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# $B$-CONVERGENCE PROPERTIES OF MULTISTEP RUNGE-KUTTA METHODS

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ABSTRACT. By using the theory of $B$-convergence for general linear methods to the special case of multistep Runge-Kutta methods, a series of $B$-convergence results for multistep Runge-Kutta methods is obtained, and it is proved that the family of algebraically stable $r$-step $s$-stage multistep Runge-Kutta methods with parameters $\alpha_1, \alpha_2, \dots, \alpha_r$ presented by Burrage in 1987 is optimally $B$-convergent of order at least $s$, and $B$-convergent of order $s+1$, provided that $r \geq s$ and $\alpha_j > 0$, $j = 1, 2, \dots, r$. Furthermore, this family of methods is optimally B-convergent of order $s+1$ if some other additional conditions are satisfied.

## 1. INTRODUCTION

Let $X$ be a real or complex Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\|\cdot\|$, $f: X \to X$ be a given sufficiently smooth mapping satisfying a one-sided Lipschitz condition

$$\mathrm{Re}\langle f(y) - f(z), y - z \rangle \leq m\|y - z\|^2 \quad \forall y, z \in X.$$

Consider the initial value problem

$$y'(t) = f(y(t)), \quad 0 \leq t \leq T; \qquad y(0) = y_0, \quad y_0 \in X \tag{1.1}$$

and the multistep Runge-Kutta method for solving (1.1):

$$Y^{(n)} = \widetilde{A} y^{(n-1)} + h\widetilde{B} F(Y^{(n)}), \tag{1.2a}$$

$$y^{(n)} = \widetilde{C} y^{(n-1)} + h\widetilde{E} F(Y^{(n)}), \tag{1.2b}$$

$$\xi_n = \widetilde{\beta} y^{(n)}. \tag{1.2c}$$

Here the problem (1.1) is assumed to have a unique solution $y(t)$ on the interval $[0, T]$. For the method (1.2) we assume that

$$Y^{(n)} = (Y_1^{(n)}, Y_2^{(n)}, \dots, Y_s^{(n)}) \in X^s, \qquad y^{(n)} = (y_1^{(n)}, y_2^{(n)}, \dots, y_r^{(n)}) \in X^r,$$
$$\xi_n \in X, \qquad F(Y^{(n)}) = (f(Y_1^{(n)}), f(Y_2^{(n)}), \dots, f(Y_s^{(n)})) \in X^s,$$

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$h > 0$ is the stepsize, $\widetilde{A}$, $\widetilde{B}$, $\widetilde{C}$, $\widetilde{E}$, and $\widetilde{\beta}$ are linear mappings corresponding respectively to the real matrices

$$A = [a_{ij}] \in \mathbb{R}^{s\times r}, \quad B = [b_{ij}] \in \mathbb{R}^{s\times s}, \quad C = \left[\frac{0I_{r-1}}{\alpha^T}\right] \in \mathbb{R}^{r\times r}, \tag{1.3}$$
$$E = \left[\frac{0}{\gamma^T}\right] \in \mathbb{R}^{r\times s}, \quad \beta = [0, \dots, 0, 1] \in \mathbb{R}^{1\times r}$$

(cf. [11]), where $\alpha = [\alpha_1, \alpha_2, \dots, \alpha_r]^T$, $\gamma = [\gamma_1, \gamma_2, \dots, \gamma_s]^T$, $I_{r-1}$ is the $(r-1)\times(r-1)$ identity matrix, $Y_i^{(n)}$, $y_i^{(n)}$, and $\xi_n$ are approximations to $y(t_n + c_i h)$, $y(t_n + ih)$, and $y(t_n + rh)$, respectively, where

$$t_n = t_0 + nh; \qquad c_i = \sum_{j=1}^{s} b_{ij} + \sum_{j=1}^{r}(j-1)a_{ij}, \quad i = 1, 2, \dots, s.$$

For simplicity, we write $c = [c_1, c_2, \dots, c_s]^T$, $\zeta = [0, 1, \dots, r-1]^T$, $e_N = [1, 1, \dots, 1]^T \in \mathbb{R}^N$ with $N \ge 1$, $Y(t) = (y(t+c_1h), y(t+c_2h), \dots, y(t+c_sh)) \in X^s$, $H(t) = (y(t+h), y(t+2h), \dots, y(t+rh)) \in X^r$, introduce the simplifying conditions (cf. [1])

$$\begin{aligned} &B(\tau): \quad p\gamma^T c^{p-1} = r^p - \alpha^T\zeta^p, \qquad p = 1, 2, \dots, \tau;\\ &C(\tau): \quad pBc^{p-1} = c^p - A\zeta^p, \qquad p = 1, 2, \dots, \tau;\\ &E(\tau): \quad pA^T \operatorname{diag}(\gamma)c^{p-1} = \operatorname{diag}(\alpha)(r^p e_r - \zeta^p), \, p = 1, 2, \dots, \tau, \end{aligned}$$

and adopt the notational convention: $M > 0$ (or $\ge 0$) for a real symmetric matrix to mean that $M$ is positive definite (or nonnegative definite).

Note that multistep Runge-Kutta methods are a subclass of the General Linear Methods of Butcher, and it is proved by Lie and Nørsett [13] that multistep collocation methods are a subclass of multistep Runge-Kutta methods.

In 1987, Burrage [1] obtained the following results:

**Theorem 1.1.** *Suppose the method* (1.2)–(1.3) *satisfies the conditions* $B(2s)$, $C(s)$, *and* $E(s)$, $c_i \ne c_j$ *whenever* $i \ne j$, $\sum_{j=1}^r \alpha_j = 1$, $\alpha_1 > 0$, *and* $\alpha_j \ge 0$, $j = 2, 3, \dots, r$. *Then this method is algebraically stable for the matrices*

$$G = \operatorname{diag}\left(\alpha_1, \alpha_1 + \alpha_2, \dots, \sum_{j=1}^{r}\alpha_j\right), \qquad D = \operatorname{diag}(\gamma_1, \gamma_2, \dots, \gamma_s), \tag{1.4}$$

*and necessarily* $G > 0$, $D > 0$.

**Theorem 1.2.** *Suppose that* $\sum_{j=1}^r \alpha_j = 1$, $\alpha_1 > 0$, $\alpha_j \ge 0$, $j = 2, 3, \dots, r$. *Then the multistep Runge-Kutta methods defined by* (1.2), (1.3) *and*

$$
(1.5)\quad \begin{cases}
\gamma_j = \int_0^r l_j(x)\,dx - \sum_{k=2}^{r} \alpha_k \int_0^{k-1} l_j(x)\,dx, & j = 1, 2, \dots, s; \\
a_{ij} = \frac{\alpha_j}{\gamma_i} \int_{j-1}^{r} l_i(x)\,dx, & i = 1, 2, \dots, s,\ j = 1, 2, \dots, r; \\
b_{ij} = \int_0^{c_i} l_j(x)\,dx - \sum_{k=2}^{r} a_{ik} \int_0^{k-1} l_j(x)\,dx, & i, j = 1, 2, \dots, s; \\
l_j(x) = \frac{P(x)}{(x - c_j)P'(c_j)}, & j = 1, 2, \dots, s; \\
P(x) = \prod_{k=1}^{s} (x - c_k) = \det \begin{bmatrix} h_1 h_2 \cdots h_s h_{s+1} \\ h_2 h_3 \cdots h_{s+1} h_{s+2} \\ \dots\dots\dots\dots\dots\dots \\ h_s h_{s+1} \cdots h_{2s-1} h_{2s} \\ 1 x \cdots x^{s-1} x^s \end{bmatrix}; \\
h_i = \frac{1}{i}(r^i - \alpha^T \zeta^i),\ i = 1, 2, \dots, 2s,
\end{cases}
$$

*satisfy all the hypotheses of Theorem* 1.1, *and they are all algebraically stable for the matrices* $G > 0$, $D > 0$ *defined by* (1.4).

In 1988, the author of the present paper [10, 11] established the theory of $B$-convergence ($B$-theory) for general linear methods. We here only recall one of the basic principles:

**Theorem 1.3.** *If a general linear method is* $BH$*-stable and* $BH$*-(resp.* $BH^*$*-) consistent of order* $p$, *then this method is optimally* $B$*-convergent of order* $p$ *(resp.* $B$*-convergent of order* $p$*).*

In the present paper, the $B$-theory for general linear methods is applied to the special case of multistep Runge-Kutta methods. We first discuss the generalized stage order and diagonal stability of the methods (see Theorems 2.1-2.3); then, in view of $B$-theory and Theorems 1.1–1.3, a series of $B$-convergence results for multistep Runge-Kutta methods is obtained (see Theorems 2.4-2.7).

## 2. Main results and their proofs

**Definition 2.1.** The method (1.2) is said to be *diagonally stable,* if there exists an $s \times s$ diagonal matrix $Q > 0$ such that $QB + B^T Q > 0$.

**Definition 2.2.** The method (1.2) is said to have *generalized stage order* $p$, if $p$ is the largest nonnegative integer which possesses the following properties:

For any given initial value problem (1.1) and stepsize $h \in (0, h_0]$, there exist abstract functions $Y^h$ and $H^h$:

$$
Y^h(t) = (Y_1^h(t), Y_2^h(t), \dots, Y_s^h(t)) \in X^s,
$$
$$
H^h(t) = (H_1^h(t), H_2^h(t), \dots, H_r^h(t)) \in X^r,
$$

such that

$$\|H^h(t) - H(t)\| \le d_0 h^p, \quad \|\Delta^h(t)\| \le d_1 h^{p+1},$$
$$\|\delta^h(t)\| \le d_2 h^{p+1}, \quad \|\sigma^h(t)\| \le d_3 h^p,$$

where $h_0 > 0$ is only required to be so small that for $h \in (0, h_0]$ all the time nodes belong to the integration interval $[0, T]$; each $d_i$ $(i = 0, 1, 2, 3)$ depends only on the method and on bounds $M_i$ of some derivatives of the exact solution $y(t)$: $\|d^i y(t)/dt^i\| \le M_i$, $t \in [0, T]$; $\Delta^h(t)$, $\delta^h(t)$, and $\sigma^h(t)$ are determined by the equations

$$\text{(2.1)} \qquad \begin{cases} Y^h(t) = \tilde{A} H^h(t-h) + h\tilde{B} F(Y^h(t)) + \Delta^h(t), \\ H^h(t) = \tilde{C} H^h(t-h) + h\tilde{E} F(Y^h(t)) + \delta^h(t), \\ y(t+rh) = \tilde{\beta} H^h(t) + \sigma^h(t); \end{cases}$$

the norm $\|\cdot\|$ on $X^N$ $(N \ge 1)$ is defined by

$$\|U\| = \left(\sum_{i=1}^{N} \|u_i\|^2\right)^{1/2} \quad \forall U = (u_1, u_2, \dots, u_N) \in X^N.$$

Furthermore, if the quantities $d_i$ $(i = 0, 1, 2, 3)$ are also allowed to depend on bounds $\kappa_i$ for certain derivatives of the mapping $f$ (but not on $\kappa_1$): $\|d^i f(y)/dy^i\| \le \kappa_i$, $y \in X$, then the aforementioned integer $p$ is known as *generalized weak stage order* of the method. For the special case where $H^h(t) \equiv H(t)$, the generalized stage order and generalized weak stage order are simply called *stage order* and *weak stage order*, respectively.

Note that these two definitions follow from related previous papers, such as [2, 5, 6, 7, 11].

**Theorem 2.1.** *The method* (1.2)–(1.3) *has stage order not smaller than* $\tau$ *if* $\sum_{j=1}^{r} \alpha_j = 1$, $\sum_{j=1}^{r} a_{ij} = 1$, $i = 1, 2, \dots, s$, *and the conditions* $B(\tau)$, $C(\tau)$ *hold.*

*Proof.* Let $H^h(t) = H(t)$, $Y^h(t) = Y(t)$. Substituting this in (2.1), we get by Taylor expansion

$$\text{(2.2)} \quad \begin{cases} [\Delta^h(t)]_i = \displaystyle\sum_{p=1}^{\tau} \frac{h^p}{p!} \left( c_i^p - \sum_{j=1}^{r} a_{ij}(j-1)^p - p\sum_{j=1}^{s} b_{ij} c_j^{p-1} \right) y^{(p)}(t) + R_{i\tau}(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad i = 1, 2, \dots, s; \\ [\delta^h(t)]_r = \displaystyle\sum_{p=1}^{\tau} \frac{h^p}{p!} \left( r^p - \sum_{j=1}^{r} \alpha_j (j-1)^p - p\sum_{j=1}^{s} \gamma_j c_j^{p-1} \right) y^{(p)}(t) + R_\tau(t); \\ [\delta^h(t)]_i = 0,\ i = 1, 2, \dots, r-1; \quad \sigma^h(t) = 0; \quad H^h(t) - H(t) = 0, \end{cases}$$

where

(2.3)

$$\begin{cases} R_{i\tau}(t) = \displaystyle\int_0^1 \Bigg[\frac{(1-\theta)^\tau}{\tau!}\Bigg(c_i^{\tau+1}y^{(\tau+1)}(t+\theta c_i h) \\ \qquad\qquad - \displaystyle\sum_{j=1}^{r} a_{ij}(j-1)^{\tau+1}y^{(\tau+1)}(t+\theta(j-1)h)\Bigg) \\ \qquad\qquad - \dfrac{(1-\theta)^{\tau-1}}{(\tau-1)!}\displaystyle\sum_{j=1}^{s} b_{ij}c_j^\tau y^{(\tau+1)}(t+\theta c_j h)\Bigg] h^{\tau+1}\, d\theta, \\ \qquad\qquad\qquad\qquad i = 1, 2, \dots, s; \\ R_\tau(t) = \displaystyle\int_0^1 \Bigg[\frac{(1-\theta)^\tau}{\tau!}\Bigg(r^{\tau+1}y^{(\tau+1)}(t+\theta r h) \\ \qquad\qquad - \displaystyle\sum_{j=1}^{r} \alpha_j(j-1)^{\tau+1}y^{(\tau+1)}(t+\theta(j-1)h)\Bigg) \\ \qquad\qquad - \dfrac{(1-\theta)^{\tau-1}}{(\tau-1)!}\displaystyle\sum_{j=1}^{s} \gamma_j c_j^\tau y^{(\tau+1)}(t+\theta c_j h)\Bigg] h^{\tau+1}\, d\theta, \end{cases}$$

and therefore

$$\|R_{i\tau}(t)\| \le k_{i\tau}h^{\tau+1}M_{\tau+1}, \qquad \|R_\tau(t)\| \le k_\tau h^{\tau+1}M_{\tau+1}, \tag{2.4}$$

where $k_{i\tau}$ $(i = 1, 2, \dots, s)$ and $k_\tau$ depend only on the method. Thus, using the conditions $B(\tau)$ and $C(\tau)$, we get the conclusion from (2.2), (2.4), and Definition 2.2. □

**Theorem 2.2.** *Suppose the method* (1.2)–(1.3) *satisfies the conditions* $B(\tau+1)$, $C(\tau)$, *and* $\sum_{j=1}^r \alpha_j = 1$, $\sum_{j=1}^r a_{ij} = 1$, $i = 1, 2, \dots, s$. *Then*

(i) *this method has weak stage order not smaller than* $\tau + 1$;

(ii) *if there exists a real number* $\nu$ *such that*

$$c^{\tau+1} - A\zeta^{\tau+1} - (\tau+1)Bc^\tau = \nu e_s, \tag{2.5}$$

*then this method has generalized stage order not smaller than* $\tau + 1$.

*Proof.* Let

$$H_i^h(t) = y(t+ih) + \delta h^{\tau+1}y^{(\tau+1)}(t), \qquad i = 1, 2, \dots, r;$$
$$Y_i^h(t) = y(t+c_i h) + \mu_i h^{\tau+1}y^{(\tau+1)}(t), \qquad i = 1, 2, \dots, s,$$

where $\mu_i$ and $\delta$ are constants to be determined. Substituting this in (2.1), expanding into Taylor series, and using the conditions $B(\tau+1)$ and $C(\tau)$, we get

(2.6a)

$$\begin{aligned}[\Delta^h(t)]_i = {} & \left[\frac{1}{(\tau+1)!}(c^{\tau+1} - A\zeta^{\tau+1} - (\tau+1)Bc^\tau) + \mu - \delta e_s\right]_i h^{\tau+1}y^{(\tau+1)}(t) \\ & + \delta h^{\tau+2}\int_0^1 y^{(\tau+2)}(t-\theta h)\, d\theta + R_{i,\tau+1}(t) \\ & + h\sum_{i=1}^{s} b_{ij}Q_j(t;\mu,\tau,h), \qquad i = 1, 2, \dots, s;\end{aligned}$$

$$[\delta^h(t)]_r = \delta h^{\tau+2}\int_0^1 y^{(\tau+2)}(t-\theta h)\,d\theta + R_{\tau+1}(t) + h\sum_{j=1}^{s}\gamma_j Q_j(t;\mu,\tau,h); \tag{2.6b}$$

$$[\delta^h(t)]_i = \delta h^{\tau+2}\int_0^1 y^{(\tau+2)}(t-\theta h)\,d\theta, \qquad i=1,2,\dots,r-1; \tag{2.6c}$$

$$\begin{gathered}\sigma^h(t) = -\delta h^{\tau+1}y^{(\tau+1)}(t);\\ [H^h(t)-H(t)]_i = \delta h^{\tau+1}y^{(\tau+1)}(t), \quad i=1,2,\dots,r,\end{gathered} \tag{2.6d}$$

where

$$\begin{gathered}\mu = [\mu_1,\mu_2,\dots,\mu_s]^T,\\ Q_j(t;\mu,\tau,h) = f(y(t+c_jh)) - f(y(t+c_jh)+\mu_j h^{\tau+1}y^{(\tau+1)}(t)),\end{gathered} \tag{2.7}$$

and $R_{i,\tau+1}(t)$, $R_{\tau+1}(t)$ are given by (2.3). Therefore, we have

$$\begin{cases}\|H^h(t)-H(t)\| \le \sqrt{r}|\delta|h^{\tau+1}M_{\tau+1}, & \|\sigma^h(t)\| \le |\delta|h^{\tau+1}M_{\tau+1},\\ \|[\delta^h(t)]_i\| \le |\delta|h^{\tau+2}M_{\tau+2}, & i=1,2,\dots,r-1,\end{cases} \tag{2.8}$$

and by Taylor expansion,

$$\begin{aligned}Q_j(t;\mu,\tau,h)&\\ = -\mu_j h^{\tau+1}\Big\{&f'(y(t))y^{(\tau+1)}(t)\\ &+\int_0^1 [f''((1-\theta)y(t)+\theta y(t+c_jh))(y(t+c_jh)-y(t))\\ &\quad + (1-\theta)\mu_j h^{\tau+1} f''(y(t+c_jh)+\theta\mu_j h^{\tau+1}y^{(\tau+1)}(t))y^{(\tau+1)}(t)]\\ &\quad \times y^{(\tau+1)}(t)\,d\theta\Big\}.\end{aligned} \tag{2.9}$$

By the technique in [7], we can easily prove that

$$\|f'(y(t))y^{(\tau+1)}(t)\| \le N_\tau \tag{2.10}$$

with $N_\tau$ depending only on some bounds $M_i$ and $\kappa_i$ (but not on $\kappa_1$). The relations (2.9) and (2.10) lead to

$$\|Q_j(t;\mu,\tau,h)\| \le N_{\mu\tau}h^{\tau+1}, \qquad 0<h\le h_0, \tag{2.11}$$

where the constant $h_0$ only need to satisfy the requirement mentioned in Definition 2.2, and $N_{\mu\tau}$ depends only on the method and on some bounds $M_i$ and $\kappa_i$ (but not on $\kappa_1$). Now choose

$$\delta = 0, \qquad \mu = -\frac{1}{(\tau+1)!}(c^{\tau+1} - A\zeta^{\tau+1} - (\tau+1)Bc^\tau).$$

Then the relations (2.4), (2.6a), (2.6b), and (2.11) lead to

$$(2.12)\quad \begin{cases} \|[\Delta^h(t)]_i\| \le \left(k_{i,\tau+1}M_{\tau+2} + N_{\mu\tau}\sum_{j=1}^{s}|b_{ij}|\right)h^{\tau+2}, & i = 1, 2, \dots, s, \\ \|[\delta^h(t)]_r\| \le \left(k_{\tau+1}M_{\tau+2} + N_{\mu\tau}\sum_{j=1}^{s}|\gamma_j|\right)h^{\tau+2}, & \end{cases}$$

provided that $h \in (0, h_0]$. Thus, it is easily seen from (2.8), (2.12), and Definition 2.2 that the method (1.2)–(1.3) has weak stage order not smaller than $\tau + 1$.

Furthermore, if the additional condition (2.5) is satisfied, then we would instead choose $\mu = 0$ and $\delta = \nu/(\tau+1)!$. In this case, (2.4), (2.6a), (2.6b), and (2.7) lead to

$$(2.13)\quad \begin{cases} \|[\Delta^h(t)]_i\| \le (|\nu|/(\tau+1)! + k_{i,\tau+1})h^{\tau+2}M_{\tau+2}, & i = 1, 2, \dots, s, \\ \|[\delta^h(t)]_r\| \le (|\nu|/(\tau+1)! + k_{\tau+1})h^{\tau+2}M_{\tau+2}, & \end{cases}$$

and it follows from (2.8), (2.13), and Definition 2.2 that the method (1.2)–(1.3) has generalized stage order not smaller than $\tau + 1$. □

**Theorem 2.3.** *Suppose the method* (1.2)–(1.3) *satisfies the conditions* $B(2s)$, $C(s)$, *and* $E(s)$, $r \ge s$, $c_i \ne c_j$ *whenever* $i \ne j$, $\sum_{j=1}^{r}\alpha_j = 1$ *and* $\alpha_j > 0$, $j = 1, 2, \dots, r$. *Then this method is diagonally stable.*

This theorem was first proved in 1989 by the author and his post-graduate student Cao Xuenian in a research report "*BH*-algebraic stability of general multivalue methods" at Xiangtan University. In the following we give an alternative proof.

Let $Q = \operatorname{diag}(\gamma_1, \gamma_2, \dots, \gamma_s)$. Then it is seen from Theorem 1.1 that $Q > 0$. Thus, we only need to prove $QB + B^TQ > 0$. Let $\rho_l(x) = \prod_{k=0}^{l-1}(x-k)$, $l = 1, 2, \dots, s$. Making a congruence transform based on the transformation matrix

$$V = \begin{bmatrix} \rho_1'(c_1) & \rho_2'(c_1) & \dots & \rho_s'(c_1) \\ \rho_1'(c_2) & \rho_2'(c_2) & \dots & \rho_s'(c_2) \\ \dots & \dots & \dots & \dots \\ \rho_1'(c_s) & \rho_2'(c_s) & \dots & \rho_s'(c_s) \end{bmatrix},$$

and using the conditions $B(2s)$, $C(s)$, and $E(s)$, with the technique in [1] we obtain

$$V^T(QB + B^TQ)V = [\delta_{l,m}],$$

where

$$
\begin{aligned}
\delta_{lm} &= \sum_{i=1}^{s} \gamma_i \rho_l'(c_i) \sum_{j=1}^{s} b_{ij} \rho_m'(c_j) + \sum_{i=1}^{s} \gamma_i \rho_m'(c_i) \sum_{j=1}^{s} b_{ij} \rho_l'(c_j) \\
&= \sum_{i=1}^{s} \gamma_i [\rho_l(x)\rho_m(x)]'_{x=c_i} \\
&\quad - \sum_{j=1}^{r} \rho_m(j-1) \sum_{i=1}^{s} \gamma_i a_{ij} \rho_l'(c_i) - \sum_{j=1}^{r} \rho_l(j-1) \sum_{i=1}^{s} \gamma_i a_{ij} \rho_m'(c_i) \\
&= \rho_l(r)\rho_m(r) - \sum_{j=1}^{r} \alpha_j \rho_l(j-1)\rho_m(j-1) \\
&\quad - \sum_{j=1}^{r} \rho_m(j-1)\alpha_j[\rho_l(r) - \rho_l(j-1)] - \sum_{j=1}^{r} \rho_l(j-1)\alpha_j[\rho_m(r) - \rho_m(j-1)] \\
&= \rho_l(r)\rho_m(r) - \sum_{j=1}^{r} \alpha_j \rho_l(r)\rho_m(j-1) - \sum_{j=1}^{r} \alpha_j \rho_m(r)\rho_l(j-1) \\
&\quad + \sum_{j=1}^{r} \alpha_j \rho_l(j-1)\rho_m(j-1), \qquad l, m = 1, 2, \ldots, s.
\end{aligned}
$$

Let

$$
R = \left[\begin{array}{cccc|c}
\alpha_2 & & 0 & & -\alpha_2 \\
 & \alpha_3 & & & -\alpha_3 \\
 & & \ddots & & \vdots \\
0 & & & \alpha_r & -\alpha_r \\
\hline
-\alpha_2 & -\alpha_3 & \ldots & -\alpha_r & 1
\end{array}\right], \qquad
U = \begin{bmatrix}
\rho_1(1) & \rho_2(1) & \ldots & \rho_s(1) \\
\rho_1(2) & \rho_2(2) & \ldots & \rho_s(2) \\
\cdots & \cdots & \cdots & \cdots \\
\rho_1(r) & \rho_2(r) & \ldots & \rho_s(r)
\end{bmatrix}.
$$

It is readily verified that the $(l, m)$-element of the matrix $U^T R U$ is also equal to $\delta_{lm}$, $l, m = 1, 2, \ldots, s$. Therefore,

$$
V^T(QB + B^T Q)V = U^T R U. \tag{2.14}
$$

Since $\sum_{j=1}^{r} \alpha_j = 1$ and $\alpha_j > 0$, $j = 1, 2, \ldots, r$, for any given

$$
x = [x_1, x_2, \ldots, x_r]^T \neq 0
$$

we have

$$
\begin{aligned}
x^T R x &= \sum_{i=1}^{r-1} \alpha_{i+1} x_i^2 + x_r^2 - 2x_r \sum_{i=1}^{r-1} \alpha_{i+1} x_i \\
&\geq \alpha_1 \sum_{i=1}^{r-1} \alpha_{i+1} x_i^2 + \left( x_r - \sum_{i=1}^{r-1} \alpha_{i+1} x_i \right)^2 > 0.
\end{aligned}
$$

Thus, $R > 0$. Since $r \geq s$ and $c_1, c_2, \ldots, c_s$ are distinct, $\operatorname{rank}(V) = \operatorname{rank}(U) = s$, and therefore the conclusion $QB + B^T Q > 0$ follows from (2.14) and $R > 0$. □

In view of the $B$-theory for general linear methods (cf. [11]), a combination of Theorems 2.1–2.3 and 1.1–1.3 yields the following results:

**Theorem 2.4.** *Suppose the method* (1.2)–(1.3) *is algebraically stable and diagonally stable, and satisfies* $B(\tau)$, $C(\tau)$, $\sum_{j=1}^{r}\alpha_j = 1$, *and* $\sum_{j=1}^{r} a_{ij} = 1$, $i = 1, 2, \dots, s$. *Then this method is optimally* $B$*-convergent of order at least* $\tau$.

**Theorem 2.5.** *Suppose the method* (1.2)–(1.3) *is algebraically stable and diagonally stable, and satisfies* $B(\tau+1)$, $C(\tau)$, $\sum_{j=1}^{r}\alpha_j = 1$, *and* $\sum_{j=1}^{r} a_{ij} = 1$, $i = 1, 2, \dots, s$. *Then*

(i) *this method is* $B$*-convergent of order* $\tau + 1$;

(ii) *if there exists a real number* $\nu$ *such that* (2.5) *holds, then this method is optimally* $B$*-convergent of order* $\tau + 1$.

**Theorem 2.6.** *Suppose the method* (1.2)–(1.3) *satisfies the conditions* $B(w)$, $C(\eta)$, *and* $E(\xi)$, $r, \eta, \xi \geq s$, $w \geq 2s$, $c_i \neq c_j$ *whenever* $i \neq j$, $\sum_{j=1}^{r}\alpha_j = 1$, *and* $\alpha_j > 0$, $j = 1, 2, \dots, r$. *Then*

(i) *this method is optimally* $B$*-convergent of order at least* $\min\{w, \eta\}$;

(ii) *this method is* $B$*-convergent of order* $\min\{w, \eta+1\}$;

(iii) *if there exists a real number* $\nu$ *such that* (2.5) *holds with* $\tau = \eta$, *then this method is optimally* $B$*-convergent of order* $\min\{w, \eta+1\}$.

**Theorem 2.7.** *Suppose* $r \geq s$, $\sum_{j=1}^{r}\alpha_j = 1$, *and* $\alpha_j > 0$, $j = 1, 2, \dots, r$. *Then the multistep Runge-Kutta methods defined by* (1.2), (1.3), *and* (1.5) *are all optimally* $B$*-convergent of order at least* $s$ *and* $B$*-convergent of order* $s+1$.

*Remark* 1. Specializing Theorems 2.4 and 2.5 to the case of $r = 1$, we obtain immediately the well-known related results for Runge-Kutta methods presented by Frank et al. [6, 7] and Burrage and Hundsdorfer [2].

*Remark* 2. Specializing Theorem 2.6 to the case of $r = 1$, we obtain immediately the well-known result that the implicit midpoint rule is optimally $B$-convergent of order 2 (cf. [9, 10]).

*Remark* 3. For existence and uniqueness of the solution to the equation (1.2a), we refer to [12]; if the space $X$ is of finite dimension, see also [3, 4, 5, 7, 8].

## 3. Some examples

**Example 1.** Consider the $r$-step one-stage multistep Runge-Kutta method

$$\tag{3.1} \begin{cases} Y = \sum_{j=1}^{r} a_j y_{n-1+j} + hbf(Y), \\ y_{n+r} = \sum_{j=1}^{r} \alpha_j y_{n-1+j} + h\gamma f(Y), \end{cases}$$

or equivalently,

$$\tag{3.2} y_{n+r} = \sum_{j=1}^{r} \alpha_j y_{n-1+j} + h\gamma f\left(\beta y_{n+r} + \sum_{j=1}^{r}(a_j - \beta\alpha_j) y_{n-1+j}\right),$$

where

$$r \geq 1, \quad \gamma = r - \sum_{j=1}^{r} \alpha_j (j-1), \qquad a_j = \frac{\alpha_j}{\gamma}(r+1-j), \qquad j = 1, 2, \ldots, r,$$

$$b = \frac{1}{2\gamma}\left[r^2 - \sum_{j=1}^{r} \alpha_j (j-1)(2r+1-j)\right], \quad \beta = \frac{b}{\gamma},$$

the real parameters $\alpha_1, \alpha_2, \ldots, \alpha_r$ satisfy $\sum_{j=1}^{r} \alpha_j = 1$ and $\alpha_j > 0$, $j = 1, 2, \ldots, r$. It is easily seen that the method satisfies the assumptions of Theorem 2.6 with $w = 2$ and $\eta = \xi = 1$, and the condition (2.5) with $\tau = 1$ is trivial since $s = 1$. Therefore, in view of Theorem 2.6, the method (3.1) or (3.2) is optimally $B$-convergent of order 2.

**Example 2.** For $r = s = 2$, the coefficients of a series of methods which satisfy the assumptions of Theorem 2.7 have been computed; some of them are as follows:

(i)

$$\alpha = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0.8570633514 \\ 0.3929366486 \end{bmatrix}, \quad A = \begin{bmatrix} 0.2352842040 & 0.7647157960 \\ 0.7592738744 & 0.2407261256 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4290266119 & 0.4402646229 \\ -0.1374873664 & 0.4682336621 \end{bmatrix}, \quad c = \begin{bmatrix} 1.634007031 \\ 0.5714724214 \end{bmatrix};$$

(ii)

$$\alpha = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0.9106438658 \\ 0.5893561342 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5049603372 & 0.4950396628 \\ 0.9165272661 & 0.08347273392 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4553667456 & 0.6609526643 \\ -0.1031014246 & 0.4991787090 \end{bmatrix}, \quad c = \begin{bmatrix} 1.611359073 \\ 0.4795500183 \end{bmatrix};$$

(iii)

$$\alpha = \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 0.9560446375 \\ 0.7939553625 \end{bmatrix}, \quad A = \begin{bmatrix} 0.7597573923 & 0.2402426077 \\ 0.9744099679 & 0.02559003213 \end{bmatrix},$$

$$B = \begin{bmatrix} 0.4782650437 & 0.8750152175 \\ -0.08644371227 & 0.5036002413 \end{bmatrix}, \quad c = \begin{bmatrix} 1.593522869 \\ 0.4427465611 \end{bmatrix}.$$

However, for all these methods, condition (2.5) with $\tau = 2$ does not seem to be satisfied, so we can only conclude that these methods are optimally $B$-convergent of order 2 and $B$-convergent of order 3.

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