ACCELERATION OF CONVERGENCE OF A FAMILY OF LOGARITHMICALLY CONVERGENT SEQUENCES

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ABSTRACT. The asymptotic behavior of several sequence transformations is investigated as $n \to \infty$ when applied to a certain family of logarithmically convergent sequences. The transformations considered are the iterations of the transformations $e_1^{(s)}(A_n)$ of Shanks and W_n of Lubkin, the θ-algorithm of Brezinski, the Levin u- and v-transforms, and generalizations of the ρ-algorithm and the Neville table. Computational results are given for both real and complex sequences.

1. INTRODUCTION

An infinite sequence A_n with limit A is defined to be logarithmically convergent if

$$\lim_{n \to \infty} \Delta A_{n+1}/\Delta A_n \quad \text{and} \quad \lim_{n \to \infty} (A_{n+1} - A)/(A_n - A)$$

both exist and are equal to unity, where $\Delta A_n = A_{n+1} - A_n$. Such sequences are slowly convergent, and therefore not convenient for numerical calculation. When these two limits are equal to $s \neq 1$ or 0, the convergence is called linear. Methods for accelerating logarithmic convergence have been investigated in [1] through [4], [9] through [15], [22], and in other references.

In the present paper, we will consider the family of sequences A_n which are asymptotically of the form

$$(1.1) \qquad A_n \sim A + n^{-\alpha} \sum_{j=0}^{\infty} a_j n^{-j/k}$$

as $n \to \infty$, where $k \geq 1$ is an integer, and A, α, and the a_j are complex. The expansion in (1.1) is assumed to be asymptotic in the sense of Poincaré, with $a_0 \neq 0$ and $\alpha \neq -m/k$ or $-m/k - 1$ for $m \geq 0$. We see that A_n is logarithmically convergent when Re $\alpha > 0$, and divergent when Re $\alpha < 0$. Sequences with the asymptotic behavior (1.1) occur in applications ([7, 8, and 18], for example), and it can be shown that all slowly convergent sequences occurring as examples in the references of the present paper satisfy (1.1) with $k = 1$. In [18], possible applications in which $k > 1$ are given.

The purpose of the present investigation is to find the asymptotic behavior of the following transformations as $n \to \infty$ when applied to the family of sequences (1.1): (a) the transformations $e_1^{(s)}(A_n)$ of Shanks and W_n of Lubkin and their iterations; (b) the θ-algorithm of Brezinski; (c) a generalization of the ρ-algorithm of Wynn; (d) the u- and v-transforms of Levin; and (e) a linear transformation which is a generalization of the Neville table. The generalization of the ρ-algorithm does not seem to have been given explicitly in the literature, but it can be shown to follow by applying results of [2] to sequences satisfying (1.1). The linear transformation in (e) appears to be new. When $k = 1$, some overlap of the present results is found with [1], where $e_1^{(s)}(A_n)$ is applied to sequences of the form (1.1), with [14] in the case of the Levin transforms, and more recently, with [12]. The generalization of the ρ-algorithm in [12] agrees with the present one when $k = 1$.

The θ-algorithm, the u- and v-transforms of Levin, and the iteration of W_n accelerate linear as well as logarithmic convergence. Because of this property, the Levin u-transform has been used in a general-purpose computer program for acceleration of convergence [5]. The θ-algorithm and the Levin transforms are among several procedures for acceleration of convergence tested in [16] and [17].

Each of the transformations (a) through (e) produces transformed sequences $B_{m,n}$, $m = 1, 2, \ldots$. The results of the present investigation show that for (a), (b), (c), and (e), $B_{m+1,n}$ converges more rapidly than $B_{m,n}$ as $n \to \infty$, $m \geq 1$, for all sequences A_n satisfying (1.1). In the case of (d), however, this acceleration is found only for $k = 1$. For $k \geq 2$, all sequences $B_{m,n}$ obtained from a given sequence by use of the u-and v-transforms are found to converge with the same rapidity.

In the present paper, acceleration of convergence is investigated only under the limiting process $\lim_{n \to \infty} B_{m,n}$ for fixed m. This process, and the limiting process $\lim_{m \to \infty} B_{m,n}$ for fixed n are called Process I and Process II, respectively, in [14] and [15]. In [14] and [15] acceleration of convergence is investigated theoretically under both Process I and Process II when Levin's transformations are applied to logarithmically and linearly convergent sequences, including sequences of the form (1.1) with $k = 1$.

The investigation of cases (a) through (e) is given in §§2 through 6 of the present paper. Section 7 includes numerical examples, with both real and complex α and A, to illustrate the acceleration of convergence obtained by the transformations considered and to test the asymptotic results. As suggested by the referee, calculations have also been carried out for the $d^{(m)}$-transformation due to Levin and Sidi [10], using the algorithm and FORTRAN program of [6]. For $m = 1$, the $d^{(m)}$-transformation reduces to the Levin u-transformation when $\sigma = \text{SIGMA} = 1D0$ and $\text{INCR} = 1$ in the FORTRAN program of [6]. When $k \geq 2$ in (1.1) and $m = k$, $\sigma = 1$, the present calculations indicate that acceleration of convergence occurs mainly under Process II.

2. THE TRANSFORMATIONS $e_1^{(s)}(A_n)$ AND $W(A_n)$

The transformation $e_1^{(s)}(A_n)$ is defined by

$$(2.1) \qquad e_1^{(s)}(A_n) = \frac{s\, e_1(A_n) - A_n}{s - 1},$$

$s \neq 1$, where

$$(2.2) \qquad e_1(A_n) = \frac{A_{n-1}A_{n+1} - A_n{}^2}{A_{n-1} + A_{n+1} - 2A_n} = A_n + [\Delta(1/\Delta A_{n-1})]^{-1}$$

is the first-order transform of Shanks [13, p. 39], also called the Aitken δ^2-process. The transformation (2.1) was introduced to accelerate the convergence of A_n when the conditions

$$(2.3) \qquad \lim_{n \to \infty} \Delta A_{n+1}/\Delta A_n = 1, \qquad \lim_{n \to \infty} \Delta e_1(A_n)/\Delta A_n = 1/s \neq 1$$

are satisfied simultaneously. A transformation equivalent to (2.1), called U_n, was also introduced by Lubkin [11, p. 232]. A second transformation due to Lubkin for acceleration of convergence of sequences satisfying (2.3), called W_n [11, p. 229], does not require knowledge of s. It will be denoted by $W(A_n)$ in the following, and is given by

$$(2.4) \qquad W(A_n) = \frac{e_1(A_n) - A_n \Delta e_1(A_n)/\Delta A_n}{1 - \Delta e_1(A_n)/\Delta A_n}.$$

We will use the notation $e_1^{(s_1)}\big(e_1^{(s_2)}(A_n)\big) = e_1^{(s_1)}e_1^{(s_2)}(A_n)$ and $W\big(W^{m-1}(A_n)\big) = W^m(A_n)$, where $W^1(A_n) = W(A_n)$. Using the second expression in (2.2), we find that (2.1) and (2.4) can be written in the alternate forms

$$(2.5) \qquad e_1^{(s)}(A_n) = A_n + \frac{s}{s-1}[\Delta(1/\Delta A_{n-1})]^{-1}$$

and

$$(2.6) \qquad W(A_n) = A_n - \frac{\Delta A_n [\Delta(1/\Delta A_{n-1})]^{-1}}{\Delta [\Delta(1/\Delta A_{n-1})]^{-1}},$$

respectively. The proofs of the following theorems will be carried out by direct substitution, noting that asymptotic expansions can always be added and multiplied, and that division can be carried out when the leading coefficient of the divisor does not vanish.

Theorem 2.1. *Let A_n satisfy the conditions of* (1.1), *and let $s \neq 1$. Then as $n \to \infty$, we have*

$$(2.7) \qquad \begin{aligned} e_1^{(\alpha+1)}(A_n) &\sim A + n^{-\alpha-2}\sum_{j=0}^{\infty} b_j n^{-j}, \quad k = 1 \\ &\sim A + n^{-\alpha-1/k}\sum_{j=0}^{\infty} c_j n^{-j/k}, \quad k \geq 2 \end{aligned}$$

and

$$(2.8) \qquad e_1^{(s)}(A_n) \sim A + n^{-\alpha}\sum_{j=0}^{\infty} d_j\, n^{-j/k}, \quad s \neq \alpha + 1,$$

where b_0 and c_0 may vanish, and $d_0 \neq 0$.

Proof. We have

(2.9) $$\Delta A_n \sim n^{-\alpha-1} \sum_{j=0}^{\infty} e_j n^{-j/k}$$

for $k \geq 1$, where $e_0 = -\alpha a_0 \neq 0$. Considering the case $k = 1$ first, we find that

(2.10) $$\left[\Delta(1/\Delta A_{n-1})\right]^{-1} \sim n^{-\alpha} \sum_{j=0}^{\infty} f_j n^{-j},$$

where

(2.11)
$$f_0 = -\frac{\alpha a_0}{\alpha + 1},$$
$$f_1 = -\frac{\alpha a_1}{\alpha + 1},$$
$$f_2 = -\frac{\alpha(\alpha-1)}{12(\alpha+1)} a_0 - \frac{(\alpha-1)(\alpha+2)}{(\alpha+1)^2} a_2 - \frac{1}{\alpha(\alpha+1)} \frac{a_1^2}{a_0}.$$

The first part of (2.7) follows by substitution of (1.1), (2.10), and (2.11) into (2.5), with

(2.12) $$b_0 = -\frac{\alpha-1}{12} a_0 + \frac{2}{\alpha(\alpha+1)} a_2 - \frac{1}{\alpha^2} \frac{a_1^2}{a_0}.$$

We see that b_0 may vanish for a given $\alpha \neq -1$ or 0.

When $k \geq 2$, we have

(2.13) $$\left[\Delta(1/\Delta A_{n-1})\right]^{-1} \sim n^{-\alpha} \sum_{j=0}^{\infty} g_j n^{-j/k},$$

where

(2.14)
$$g_0 = -\frac{\alpha a_0}{\alpha + 1},$$
$$g_1 = -\frac{(\alpha+1/k)(\alpha+1-1/k)}{(\alpha+1)^2} a_1.$$

The statement of (2.7) for $k \geq 2$ now follows from (1.1), (2.13), and (2.14), with

(2.15) $$c_0 = \frac{(1-1/k)/k}{\alpha(\alpha+1)} a_1.$$

We see that c_0 vanishes when $a_1 = 0$ for $\alpha \neq -1$ or 0.

To prove the last part of the theorem, we find from (2.10) through (2.14) that (2.8) holds with

(2.16) $$d_0 = \frac{s-\alpha-1}{(s-1)(\alpha+1)} a_0$$

when $s \neq 1$ and $s \neq \alpha + 1$. We see that $d_0 \neq 0$. $\quad\square$

Corollary 2.1.1. *For fixed* $m \geq 0$ *and* $k \geq 1$, *we have*

(2.17)
$$e_1^{(\alpha+1+m/k)} e_1^{(\alpha+1+(m-1)/k)} \cdots e_1^{(\alpha+1)}(A_n)$$
$$\sim A + n^{-\alpha-(m+1)/k} \sum_{j=0}^{\infty} b_{m,j}\, n^{-j/k},$$

where $b_{2m,0} = 0$ *for* $m \geq 0$ *when* $k = 1$, *and where* $b_{m,0}$ *may also vanish for other values of* m *when* $k \geq 1$.

Proof. It follows from Theorem 2.1 that (2.17) holds for $m = 0$, since $a_0 \neq 0$, with $b_{0,0} = 0$ when $k = 1$. Assuming that it holds for $0 \leq m \leq m_1$, let $b_{m_1,j} = 0$ for $0 \leq j \leq p-1$ for some integer $p \geq 1$, and let $b_{m_1,p} \neq 0$. It then follows from Theorem 2.1 that (2.17) holds for $m_1 + 1 \leq m \leq m_1 + p$, with the first $p - (m - m_1) - 1$ coefficients equal to zero for each m in this interval. We can now repeat this process starting from $m = m_1 + 1$, since $b_{m_1,0} \neq 0$. Hence, (2.17) holds for all $m \geq 0$. When $k = 1$, it then follows from (2.7) and (2.8) that $b_{2m,0} = 0$ for $m \geq 0$. $\square$

Corollary 2.1.2. *When* $k = 1$,

(2.18)
$$e_1^{(\alpha+2m+1)} e_1^{(\alpha+2m-1)} \cdots e_1^{(\alpha+1)}(A_n)$$
$$\sim A + n^{-\alpha-2m-2} \sum_{j=0}^{\infty} b_{m,j}\, n^{-j}, \quad k = 1,$$

for $0 \leq m \leq m_1$, *where* m_1 *is the first value of* m *for which* $b_{m_1+1,0}$ *vanishes.*

This corollary is an immediate consequence of Theorem 2.1. We see that Corollary 2.1.1 holds for all $m \geq 0$, while Corollary 2.1.2 may hold only for a finite number of values of m. When $k = 1$ and (2.18) is applicable, however, the latter should be both more efficient and more accurate than (2.17).

Theorem 2.2. *Let* A_n *satisfy the conditions of* (1.1). *Then as* $n \to \infty$,

(2.19)
$$W(A_n) \sim A + n^{-\alpha-2} \sum_{j=0}^{\infty} b_j n^{-j}, \quad k = 1$$
$$\sim A + n^{-\alpha-1/k} \sum_{j=0}^{\infty} c_j n^{-j/k}, \quad k \geq 2,$$

where b_0 *and* c_0 *may vanish.*

Proof. Considering the case $k = 1$ first, we find from (2.9) through (2.12) that

(2.20)
$$\frac{\Delta A_n \left[\Delta(1/\Delta A_{n-1}) \right]^{-1}}{\Delta \left[\Delta(1/(\Delta A_{n-1}) \right]^{-1}} \sim n^{-\alpha} \sum_{j=0}^{\infty} d_j n^{-j},$$

where

(2.21)
$$d_0 = a_0,$$
$$d_1 = a_1,$$
$$d_2 = \frac{1-\alpha}{6\alpha} a_0 + \frac{(\alpha+2)(\alpha^2 - \alpha + 2)}{\alpha^2(\alpha+1)} a_2 - \frac{2}{\alpha^3} \frac{a_1^2}{a_0}.$$

We see that the first part of (2.19) follows from (1.1), (2.6), (2.20), and (2.21), with

$$(2.22) \qquad b_0 = \frac{\alpha - 1}{6\,\alpha}\, a_0 - \frac{4}{\alpha^2(\alpha + 1)}\, a_2 + \frac{2}{\alpha^3}\, \frac{a_1^2}{a_0}.$$

We see that the ratio of (2.22) to (2.12) is equal to $2/\alpha$. Hence, if b_0 vanishes in (2.12), it also vanishes in (2.22). When $k \geq 2$, we have

$$(2.23) \qquad \frac{\Delta A_n \left[\Delta(1/\Delta A_{n-1})\right]^{-1}}{\Delta \left[\Delta(1/\Delta A_{n-1})\right]^{-1}} \sim n^{-\alpha} \sum_{j=0}^{\infty} e_j n^{-j/k},$$

where

$$(2.24) \qquad \begin{aligned} e_0 &= a_0, \\ e_1 &= \frac{(\alpha + 1/k)\left[\alpha^2 + (1 - 1/k)\alpha - (1 - 1/k)/k\right]}{\alpha^2(\alpha + 1)}\, a_1. \end{aligned}$$

The second part of (2.19) follows from (1.1), (2.6), (2.23), and (2.24), with

$$(2.25) \qquad c_0 = \frac{(1 - 1/k)/k^2}{\alpha^2(\alpha + 1)}\, a_1.$$

We see that c_0 vanishes when $a_1 = 0$ for $\alpha \neq -1$ or 0. $\square$

Corollary 2.2. *As $n \to \infty$ for fixed $m \geq 1$, we have*

$$(2.26) \qquad \begin{aligned} W^m(A_n) &\sim A + n^{-\alpha - 2m} \sum_{j=0}^{\infty} b_{m,j}\, n^{-j}, \quad k = 1 \\ &\sim A + n^{-\alpha - m/k} \sum_{j=0}^{\infty} c_{m,j}\, n^{-j/k}, \quad k \geq 2, \end{aligned}$$

where $b_{m,0}$ and $c_{m,0}$ may vanish for some values of m.

Proof. When $k = 1$, it follows from Theorem 2.2 that (2.26) holds through the first value m_1 of m for which $b_{m_1,0} = 0$. Let $b_{m_1,j} = 0$ for $0 \leq j \leq p - 1$, with $b_{m_1,p} \neq 0$. Then we have

$$(2.27) \qquad W^{m_1}(A_n) \sim A + n^{-\alpha - p - 2m_1} \sum_{j=0}^{\infty} b'_{m_1,j}\, n^{-j},$$

where $b'_{m_1,j} = b_{m_1,j+p}$, and $b'_{m_1,0} \neq 0$. We can therefore apply Theorem 2.2 again with α replaced by $\alpha + p$. Continuing in this way, we obtain the first part of (2.26) with some of the coefficients equal to zero. Similarly, when $k \geq 2$, let $c_{m_1,j} = 0$ for $0 \leq j \leq p - 1$, with $c_{m_1,0} \neq 0$. Then we have

$$(2.28) \qquad W^{m_1}(A_n) \sim A + n^{-\alpha - p/k - m_1/k} \sum_{j=0}^{\infty} c'_{m_1,j}\, n^{-j/k},$$

where $c'_{m_1,j} = c_{m_1,j+p}$, and $c'_{m_1,0} \neq 0$. As before, we obtain the second part of (2.26) with some of the coefficients equal to zero. $\square$

3. The θ-algorithm

The θ-algorithm of Brezinski is defined by the equations

(3.1)
$$\theta_{-1}^{(n)} = 0, \quad \theta_0^{(n)} = A_n,$$

and

(3.2)
$$\theta_{2m+1}^{(n)} = \theta_{2m-1}^{(n)} + 1/\Delta\theta_{2m}^{(n)},$$

(3.3)
$$\theta_{2m+2}^{(n)} = \theta_{2m}^{(n+1)} - \frac{\Delta\theta_{2m}^{(n+1)}\left[\Delta(1/\Delta\theta_{2m+1}^{(n)})\right]^{-1}}{\Delta\left[\Delta(1/\Delta\theta_{2m+1}^{(n)})\right]^{-1}}$$

for $m \geq 0$ [4, p. 34], where Δ is the forward difference operator acting on n. The following theorem will now be proved by use of Theorem 2.2.

Theorem 3.1. *Let A_n satisfy the conditions of (1.1). Then as $n \to \infty$ for fixed $m \geq 1$, we have*

(3.4)
$$\theta_{2m}^{(n)} \sim A + n^{-\alpha-2m}\sum_{j=0}^{\infty} b_{m,j}n^{-j}, \quad k = 1$$

$$\sim A + n^{-\alpha-m/k}\sum_{j=0}^{\infty} c_{m,j}n^{-j/k}, \quad k \geq 2$$

and

(3.5)
$$\theta_{2m-1}^{(n)} \sim n^{\alpha+2m-1}\sum_{j=0}^{\infty} d_{m,j}n^{-j}, \quad k = 1$$

$$\sim n^{\alpha+(m-1)/k+1}\sum_{j=0}^{\infty} e_{m,j}n^{-j/k}, \quad k \geq 2,$$

where $b_{m,0}$, $c_{m,0}$, $d_{m,0}$, and $e_{m,0}$ may vanish for some values of m.

Proof. From (3.1) through (3.3) together with (2.9), we have

(3.6)
$$\theta_1^{(n)} = 1/\Delta A_n$$

and

(3.7)
$$\theta_2^{(n)} = A_{n+1} - \frac{\Delta A_{n+1}\left[\Delta(1/\Delta A_n)\right]^{-1}}{\Delta\left[\Delta(1/\Delta A_n)\right]^{-1}} = W(A_{n+1}).$$

It follows from Theorem 2.2 and from (3.6) and (2.9) that (3.4) and (3.5) hold for $m = 1$.

Assuming that (3.4) and (3.5) hold for a given m with $b_{m,0}$ and $c_{m,0} \neq 0$, we find from (3.2) and (3.4) that (3.5) holds for $m+1$ with $d_{m+1,0} = 1/b_{m,0}$, $e_{m,0} = 1/c_{m,0}$. In order to show that (3.4) holds for $m+1$, we find from (3.2) and (3.3) that

(3.8)
$$\theta_{2m+2}^{(n)} = \theta_{2m}^{(n+1)} - \frac{\Delta\theta_{2m}^{(n+1)}\left[\Delta\theta_{2m-1}^{(n+1)} + \Delta(1/\Delta\theta_{2m}^{(n)})\right]^{-1}}{\Delta\left[\Delta\theta_{2m-1}^{(n)} + \Delta(1/\Delta\theta_{2m}^{(n)})\right]^{-1}}.$$

Referring to (3.4) and (3.5), we have

$$[\Delta\theta_{2m-1}^{(n+1)} + \Delta(1/\Delta\theta_{2m}^{(n)})]^{-1}$$

$$= [\Delta(1/\Delta\theta_{2m}^{(n)})]^{-1}\left\{1 + \Delta\theta_{2m-1}^{(n+1)}[\Delta(1/\Delta\theta_{2m}^{(n)})]^{-1}\right\}^{-1}$$

(3.9)

$$\sim [\Delta(1/\Delta\theta_{2m}^{(n)})]^{-1}\left[1 + n^{-2}\sum_{j=0}^{\infty}\overline{c}_{m,j}n^{-j}\right], \quad k = 1$$

$$\sim [\Delta(1/\Delta\theta_{2m}^{(n)})]^{-1}\left[1 + n^{-1/k}\sum_{j=0}^{\infty}\overline{d}_{m,j}n^{-j/k}\right], \quad k \geq 2,$$

where

(3.10)
$$\overline{c}_{m,0} = \frac{(\alpha+2m)(\alpha+2m-1)}{\alpha+2m+1}b_{m,0}d_{m,0},$$

$$\overline{d}_{m,0} = \frac{(\alpha+m/k)[\alpha+(m-1)/k+1]}{\alpha+m/k+1}c_{m,0}e_{m,0}.$$

Substituting (3.9) into (3.8) and referring to (2.7), we obtain

$$\theta_{2m+2}^{(n)} \sim W(\theta_{2m}^{(n+1)}) + n^{-\alpha-2m-2}\sum_{j=0}^{\infty}\overline{e}_{m,j}n^{-j}, \quad k = 1$$

(3.11)

$$\sim W(\theta_{2m}^{(n+1)}) + n^{-\alpha-(m+1)/k}\sum_{j=0}^{\infty}\overline{f}_{m,j}n^{-j/k}, \quad k \geq 2,$$

where

(3.12)
$$\overline{e}_{m,0} = -2\left[\frac{\alpha+2m-1}{\alpha+2m+1}\right]b_{m,0}^2 d_{m,0},$$

$$\overline{f}_{m,0} = -2\left[\frac{\alpha+(m-1)/k+1}{\alpha+m/k+1}\right]c_{m,0}^2 e_{m,0}.$$

Finally, it follows from Theorem 2.2 and from (3.9) through (3.12) that (3.4) holds for $m+1$. Since (3.4) and (3.5) hold for $m = 1$, it follows that they hold up to the first value of m for which $b_{m,0}$ and $c_{m,0}$ vanish. We note that the conditions $b_{m,0} \neq 0$ and $c_{m,0} \neq 0$ are necessary only for the application of Theorem 2.2, and that $d_{m,0}$ and $e_{m,0}$ may vanish.

To complete the proof, we first consider the case $k = 1$. Assuming that (3.4) and (3.5) hold for $1 \leq m \leq m_1$, let $b_{m_1,j} = 0$ for $0 \leq j \leq p-1$ for some integer $p \geq 1$, and let $b_{m_1,p} \neq 0$. Then writing $\alpha_1 = \alpha+p$ when $m > m_1$, we have

(3.13)
$$\theta_{2m}^{(n)} \sim A + n^{-\alpha_1-2m}\sum_{j=0}^{\infty}b'_{m,j}n^{-j}$$

and

(3.14)
$$\theta_{2m-1}^{(n)} \sim n^{\alpha_1+2m-1}\sum_{j=p}^{\infty}d'_{m,j}n^{-j},$$

where $b'_{m,j} = b_{m,j+p}$ and $d'_{m,j} = d_{m,j-p}$. Since $b'_{m,0} \neq 0$, and since it is permissible to have $d'_{m,0}$ in the first part of the proof as noted above, it follows

that a continuation of the calculation of $\theta_m^{(n)}$ leads to results of the form (3.4) and (3.5) with α replaced by α_1. This asymptotic behavior then holds until $b'_{m,0}$ vanishes. Repeating the preceding discussion as often as necessary, we obtain results which can be written in the form (3.4) and (3.5) with certain of the coefficients equal to 0.

Similarly, when $k \geq 2$, let $d_{m,j} = 0$ for $0 \leq j \leq p - 1$ with $d_{m,p} \neq 0$. Then

$$(3.15) \qquad \theta_{2m}^{(n)} \sim A + n^{-\alpha_1 - m/k} \sum_{j=0}^{\infty} c'_{m,j}\, n^{-j/k}$$

and

$$(3.16) \qquad \theta_{2m-1}^{(n)} \sim n^{\alpha_1 + (m-1)/k + 1} \sum_{j=p}^{\infty} e'_{m,j}\, n^{-j/k},$$

for $m > m_1$, with $c'_{m,j} = c_{m,j+p}$, $e'_{m,j} = e_{m,j-p}$, and $\alpha_1 = \alpha + p/k$. The remainder of the proof is now the same as that for the case $k = 1$. $\square$

4. A GENERALIZATION OF THE ρ-ALGORITHM

Wynn's ρ-algorithm [22] is defined by

$$(4.1) \qquad \begin{aligned} \rho_{-1}^n &= 0, \quad \rho_0^n = A_n, \\ \rho_s^n &= \rho_{s-2}^{n+1} + \frac{s}{\Delta \rho_{s-1}^n}, \end{aligned}$$

$s \geq 1$, where as before, Δ is the forward difference operator acting on n. An analysis similar to that in §3 shows that a theorem for ρ_s^n of the same form as Theorem 3.1 holds only when $\alpha = k = 1$. In order to obtain acceleration of convergence of A_n for general α and for $k \geq 1$, we introduce a generalization of (4.1) given by

$$(4.2) \qquad \begin{aligned} \rho_{-1,n}^{(\alpha,k)} &= 0, \quad \rho_{0,n}^{(\alpha,k)} = A_n, \\ \rho_{s,n}^{(\alpha,k)} &= \rho_{s-2,n+1}^{(\alpha,k)} + C_s^{(\alpha,k)}/\Delta \rho_{s-1,n}^{(\alpha,k)}, \end{aligned}$$

$s \geq 1$, where

$$(4.3) \qquad C_s^{(\alpha,1)} = \alpha + s - 1$$

and

$$(4.4) \qquad C_{2m-1}^{(\alpha,k)} = \alpha + (m-1)/k, \qquad C_{2m}^{(\alpha,k)} = \alpha + (m-1)/k + 1$$

for $m \geq 1$, $k \geq 2$. We see that the algorithm (4.2) through (4.4) reduces to the ρ-algorithm when $\alpha = k = 1$, and hence, that $\rho_{s,n}^{(1,1)} = \rho_s^n$. This algorithm can be arrived at by inspection, starting from (2.5) and Theorem 2.1. Alternatively, it can be obtained by the procedure of [2, p. 158], leading to

$$(4.5) \qquad \omega_{2m} = C_{2m-1}^{(\alpha,k)}, \qquad \omega_{2m+1} = C_{2m}^{(\alpha,k)}.$$

The following theorem is seen to be of the same form as Theorem 3.1.

Theorem 4.1. *Let* A_n *satisfy the conditions of* (1.1). *Then as* $n \to \infty$ *for fixed* $m \geq 1$, *we have*

(4.6)
$$\rho_{2m,n}^{(\alpha,k)} \sim A + n^{-\alpha-2m} \sum_{j=0}^{\infty} b_{m,j} n^{-j}, \quad k = 1$$

$$\sim A + n^{-\alpha-m/k} \sum_{m=0}^{\infty} c_{m,j} n^{-j/k}, \quad k \geq 2,$$

and

(4.7)
$$\rho_{2m-1,n}^{\alpha,k)} \sim n^{\alpha+2m-1} \sum_{j=0}^{\infty} d_{m,j} n^{-j}, \quad k = 1$$

$$\sim n^{\alpha+(m-1)/k+1} \sum_{j=0}^{\infty} e_{m,j} n^{-j}, \quad k \geq 2,$$

where $b_{m,0}$, $c_{m,0}$, $d_{m,0}$, *and* $e_{m,0}$ *may vanish for some values of* m.

Proof. From (4.2) and (2.5), we find that

(4.8)
$$\rho_{1,n}^{(\alpha,k)} = \alpha/\Delta A_n$$

and

(4.9)
$$\rho_{2,n}^{(\alpha,k)} = A_{n+1} + \frac{\alpha+1}{\alpha} \left[\Delta(1/\Delta A_n) \right]^{-1} = e_1^{(\alpha+1)}(A_{n+1}).$$

It follows from Theorem 2.1 and from (2.8), (4.8), and (4.9) that (4.6) and (4.7) hold for $m = 1$.

Let (4.6) and (4.7) hold for a given $m \geq 1$ with $b_{m,0} \neq 0$ and $c_{m,0} \neq 0$. Then it follows from (4.2) and (4.6) that (4.7) holds for $m + 1$ with $d_{m+1,0} = 1/b_{m,0}$ and $e_{m+1} = 1/c_{m,0}$. In order to show that (4.6) holds for $m + 1$, we have

(4.10)
$$\rho_{2m+2,n}^{(\alpha,k)} = \rho_{2m,n}^{(\alpha,k)} + C_{2m+2}^{(\alpha,k)} \left[\Delta\rho_{2m-1,n}^{(\alpha,k)} + C_{2m+1}^{(\alpha,k)} \Delta(1/\Delta\rho_{2m,n}^{(\alpha,k)}) \right]^{-1}$$

from (4.2). From (4.6) and (4.7), we find that

$$\left[\Delta\rho_{2m-1,n}^{(\alpha,k)} + C_{2m+1}^{(\alpha,k)} \Delta(1/\Delta\rho_{2m,n}^{(\alpha,k)}) \right]^{-1}$$

$$= \left[C_{2m+1}^{(\alpha,k)} \Delta(1/\Delta\rho_{2m,n}^{(\alpha,k)}) \right]^{-1} \left\{ 1 + \Delta\rho_{2m-1,n}^{(\alpha,k)} \left[C_{2m+1}^{(\alpha,k)} \Delta(1/\rho_{2m,n}^{(\alpha,k)}) \right]^{-1} \right\}^{-1}$$

(4.11)
$$\sim \left[C_{2m+1}^{(\alpha,k)} \Delta(1/\Delta\rho_{2m,n}^{(\alpha,k)}) \right]^{-1} \left[1 + n^{-2} \sum_{j=0}^{\infty} \overline{c}_{m,j} n^{-j} \right], \quad k = 1$$

$$\sim \left[C_{2m+1}^{(\alpha,k)} \Delta(1/\Delta\rho_{2m,n}^{(\alpha,k)}) \right]^{-1} \left[1 + n^{-1/k} \sum_{j=0}^{\infty} \overline{d}_{m,j} n^{-j/k} \right], \quad k \geq 2,$$

where

(4.12)
$$\overline{c}_{m,0} = \frac{\alpha+2m-1}{\alpha+2m+1} b_{m,0} d_{m,0}, \quad k = 1,$$

$$\overline{d}_{m,0} = \frac{\alpha+(m-1)/k+1}{\alpha+m/k+1} c_{m,0} e_{m,0}, \quad k \geq 2.$$

From (4.10), (4.11), and (2.6), we have

$$\rho_{2m+2,n}^{(\alpha,k)} \sim e_1^{(\alpha+2m+1)}(\rho_{2m,n+1}^{(\alpha,k)}) + n^{-\alpha-2m-2}\sum_{j=0}^{\infty} \overline{e}_{m,j}\, n^{-j}, \quad k = 1$$

(4.13)

$$\sim e_1^{(\alpha+m/k+1)}(\rho_{2m,n+1}^{(\alpha,k)}) + n^{-\alpha-(m+1)/k}\sum_{j=0}^{\infty} \overline{f}_{m,j}\, n^{-j/k}, \quad k \geq 2,$$

where

(4.14)
$$\overline{e}_{m,0} = -\frac{\alpha+2m-1}{\alpha+2m}\, b_{m,0}^2\, d_{m,0},$$

$$\overline{f}_{m,0} = -\frac{\alpha+(m-1)/k+1}{\alpha+m/k+1}\, c_{m,0}^2\, e_{m,0}.$$

Finally, it follows from (4.13) and Theorem 2.1 that (4.6) holds for $m+1$. Since (4.6) and (4.7) hold for $m = 1$, they also hold up to the first value of m for which $b_{m,0}$ and $c_{m,0}$ vanish. The remainder of the proof is now the same as that of Theorem 3.1 when θ_m^n is replaced by $\rho_{m,n}^{(\alpha,k)}$. $\quad\square$

5. THE LEVIN TRANSFORMATIONS

Transformations of the Levin type [9] can be written in the form

(5.1)
$$T_{m,n} = \frac{\Delta^m(n^{m-1}A_n/R_n)}{\Delta^m(n^{m-1}/R_n)}$$

[14], where $R_0 = 1$ and $R_n = \Delta A_{n-1}$, $n \geq 1$, in the case of the u-transform, and $R_n = -[\Delta(1/\Delta A_{n-1})]^{-1}$, $n \geq 1$, in the case of the v-transform. When A_n satisfies the conditions of (1.1), we find that these choices of R_n are both asymptotically of the form

(5.2)
$$R_n \sim n^{-\alpha}\sum_{j=0}^{\infty} r_j\, n^{-j/k}$$

as $n \to \infty$, with $r_0 = -\alpha a_0$ and $r_0 = -\alpha a_0/(\alpha+1)$, respectively. The following theorem will be proved for general R_n satisfying (5.2), with $r_0 \neq 0$. Hence, the u- and v-transforms are included as special cases. We see that (5.1) is a nonlinear transformation when the r_j are functions of the coefficients a_j, as in the cases of the u- and v-transforms, and that it is a linear transformation when the r_j are given. When $k = 1$, the statement of Theorem 5.1 is the same as that of Theorem 4.2 of Sidi in [14]. When $k = 2$, however, we find that all sequences $T_{m,n}$ have the same rate of convergence for $m \geq 2$ as $n \to \infty$. We note that the differentiability conditions on the function $f(z)$ in [14] are needed only for Process II.

Theorem 5.1. *Let A_n satisfy the conditions of* (1.1), *and let R_n satisfy* (5.2) *with $r_0 \neq 0$. Then as $n \to \infty$ for fixed $m \geq 1$, we have*

$$T_{m,n} \sim A + n^{-\alpha-m}\sum_{j=0}^{\infty} b_{m,j}\, n^{-j}, \quad k = 1$$

(5.3)

$$\sim A + n^{-\alpha-1/k}\sum_{j=0}^{\infty} c_{m,j}\, n^{-j/k}, \quad k \geq 2,$$

where $b_{m,0}$ and $c_{m,0}$ may vanish for some values of m.

Proof. From (1.1) and (5.2), we have

$$(5.4) \qquad n^{m-1} A_n/R_n \sim n^{m-1} A/R_n + n^{m-1} \sum_{j=0}^{\infty} d_j n^{-j/k}$$

and

$$(5.5) \qquad n^{m-1}/R_n \sim n^{\alpha+m-1} \sum_{j=0}^{\infty} e_j n^{-j/k}$$

for $k \geq 1$, where $d_0 = a_0/r_0$ and $e_0 = 1/r_0$. It follows that

$$(5.6) \qquad T_{m,n} \sim A + \frac{\Delta^m \left[n^{m-1} \sum_{j=0}^{\infty} d_j n^{-j/k} \right]}{\Delta^m \left[n^{\alpha+m-1} \sum_{j=0}^{\infty} e_j n^{-j/k} \right]}.$$

We find that

$$(5.7) \qquad \begin{aligned} \Delta^m \left[n^{m-1} \sum_{j=0}^{\infty} d_j n^{-j/k} \right] &\sim n^{-m-1} \sum_{j=0}^{\infty} f_{m,j} n^{-j}, \quad k = 1 \\ &\sim n^{-1-/k} \sum_{j=0}^{\infty} g_{m,j} n^{-j/k}, \quad k \geq 2, \end{aligned}$$

and that

$$(5.8) \qquad \Delta^m \left[n^{\alpha+m-1} \sum_{j=0}^{\infty} e_j n^{-j/k} \right] \sim n^{\alpha-1} \sum_{j=0}^{\infty} h_{m,j} n^{-j/k}$$

for $k \geq 1$, where

$$(5.9) \qquad \begin{aligned} f_{m,0} &= (-1)^m m! \, d_m, \\ g_{m,0} &= -(m - 1/k - 1)(m - 1/k - 2) \cdots (1/k) d_1, \\ h_{m,0} &= (\alpha + m - 1)(\alpha + m - 2) \cdots \alpha e_0. \end{aligned}$$

Finally, (5.3) follows from (5.6) through (5.8) with

$$(5.10) \qquad b_{m,0} = f_{m,0}/h_{m,0}, \quad c_{m,0} = g_{m,0}/h_{m,0}.$$

We see that $b_{m,0} = 0$ when d_m vanishes, and that $c_{m,0} = 0$ when d_1 vanishes. $\square$

6. A GENERALIZATION OF THE NEVILLE TABLE

The Neville table for a sequence A_n is defined by

$$(6.1) \qquad e_n^0 = A_n, \quad n \geq 0,$$

and

$$(6.2) \qquad e_n^m = [n e_n^{m-1} - (n - m) e_{n-1}^{m-1}]/m,$$

$m \geq 1$, $n \geq m$ [7, p. 191]. As in [7], the Neville table is found to accelerate the convergence of sequences which are asymptotically of the form (1.1) with $\alpha = 1$

and $k = 1$ as $n \to \infty$ for fixed m. Noting that $(n - \beta)(n - 1)^{-\beta} = n^{1-\beta} + O(n^{-1-\beta})$ for $\beta \neq 0$, we arrive at the more general sequence transformation

$$(6.3) \qquad e_{0,n}^{(\alpha,k)} = A_n, \quad n \geq 0,$$

and

$$(6.4) \quad e_{m,n}^{(\alpha,k)} = [n\, e_{m-1,n}^{(\alpha,k)} - (n - \alpha - (m-1)/k)e_{m-1,n-1}^{(\alpha,k)}]/(\alpha + (m-1)/k),$$

$m \geq 1$, $n \geq m$, and $k \geq 1$. We see that (6.4) reduces to (6.2) for $\alpha = 1$, $k = 1$. We will now show that this transformation accelerates the convergence of the family (1.1) when $k \geq 1$.

Theorem 6.1. *Let A_n satisfy the conditions of (1.1). Then as $n \to \infty$ for fixed $m \geq 0$, we have*

$$(6.5) \qquad e_{m,n}^{(\alpha,k)} \sim A + n^{-\alpha-m/k} \sum_{j=0}^{\infty} b_{m,j}\, n^{-j/k}$$

for $k \geq 1$, where $b_{m,0}$ may vanish for some values of m.

Proof. Assuming that (6.5) holds for a given $m \geq 0$, we find from (6.4) that

$$e_{m+1,n}^{(\alpha,k)} \sim A + \left\{ b_{m,0}\left[n^{-\alpha-m/k+1} - (n - \alpha - m/k)(n-1)^{-\alpha-m/k}\right] \right.$$

$$(6.6) \qquad + \sum_{j=1}^{\infty} b_{m,j}\left[n^{-\alpha-m/k-j/k+1} - (n-1)^{-\alpha-m/k-j/k+1}\right]$$

$$\left. + (\alpha + m/k - 1)\sum_{j=1}^{\infty} b_{m,j}\,(n-1)^{-\alpha-m/k-j/k} \right\}/(\alpha + m/k).$$

We have

$$n^{-\alpha-m/k+1} - (n - \alpha - m/k)(n-1)^{-\alpha-m/k}$$

$$(6.7) \qquad \sim n^{-\alpha-m/k-1} \sum_{j=0}^{\infty} c_{m,j}\, n^{-j}$$

$$\sim n^{-\alpha-(m+1)/k} \sum_{j=0}^{\infty} c_{m,j}\, n^{-[(j+1)k-1]/k},$$

where

$$(6.8) \qquad c_{m,0} = (\alpha + m/k)(\alpha + m/k - 1)/2\,,$$

and we can verify that the second and third terms on the right-hand side of (6.6) are asymptotically of the form of the second term on the right of (6.5) with m replaced by $m + 1$. It follows that $e_{m+1,n}^{(\alpha,k)}$ is asymptotically of the form of (6.5) with

$$(6.9) \qquad \begin{aligned} b_{m+1,0} &= (b_{m,0}\,c_{m,0} - b_{m,1})/(\alpha + m), \quad k = 1 \\ &= -b_{m,1}/(\alpha k + m), \quad k \geq 2. \end{aligned}$$

Hence, since $\alpha + m/k \neq 0$, $b_{m+1,0}$ may be either nonzero or zero, and the statement of Theorem 6.1 holds. $\square$

7. Numerical examples

In order to test the preceding asymptotic results, the six sequence transformations considered have been applied to the sequence

$$(7.1) \qquad\qquad A_n = A - \frac{n^{-\alpha}}{1 + n^{-1/k}}$$

for $k = 1, 2$, and 3, and for both real and complex values of α and A. Also, calculations have been carried out by the algorithm and FORTRAN program of [6] with $\sigma = 1, 1.1$, and 1.2, and $m = 1, 2$, and 3. Calculations were carried out on the CDC Cyber 875 in double precision, or 28 significant figures, when A_n is real. When A_n is complex, calculations were carried out on a personal computer in double precision, or about 16 significant figures, using RM/FORTRAN. The number of terms used in each calculation was 49, but the later terms were not always useful because of loss of accuracy. This loss of accuracy may not be present in the case of the $d^{(m)}$-transformation with $\sigma > 1$, however, since the latter is then found to become increasingly more stable numerically as $\sigma > 1$ increases, and to give more accurate results than for $\sigma = 1$.

Table 7.1 gives the number of significant figures obtained by each of the six transformations when applied to (7.1) for $\alpha = 1/2$ and $k = 1, 2$, and 3, and for the algorithm of [6]. In the iteration of $e_1^{(s)}$, (2.18) was used when $k = 1$, and (2.17) when $k = 2$ and 3. In the algorithm of [6], $\sigma = 1$ and $m = k$ for $k = 1, 2$, and 3. More accurate results can be obtained for $\sigma = 1.1$ and 1.2 than for $\sigma = 1$, but more terms of A_n are then used in the calculation.

The numerical behavior of the first six of the transformed sequences in Table 7.1 was found to be consistent with Theorems 2.1 through 6.1 when m is not too large. For larger values of m, the maximum accuracy for fixed m does not necessarily occur at the largest value of n due to loss of accuracy. Also, the maximum accuracy does not necessarily occur at the largest value of m. We see that the algorithm of [6] gives good accuracy for each of the values of k.

Tables 7.2 through 7.5 illustrate the asymptotic behavior of $W^m(A_n)$ and of the u-transform as given in Theorems 2.2 and 5.1, respectively. As in §5,

TABLE 7.1. Number of significant digits obtained in acceleration of convergence of (7.1) with $\alpha = 1/2$ and $A = 1$

Method	$k = 1$	$k = 2$	$k = 3$
Iter. of $e_1^{(s)}$	18	11	9
$W^m(A_n)$	15	12	7
$\theta_m^{(n)}$	14	9	5
$\rho_{m,n}^{(\alpha,k)}$	22	11	9
u-transform	16	3	3
$e_{m,n}^{(\alpha,k)}$	19	12	8
$d^{(k)}$-transformation	16	13	9

TABLE 7.2. Calculated values of $B_{m,n} = [W^m(A_n) - 1] n^{1/2+2m}$ for (7.1) with $k = 1$

n	$B_{1,n-3}$	$B_{2,n-6}$	$B_{3,n-9}$
47	-4.92723	-4.22842	-3.66563
48	-4.93275	-4.23681	-3.77912
49	-4.93803	-4.24470	-3.89092

TABLE 7.3. Calculated values of $C_{m,n} = [W^m(A_n) - 1] n^{1/2+m/k}$ for (7.1) with $k = 2$

n	$C_{1,n-3}$	$C_{2,n-6}$	$C_{3,n-9}$
47	0.318003	-0.0035669	-0.0000664
48	0.318328	-0.0034819	-0.0000651
49	0.318638	-0.0034005	-0.0000639

TABLE 7.4. Calculated values of $D_{m,n} = (T_{m,n} - 1) n^{1/2+m}$ for (7.1) with $k = 1$

n	$D_{1,n-1}$	$D_{2,n-2}$	$D_{3,n-3}$
47	2.69032	-5.21871	9.48604
48	2.68576	-5.21782	9.49511
49	2.68142	-5.21695	9.50380

TABLE 7.5. Calculated values of $E_{m,n} = (T_{m,n} - 1) n^{1/2+1/k}$ for (7.1) with $k = 2$

n	$E_{1,n-1}$	$E_{2,n-2}$	$E_{3,n-3}$
47	0.737337	0.325399	0.190261
48	0.740541	0.325562	0.190261
49	0.743638	0.325719	0.190674

the u-transform is denoted by $T_{m,n}$. We see that the calculated quantities in a given column become nearly constant as n increases, in agreement with these theorems.

In order to investigate the acceleration of convergence of the algorithm of [6] numerically when applied to (7.1), we repeat the computations of Table 7.5 for the latter when $m = k = 2$ and $\sigma = 1$. Denoting this transformation by $U_{m,n}$, we obtain the results shown in Table 7.6. This table indicates that acceleration of convergence under Process I is not significant, and hence, that the acceleration of convergence found in Table 7.1 for $k = 2$ appears to occur mainly under Process II. The corresponding calculations for $m = k = 3$ and $\sigma = 1$ lead to these same conclusions when $k = 3$.

TABLE 7.6. Calculated values of $F_{m,n} = (U_{m,n} - 1)\, n^{1/2+1/k}$ for (7.1) with $k = 2$

n	$F_{1,n-1}$	$F_{2,n-2}$	$F_{3,n-3}$
47	0.737337	0.115339	0.282021
48	0.740541	0.114142	0.280211
49	0.743638	0.112984	0.278329

TABLE 7.7. Number of significant digits obtained in acceleration of convergence of (7.1) with $\alpha = 1/2 + i$ and $A = 1 + i$

Method	$k = 1$	$k = 2$	$k = 3$
Iter. of $e_1^{(s)}$	12	7	6
$W^m(A_n)$	9	5	5
$\theta_m^{(n)}$	9	5	5
$\rho_{m,n}^{(\alpha,k)}$	13	7	6
u-transform	10	4	5
$e_{m,n}^{(\alpha,k)}$	11	7	6
$d^{(k)}$-transformation	11	7	6

TABLE 7.8. Number of significant digits obtained in the acceleration of convergence of (7.1) with $A = 1 + i$, $k = 1$

Method	$\alpha = 1/2 + 10i$	$\alpha = 1/2 + 25i$	$\alpha = 1/2 + 50i$
Iter. of $e_1^{(s)}$	16	16	16
$W^m(A_n)$	12	15	16
$\theta_m^{(n)}$	13	15	16
$\rho_{m,n}^{(\alpha,k)}$	14	15	16
u-transform	13	15	16
$e_{m,n}^{(\alpha,k)}$	9	9	4
$d^{(1)}$-transformation	13	14	16

Table 7.7 gives the number of significant figures obtained when the six transformations considered and the algorithm of [6] are applied to (7.1) with the complex parameters $\alpha = 1/2 + i$ and $A = 1 + i$ and with $k = 1$, 2, and 3. The smaller of the number of significant digits in the real and imaginary parts is given. As before, the $d^{(k)}$-transformation gives good accuracy for each of the values of k. Similarly, Table 7.8 gives the accuracy obtained with these same transformations when $A = 1 + i$, $k = 1$, and $\alpha = 1/2 + 10i$, $1/2 + 25i$, and $1/2 + 50i$. The difference between the u-transform and the $d^{(1)}$-transformation is due only to differences in loss of accuracy.

Table 7.8 shows that the generalized Neville table is the most sensitive of the six transformations to the magnitude of α in this example, and suggests that it should be used only for small or moderate values of $|\alpha|$.

As additional examples, we apply the generalized ρ-algorithm to the sequences

$$(7.2) \qquad A_n = \sum_{j=1}^{n} (j + e^{1/j})^{-\sqrt{2}}$$

and

$$(7.3) \qquad A_n = \sum_{j=1}^{n} \binom{-\frac{1}{2}}{j-1} \frac{(-1)^{j+1}}{4j-3}$$

of [16], where they were found not to be accelerated by the ρ-algorithm. By use of the Euler-Maclaurin summation formula, it can be shown that these sequences satisfy the conditions of (1.1) with $k = 1$ and with $\alpha = \sqrt{2} - 1$ in (7.2) and $\alpha = 1/2$ in (7.3). It follows from Theorem 4.1 that the convergence of these sequences is accelerated by the generalized ρ-algorithm with $k = 1$ and with $\alpha = \sqrt{2} - 1$ and $1/2$, respectively, and that their convergence is not accelerated by the original ρ-algorithm under the limiting process $n \to \infty$ for constant m. Calculations for (7.2) and (7.3) have been carried out in double precision on the CDC 875, using 49 terms of each sequence. The results obtained converged to 19 and 20 significant figures, respectively, in the case of the generalized ρ-algorithm, and to 2 and 3 significant figures, respectively, in the case of the original ρ-algorithm.

Finally, we note that Example 2 of [12] satisfies (1.1) with $\alpha = 1/2$ and $k = 2$. Applying the generalized ρ-algorithm to 49 terms of the sequence, we obtain 11 significant figures when $\alpha = 1/2$ and $k = 2$, and 5 significant figures when $\alpha = 1/2$ and $k = 1$.

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