, an improvement can only be obtained by a polynomial factor, since it follows that

$$\frac{\bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK})}{c_{3n+2+\kappa}(R_{2n+1}^{GK})} = O(n^5)$$

if we replace $c_{3n+2+\kappa}(R_{2n+1}^{GK})$ by the lower bounds (1). A result of Brass and Schmeisser [3, Theorem 7] implies that amongst all quadrature formulae Q_m with positive weights and degree of exactness $\text{deg}(Q_m) \geq 3n+1+\kappa$, the Lobatto formula $Q_{m^*}^{L_0}$ for $m^* = \frac{3n+2+\kappa}{2}$ is worst with respect to the error constant $c_{3n+2+\kappa}(R_{2n+1}^{GK})$, i.e. (cf. [1, p. 149])

$$c_{3n+2+\kappa}(R_m) \leq c_{3n+2+\kappa}(R_{m^*}^{L_0}) = \pi 2^{-(3n+2+\kappa)}(1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

The lower bounds (1) now prove that Q_{2n+1}^{GK} can be better than this upper bound only by a polynomial factor $(O(n^{5.5}))^{-1}$. Note that Brass and Schmeisser [3, Remark 4] explicitly construct a positive quadrature formula $Q_{3n+2+\kappa}^{BS}$ that satisfies

$$c_{3n+2+\kappa}(R_{3n+2+\kappa}^{BS}) = \pi 4^{-(3n+2+\kappa)}(1 + o(1)) \quad \text{as } n \rightarrow \infty.$$

In the following corollary we will compare Q_{2n+1}^{GK} and Q_{2n+1}^G in order to show that there also exist quadrature formulae using the same number of nodes but having considerably better error constants.

Corollary. *Let $w \equiv 1$, $n \geq 1$. Then $c_{3n+2+\kappa}(R_{2n+1}^G) \leq c_{3n+2+\kappa}(R_{2n+1}^{GK})$, and we have equality only in the case $n = 1$, where the Gaussian formula and the Gauss-Kronrod formula are identical. Furthermore, for $n \geq 15$ there holds the sharper bound*

$$\frac{c_{3n+2+\kappa}(R_{2n+1}^G)}{c_{3n+2+\kappa}(R_{2n+1}^{GK})} \leq 3^{-n+1},$$

while asymptotically we find

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{c_{3n+2+\kappa}(R_{2n+1}^G)}{c_{3n+2+\kappa}(R_{2n+1}^{GK})}} = \sqrt[6]{\frac{6}{7}} = \frac{1}{4.2013\dots}$$

2. PROOFS OF THE RESULTS

In the sequel let $m = \lfloor \frac{n+1}{2} \rfloor$, $\kappa = m - \lfloor \frac{n}{2} \rfloor$, and let \mathcal{P}_s denote the space of polynomials of degree less than or equal to s .

Proof of the Theorem. We will first prove the lower bounds (1). Let $E_{n+1} \in \mathcal{P}_{n+1}$ be defined by

$$E_{n+1} = \sum_{\nu=0}^{m-1} \alpha_{\nu,n} T_{n+1-2\nu} + \begin{cases} \alpha_{m,n} T_1, & n \text{ even,} \\ \frac{1}{2} \alpha_{m,n}, & n \text{ odd,} \end{cases}$$

where T_ν denotes the ν th Chebyshev Polynomial of the first kind and

$$(2) \quad \alpha_{0,n} = 1, \quad \alpha_{\nu,n} = -f_{\nu,n} - \sum_{\mu=1}^{\nu-1} f_{\mu,n} \alpha_{\nu-\mu,n}, \quad \nu = 2, 3, \dots,$$

$$f_{0,n} = 1, \quad f_{\nu,n} = \sigma_{\nu,n} f_{\nu-1,n}, \quad \sigma_{\nu,n} = \left(1 - \frac{1}{2\nu}\right) \left(1 - \frac{1}{2n+2\nu+1}\right).$$

For reasons of simplicity, when no ambiguity arises, we do not indicate the dependence on n , i.e., $\alpha_\nu := \alpha_{\nu,n}$, $f_\nu := f_{\nu,n}$, and $\sigma_\nu := \sigma_{\nu,n}$. The zeros of E_{n+1} are the additionally chosen nodes $\xi_{1,2n+1}, \dots, \xi_{n+1,2n+1}$ of Q_{2n+1}^{GK} (cf. [7]). Rabinowitz [12] showed that with $g_\kappa = P_n E_{n+1} P_{n+1+\kappa}$ (P_ν denoting the ν th Legendre Polynomial) there holds, for n even,

$$R_{2n+1}^{GK}[g_0] = \frac{1}{2n+2}(\alpha_m - \alpha_{m+1}),$$

while for n odd, there holds

$$R_{2n+1}^{GK}[g_1] = \frac{2n+3}{(2n+2)(2n+4)}(\alpha_{m-1} - \alpha_{m+1}).$$

From the definition of the error constants $c_s(R_{2n+1}^{GK})$ we get

$$c_s(R_{2n+1}^{GK}) \geq \frac{|R_{2n+1}^{GK}[f]|}{\|f^{(s)}\|_\infty} \quad \text{for all } f \in C^s[-1, 1] \setminus \mathcal{P}_{s-1}.$$

Defining $p_\nu(x) := x^\nu$, we conclude

$$c_{3n+2+\kappa}(R_{2n+1}^{GK}) \geq \frac{|R_{2n+1}^{GK}[p_{3n+2+\kappa}]|}{(3n+2+\kappa)!}.$$

Taking into account the leading coefficients of P_ν and E_{n+1} , one readily verifies

$$g_\kappa(x) = \frac{(2n)!}{2^n(n!)^2} 2^n \frac{(2n+2+2\kappa)!}{2^{n+1+\kappa}[(n+1+\kappa)!]^2} x^{3n+2+\kappa} + p(x), \quad p \in \mathcal{P}_{3n+1+\kappa},$$

and, using the linearity and the degree of exactness of R_{2n+1}^{GK} ,

$$(3) \quad R_{2n+1}^{GK}[p_{3n+2+\kappa}] = \frac{2^{n-1-\kappa}[n!]^4}{(2n)!(2n+1)!}(\alpha_{m-\kappa} - \alpha_{m+1}).$$

We will now derive lower bounds for the difference $|\alpha_{m-\kappa} - \alpha_{m+1}|$. Szegő [14] proved that the sequence $(-\alpha_{\nu+1})$ is positive throughout, and completely monotonic, i.e.,

$$(-1)^{\nu+1} \Delta^\nu \alpha_{\nu+1} > 0, \quad \nu = 0, 1, 2, \dots,$$

while for its sum he proved $\sum_{\nu=0}^\infty \alpha_{\nu+1} = -1$. In the following Lemma 1 we state some further properties of (α_ν) needed here; the proof of Lemma 1 will be given later.

Lemma 1. *The sequence (α_ν) satisfies*

$$\alpha_0 = 1, \quad \alpha_1 = -f_1, \quad \alpha_\nu = \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-1-\mu}(\sigma_\nu - \sigma_{\nu-\mu}), \nu \geq 2.$$

Bounds for (α_ν) are given by

$$f_{\nu-1}(\sigma_\nu - \sigma_{\nu-1}) < |\alpha_\nu| < f_{\nu-1}(\sigma_\nu - \sigma_1).$$

Lower bounds for the differences of (α_ν) are, for every $p \in \mathbb{N}$,

$$p(\alpha_{\nu+1} - \alpha_\nu) \geq \alpha_{\nu+p} - \alpha_\nu, \\ p(\alpha_{\nu+1} - \alpha_{\nu-1}) \geq \alpha_{\nu+2p+1} - \alpha_{\nu-1}.$$

According to the lower bounds for the differences in Lemma 1, we are now looking for $p = p(n)$ such that the right-hand sides of

$$|\alpha_m - \alpha_{m+1}| \geq \frac{1}{p(n)} (f_{m-1}(\sigma_m - \sigma_{m-1}) - f_{m+p(n)-1}(1 - \sigma_1))$$

for n even, and

$$|\alpha_{m-1} - \alpha_{m+1}| \geq \frac{1}{p(n)} (f_{m-2}(\sigma_{m-1} - \sigma_{m-2}) - f_{m+2p(n)}(1 - \sigma_1))$$

for n odd, be positive and as big as possible. (Recall the definition of m at the beginning of this section.) By (2) we calculate $1 - \sigma_1 = \frac{n+2}{2n+3}$, and for even $n > 2$,

$$\sigma_m - \sigma_{m-1} = \frac{4(5n - 3)}{(n - 2)(3n - 1)(3n + 1)}$$

while for odd $n > 3$,

$$\sigma_{m-1} - \sigma_{m-2} = \frac{4(5n^2 - 7n + 3)}{3n(n - 3)(n - 1)(3n - 2)}.$$

Now let n be even. The following representation of the f_ν can easily be proved by induction:

$$(4) \quad f_\nu = \frac{(2\nu)!}{[\nu!]^2} \left[\frac{(n + \nu)!}{n!} \right]^2 \frac{(2n + 1)!}{(2n + 2\nu + 1)!}.$$

We determine $p = p(n)$ such that

$$f_{m-1} \frac{4(5n - 3)}{(n - 2)(3n - 1)(3n + 1)} - f_{m+p(n)-1} \frac{n + 2}{2n + 3} > 0$$

holds, which is equivalent to

$$(5) \quad \frac{4(2n + 3)(5n - 3)}{(n - 2)(n + 2)(3n - 1)(3n + 1)} \\ > \frac{(2m + 2p(n) - 2)! ((m - 1)!)^2 (2n + 2m - 2)!}{[(m + p(n) - 1)!]^2 (2m - 2)! ((n + m - 1)!)^2} \\ \cdot \frac{((n + m + p(n) - 1)!)^2}{(2n + 2m + 2p(n) - 2)!}.$$

For the right side of the inequality we find by some elementary calculation, using Stirling’s formula,

$$\frac{(2m + 2p(n) - 2)! ((m - 1)!)^2 (2n + 2m - 2)! ((n + m + p(n) - 1)!)^2}{[(m + p(n) - 1)!]^2 (2m - 2)! ((n + m - 1)!)^2 (2n + 2m + 2p(n) - 2)!} < \sqrt{e} \frac{2n + 2m - 1}{2n + 2m + 2p(n) - 1}.$$

Replacing the right side of (5) by this simpler estimate, the following condition for $p(n)$ arises:

$$2p(n) \geq \frac{\sqrt{e}(n - 2)(n + 2)(3n - 1)^2(3n + 1)}{4(5n - 3)(2n + 3)} - (3n - 1).$$

We may now choose $p(n)$ for n even as

$$p(n) := \left\lceil c \cdot \frac{\sqrt{e}(n - 2)(n + 2)(3n - 1)^2(3n + 1)}{8(2n + 3)(5n - 3)} - \frac{3n - 1}{2} \right\rceil, \quad c \geq 1,$$

leading to

$$|\alpha_m - \alpha_{m+1}| > \frac{1}{p(n)} f_{m-1} \frac{4(5n - 3)}{(n - 2)(3n - 1)(3n + 1)} \left(1 - \frac{1}{c}\right),$$

where for $n \geq 4$ from (4), using Stirling’s formula, we can show

$$f_{m-1} \geq \frac{2}{3\sqrt{e\pi}} \sqrt{\frac{3n - 2}{n(n - 2)}}.$$

Since by definition of $p(n)$,

$$p(n) \leq c \cdot \frac{\sqrt{e}(n - 2)(n + 2)(3n - 1)^2(3n + 1)}{8(5n - 3)(2n + 3)},$$

we readily conclude

$$|\alpha_m - \alpha_{m+1}| > \frac{64}{3e\sqrt{\pi}} \sqrt{\frac{3n - 2}{n(n - 2)}} \frac{(2n + 3)(5n - 3)^2}{(n - 2)(n + 2)(3n - 1)^3(3n + 1)^2} \left(\frac{c - 1}{c^2}\right).$$

Maximizing the right side with respect to $c \geq 1$ leads to $c = 2$ and therefore to the asserted inequality for n even. For the proof of the lower bound for $|\alpha_{m-1} - \alpha_{m+1}|$, n odd, we can proceed in an analogous way. This time, a sufficient condition for the $p(n)$ is

$$\frac{4(2n + 3)(5n^2 - 7n + 3)}{3(n - 3)(n - 1)n(n + 2)(3n - 2)} > \sqrt{e} \frac{2n + 2m - 4}{2n + 2m + 4p(n) + 1}.$$

For n odd, we then choose

$$p(n) := \left\lceil c \cdot \frac{3\sqrt{e}(n - 3)(n - 1)n(n + 2)(3n - 3)(3n - 2)}{16(2n + 3)(5n^2 - 7n + 3)} - \frac{3n + 2}{4} \right\rceil, \quad c \geq 1.$$

Using the inequality

$$f_{m-2} \geq \frac{2}{3\sqrt{e\pi}} \sqrt{\frac{3n - 1}{n(n - 3)}}$$

leads again, after maximization with respect to c , to the lower bound for n odd.

We will now prove the upper bounds stated in the Theorem with the help of the following lemma, which is contained in [2, Theorem 1].

Lemma 2. Let R be a continuous, linear functional on $C[-1, 1]$ satisfying $R[\mathcal{P}_{m-1}] = 0$, and let

$$K_{m,s}(x) := \frac{2}{\pi} \frac{m!2^m}{(2m)!} \sum_{\mu=0}^{s-1} (1-x^2)^{m-1/2} R[T_{m+\mu}] \frac{P_{\mu}^{(m)}(x)}{P_{\mu}^{(m)}(1)},$$

where $P_{\mu}^{(m)}$ denotes the μ th ultraspherical polynomial of order m . The limit

$$K_m(x) := \lim_{s \rightarrow \infty} K_{m,s}(x), \quad x \in [-1, 1],$$

exists. If $f \in C^m[-1, 1]$, then

$$R[f] = \int_{-1}^1 f^{(m)}(x) K_m(x) dx.$$

If $s = 0, 1, 2, \dots$, then

$$c_m(R) = \int_{-1}^1 |K_{m,s}(x)| dx + \rho_s,$$

where

$$|\rho_s| \leq \frac{1}{m!2^{m-1}} \left\{ \frac{(2m)!s!}{(2m+s)!} \frac{2m+s}{2m-1} \right\}^{1/2} \sup_{\mu \geq s} |R[T_{\mu+m}]|.$$

Using (3) and $T_{\nu}(x) = 2^{\nu-1}x^{\nu} + p(x)$, $p \in \mathcal{P}_{\nu-1}$, we find

$$(6) \quad R_{2n+1}^{GK}[T_{3n+2+\kappa}] = \frac{2^{4n}[n!]^4}{(2n)!(2n+1)!} (\alpha_{m-\kappa} - \alpha_{m+1}).$$

Taking advantage of the monotonicity of $(-\alpha_{\nu+1})$, from Lemma 1 we get

$$(7) \quad |\alpha_{m-\kappa} - \alpha_{m+1}| < |\alpha_{m-\kappa}| < f_{m-1-\kappa}(\sigma_{m-\kappa} - \sigma_1).$$

Using (4), we obtain by the use of Stirling's formula

$$(8) \quad f_{m-1-\kappa} \leq \frac{2n+2-\kappa}{3n} \sqrt{\frac{3n-2+\kappa}{n(n-2-\kappa)}}.$$

According to (2) we get for n even

$$(9) \quad \sigma_m - \sigma_1 = \frac{3n^2 - n - 10}{(2n+3)(3n+1)}$$

and for n odd

$$(10) \quad \sigma_{m-1} - \sigma_1 = \frac{3n^3 - 5n^2 - 14n + 6}{(n-1)(2n+3)(3n-2)}.$$

Lemma 2 implies for $R = R_{2n+1}^{GK}$, $s = 1$, and $m = 3n + 2 + \kappa$

$$\begin{aligned} & c_{3n+2+\kappa}(R_{2n+1}^{GK}) \\ & \leq |R_{2n+1}^{GK}[T_{3n+2+\kappa}]| \int_{-1}^1 \left| \frac{2}{\pi} \frac{(3n+2+\kappa)!2^{3n+2+\kappa}}{(6n+4+2\kappa)!} (1-x^2)^{3n+2+\kappa-1/2} \right| dx \\ & \quad + \frac{1}{(3n+2+\kappa)!2^{3n+1+\kappa}} \left\{ \frac{2(6n+4+2\kappa)!}{(6n+6+2\kappa)!} \frac{6n+6+2\kappa}{6n+3+2\kappa} \right\}^{1/2} \\ & \quad \cdot \sup_{\mu \geq 2} |R_{2n+1}^{GK}[T_{3n+2+\kappa+\mu}]|. \end{aligned}$$

Since

$$\int_{-1}^1 |(1-x^2)^{3n+2+\kappa-1/2}| dx = \frac{\Gamma(\frac{1}{2})\Gamma(3n+2+\kappa+\frac{1}{2})}{\Gamma(3n+3+\kappa)}$$

and $\sup_{\mu \geq 2} |R_{2n+1}^{GK}[T_{3n+2+\mu}]| \leq 4$, we find

$$c_{3n+2+\kappa}(R_{2n+1}^{GK}) \leq \frac{1}{(3n+2+\kappa)!2^{3n+1+\kappa}} \cdot \left(|R_{2n+1}^{GK}[T_{3n+2+\kappa}]| + \frac{4\sqrt{2}}{\sqrt{(6n+3+2\kappa)(6n+5+2\kappa)}} \right).$$

Using (6), (7), (8), (9), and (10), we conclude the result after some simplifying calculation. \square

Proof of Lemma 1. The recursion formula (2) for α_ν and $\alpha_{\nu-1}$, respectively, is

$$\alpha_\nu = -f_\nu - \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-\mu}, \quad 0 = -f_{\nu-1} - \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-1-\mu}.$$

Multiplication of the first by $f_{\nu-1}$ and the second by $-f_\nu$, and summation, yields

$$\begin{aligned} f_{\nu-1}\alpha_\nu &= \sum_{\mu=1}^{\nu-1} \alpha_\mu [f_{\nu-1-\mu}f_\nu - f_{\nu-\mu}f_{\nu-1}] = \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-1}f_{\nu-1-\mu}(\sigma_\nu - \sigma_{\nu-\mu}) \\ &= f_{\nu-1} \sum_{\mu=1}^{\nu-1} \alpha_\mu f_{\nu-1-\mu}(\sigma_\nu - \sigma_{\nu-\mu}), \end{aligned}$$

leading to the first assertion (see also [6] for this method). Using $f_\nu > 0$ and $\sigma_\nu - \sigma_{\nu-\mu} > 0$ for $\mu = 1, \dots, \nu - 1$, we find that all terms in the right-hand sum have the same sign, and we conclude the second assertion, using the monotonicity of (σ_ν) . Since $(-\alpha_{\nu+1})$ is completely monotonic, the last two inequalities of Lemma 1 follow readily. \square

Proof of the Corollary. Theorem 4 of [2] states that

$$c_{3n+2+\kappa}(R_{2n+1}^G) \leq \bar{c}_{3n+2+\kappa}(R_{2n+1}^G),$$

where

$$\begin{aligned} \bar{c}_{3n+2+\kappa}(R_{2n+1}^G) &:= \frac{\pi}{(3n+2+\kappa)!2^{3n+2+\kappa}} \\ &\cdot \left\{ \frac{(n-\kappa)!(6n+4+2\kappa)!}{(7n+4+\kappa)!} \frac{7n+4+\kappa}{8n+4} \right. \\ &\left. \cdot \left(1 + \frac{5}{3} \frac{(n+1-\kappa)(n+2-\kappa)}{(6n+3+2\kappa)(7n+5+\kappa)} \right) \right\}^{1/2}. \end{aligned}$$

Using the lower bounds (1) for $c_{3n+2+\kappa}(R_{2n+1}^{GK})$, the following inequality can be proved for $n \geq 4$ after some (elementary) calculation involving Stirling's formula:

$$(11) \quad \frac{c_{3n+2+\kappa}(R_{2n+1}^G)}{c_{3n+2+\kappa}(R_{2n+1}^{GK})} < c \cdot n^{23/4} \left(\sqrt{\frac{6^6}{7^7}} \right)^n,$$

with $c = 0.62$. Inequality (11) yields $c_{3n+2+\kappa}(R_{2n+1}^G) \leq c_{3n+2+\kappa}(R_{2n+1}^{GK})$ for $n \geq 10$ and $c_{3n+2+\kappa}(R_{2n+1}^G)/c_{3n+2+\kappa}(R_{2n+1}^{GK}) \leq 3^{-n+1}$ for $n \geq 68$, where for $1 \leq n \leq 9$ and $15 \leq n \leq 68$, respectively, the results can be proved numerically (cf. [4]). In [2, Theorem 4] it is stated that the upper bound $\bar{c}_{3n+2+\kappa}(R_{2n+1}^G)$ can only be improved by $O(n^{3/4})$. Using this result and the remark in §1 to obtain also a lower bound for the ratio in (11), the corollary follows. \square

3. FURTHER REMARKS

1. Using similar methods as described above, respective results can be obtained in the more general case of an ultraspherical weight function (cf. [4]). For $w(x) = (1 - x^2)^{\lambda-1/2}$, $\lambda \in (0, 1)$, $n \geq 4$, $\kappa = \lfloor \frac{n+1}{2} \rfloor - \lfloor \frac{n}{2} \rfloor$ there holds $c_{3n+2+\kappa}(R_{2n+1}^{GK}) \leq \bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK})$, where

$$\begin{aligned} &\bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK}) \\ &:= \frac{1}{(3n + 2 + \kappa)!2^{3n+1+\kappa}} \left(\frac{\pi\lambda}{2^{2\lambda-1}\Gamma(1-\lambda)} \frac{9n + 4\lambda}{9n - 3\kappa + 4\lambda} \right. \\ &\quad \cdot \left. \left\{ \frac{6(2n + 2 + \lambda)}{(n + \lambda - 1 + \kappa)(9n + 4\lambda)} \right\}^\lambda \right. \\ &\quad \left. + \frac{2\sqrt{\pi}(\lambda + (1 + \kappa)/4)^{\lambda-1}}{\sqrt{(6n + 3 + 2\kappa)(6n + 5 + 2\kappa)}} \right). \end{aligned}$$

Lower bounds for $c_{3n+2+\kappa}(R_{2n+1}^{GK})$ can be found in [4], which for the quality of the upper bounds yield

$$\frac{\bar{c}_{3n+2+\kappa}(R_{2n+1}^{GK})}{c_{3n+2+\kappa}(R_{2n+1}^{GK})} = O(n^{3+1/\lambda}).$$

The corresponding error constants of Q_{2n+1}^G are again significantly smaller (cf. [4]).

2. Using the methods derived in the proof of the theorem in §1, we can also prove bounds for the error constants $c_{3n+2+\kappa-s}(R_{2n+1}^{GK})$ of nonmaximum order (cf. [4] for details). In the case of constant $s \in \mathbb{N}$, these bounds are again better than the hitherto best-known bounds. For their quality we can prove

$$\frac{\bar{c}_{3n+2+\kappa-s}(R_{2n+1}^{GK})}{c_{3n+2+\kappa-s}(R_{2n+1}^{GK})} = O(n^{5+(s/2)}).$$

In the case $s = s(n) \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \frac{3n+2+\kappa-s}{n} = A > 0$, the new bounds are of quality

$$\lim_{n \rightarrow \infty} \sqrt[n]{\frac{\bar{c}_{3n+2+\kappa-s}(R_{2n+1}^{GK})}{c_{3n+2+\kappa-s}(R_{2n+1}^{GK})}} = \frac{A^A(3-A)^{(3-A)/2}(3+A)^{(3+A)/2}}{3^3(2A)^A}.$$

The corresponding error constants of Q_{2n+1}^G can be proved to be significantly smaller for $2 < A \leq 3$.

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