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NUMERICAL EVALUATION OF SOME TRIGONOMETRIC SERIES

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ABSTRACT. We present a method for the accurate numerical evaluation of a family of trigonometric series arising in the design of special-purpose quadrature rules for boundary element methods. The series converge rather slowly, but can be expressed in terms of Fourier-Chebyshev series that converge rapidly.

Let r be a positive integer, and define the functions G_r and H_r by

$$G_r(t) = 2\sum_{m=1}^{\infty} \frac{1}{m^r} \cos 2\pi mt,$$
$$H_r(t) = 2\sum_{m=1}^{\infty} \frac{1}{m^r} \sin 2\pi mt.$$

This note describes an efficient method for evaluating G_r and H_r to high accuracy. When r is small, direct evaluation of the trigonometric series is impractical because they converge too slowly.

The author's interest in G_r and H_r stems from their role in the analysis and design of numerical integration techniques for boundary element methods. That application leads to systems of nonlinear equations involving G_r and H_r (for small values of r), the solutions of which yield the weights and integration points of nonstandard, Gauss-like rules. See Chandler and Sloan [4, §5] or the survey article [11, §7] for a particularly simple example, where the integration points are the solutions of just a single equation, $G_r(t) = 0$. Brown et al. [3] discuss some important analytical properties of G_r and H_r.

Other authors have considered the numerical evaluation of certain closely related trigonometric series, arising from plate contact problems. For some methods quite different from the one presented here, see Boersma and Dempsey [2] and papers cited therein.

To evaluate G_r and H_r, it suffices to evaluate

$$C_r(t) = -2\sum_{m=1}^{\infty} \frac{1}{m^r} \cos(2\pi mt - r\pi/2),$$
$$S_r(t) = -2\sum_{m=1}^{\infty} \frac{1}{m^r} \sin(2\pi mt - r\pi/2),$$

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because for $k \geq 1$,

$$G_{2k-1}(t) = (-1)^{k+1} S_{2k-1}(t), \qquad G_{2k}(t) = (-1)^{k+1} C_{2k}(t),$$
$$H_{2k-1}(t) = (-1)^{k} C_{2k-1}(t), \qquad H_{2k}(t) = (-1)^{k+1} S_{2k}(t).$$

We shall work with C_r and S_r instead of with G_r and H_r because the relations

$$(1) \qquad C_r'(t) = 2\pi C_{r-1}(t) \quad \text{and} \quad S_r'(t) = 2\pi S_{r-1}(t)$$

are more convenient than the corresponding ones involving G_r', H_r', G_{r-1}, and H_{r-1}.

An elementary calculation reveals that

$$C_1(t) = 2\pi(t - \tfrac{1}{2}) \quad \text{for } 0 < t < 1,$$

so for all $r \geq 1$ the restriction of C_r to the unit interval is just a polynomial of degree r. In fact,

$$C_r(t) = \frac{(2\pi)^r}{r!} B_r(t) \quad \text{for } 0 < t < 1,$$

where B_r is the Bernoulli polynomial of degree r; cf. [1, p. 805]. Thus, numerical evaluation of H_1, G_2, H_3, G_4, ... presents no difficulties, but dealing with the remaining cases G_1, H_2, G_3, H_4, ... requires an efficient method of computing S_r. When $r = 1$ there is a convenient, closed form,

$$(2) \qquad S_1(t) = -2 \log |2 \sin \pi t|;$$

however, in general, S_r is a rescaled version of the Clausen function of order r:

$$S_r(t) = (-1)^{k+1} 2\,\mathrm{Cl}_r(2\pi t) \quad \text{for } r = 2k - 1 \text{ or } 2k,$$

following the notation of Lewin [9]. Our strategy for computing S_2, S_3, ... uses (2) and the second formula in (1).

In order to split S_r into a sum of singular and regular terms, we define a function Φ_r on the interval $[-1, 1]$, by writing

$$(3) \quad S_r(t) = -2\frac{(2\pi)^{r-1}}{(r-1)!}[t^{r-1}\log t + (-1)^{r-1}(1-t)^{r-1}\log(1-t)] + \Phi_r(2t-1)$$

for $0 < t < 1$. The problem now reduces to evaluating Φ_r. When $r = 1$, the closed form (2) implies that

$$(4) \qquad \Phi_1(x) = \begin{cases} -2\log \dfrac{8\cos(\pi x/2)}{1-x^2} & \text{if } -1 < x < 1, \\ -2\log 2\pi & \text{if } x = \pm 1, \end{cases}$$

and by differentiating (3) and using (1), we find that

$$(5) \qquad \Phi_r'(x) = \pi[\Phi_{r-1}(x) + Q_{r-2}(x)] \quad \text{for } r \geq 2,$$

where

$$Q_r(x) = 2\frac{\pi^r}{(r+1)!}[(1+x)^r + (-1)^r(1-x)^r] \quad \text{for } r \geq 0.$$

We generate Φ_2, Φ_3, Φ_4, ... by repeated integration of Φ_1, using the formula (5). The constants of integration follow from the relation

$$(6) \qquad \int_0^1 S_r(t)\,dt = 0 \quad \text{for } r \geq 1,$$

and the numerical calculations are easily performed using the method of Clenshaw and Curtis [6], as we now demonstrate.

Let T_k denote the first-kind Chebyshev polynomial of degree k, i.e.,

$$T_k(\cos\theta) = \cos k\theta,$$

and put

$$a_{rk} = \frac{2}{\pi}\int_{-1}^{1} \frac{\Phi_r(x)T_k(x)}{\sqrt{1-x^2}}\,dx \quad \text{for } k \geq 0,$$

so that

$$\Phi_r(x) = {\sum_{k=0}^{\infty}}' a_{rk}T_k(x) = \frac{a_{r0}}{2}T_0(x) + \sum_{k=1}^{\infty} a_{rk}T_k(x). \tag{7}$$

(The prime on the summation sign indicates that the coefficient of $T_0(x)$ is multiplied by $1/2$.) We see from (4) that Φ_1, and hence Φ_r for all $r \geq 1$, has an analytic continuation to the strip $-3 < \operatorname{Re} z < 3$. In particular, if

$$1 < \rho < 3 + 2\sqrt{2},$$

then Φ_r is analytic inside and on the ellipse $\{(z+z^{-1})/2 : |z| = \rho\}$, and therefore

$$|a_{rk}| \leq \frac{\text{const}_{r,\rho}}{\rho^k} \quad \text{for } k \geq 0; \tag{8}$$

see Rivlin [10, p. 143]. This estimate shows that the series (7) converges rapidly, and so is suitable for numerical evaluation of Φ_r. Another attractive feature of the representation (7) is that, because of the parity properties

$$S_r(1-t) = (-1)^{r+1}S_r(t), \quad \Phi_r(-x) = (-1)^{r+1}\Phi_r(x), \quad Q_r(-x) = (-1)^r Q_r(x),$$

half of the coefficients vanish:

$$a_{rk} = 0 \quad \text{if } r \text{ is } \left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\} \text{ and } k \text{ is } \left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}.$$

Thus, Φ_r can be evaluated using standard methods for the summation of even and odd Fourier-Chebyshev series; see, e.g., Clenshaw [5, pp. 9–10]. We shall also require the expansion

$$Q_r(x) = {\sum_{k=0}^{\infty}}' b_{rk}T_k(x) = {\sum_{k=0}^{r}}' b_{rk}T_k(x),$$

in which $b_{rk} = 0$ for $k > r$ because Q_r is a polynomial of degree r.

To compute a_{rk} when $r = 1$, we apply Gauss-Chebyshev quadrature, defining

$$a_{1k}^{(N)} = \frac{2}{N}\sum_{n=1}^{N} \Phi_1(x_n^{(N)})T_k(x_n^{(N)}) \quad \text{where} \quad x_n^{(N)} = \cos\frac{(2n-1)\pi}{2N},$$

and using the explicit formula (4) for Φ_1. By [10, equation (3.58)],

$$a_{1k}^{(N)} = a_{1k} + \sum_{p=1}^{\infty}(-1)^p(a_{1,2pN-k} + a_{1,2pN+k}) \quad \text{for } 0 \leq k < N,$$

so the estimate (8) implies that

$$|a_{1k}^{(N)} - a_{1k}| \le \frac{\text{const}_\rho}{\rho^{2N-k}} \quad \text{for } N > k \ge 0\,.$$

It follows from (5) that the Chebyshev coefficients of Φ_r, Φ_{r-1}, and Q_{r-2} satisfy

$$a_{rk} = \frac{\pi}{2k}[a_{r-1,k-1} + b_{r-2,k-1} - a_{r-1,k+1} - b_{r-2,k+1}] \quad \text{for } k \ge 1 \text{ and } r \ge 2\,;$$

cf. [5] or [6] or [7, p. 59]. The case $k = 0$ is handled by observing that (3) and (6) imply

$$a_{r0} = \begin{cases} 0 & \text{if } r \text{ is even,} \\ -\frac{8}{r}\frac{(2\pi)^{r-1}}{r!} + 2\sum_{k=1}^{\infty} \frac{a_{r,2k}}{4k^2-1} & \text{if } r \text{ is odd.} \end{cases}$$

The coefficients b_{rk} can be evaluated in closed form: after using the parity property $T_k(-x) = (-1)^k T_k(x)$ and the substitution $x = \cos\theta$, we find with the help of Gradshteyn and Ryzhik [8, p. 372, formula 9] that

$$b_{rk} = \pi^r \frac{1+(-1)^{r+k}}{2^{r-2}} \frac{(2r)!}{(r+1)!(r+k)!(r-k)!} \quad \text{for } 0 \le k \le r\,.$$

TABLE 1. Chebyshev coefficients of Φ_r for $1 \le r \le 6$

k	a_{1k}	a_{3k}	a_{5k}
0	-7.83870 74961 83803	-14.38120 33723 33972	-15.40706 80937 28069
2	0.24152 87859 01736	4.73732 65271 83045	7.63161 01374 83021
4	0.00203 43769 84711	0.04899 74836 10452	2.14176 52620 83497
6	0.00003 56715 82236	0.00016 23361 08813	0.00400 70818 86336
8	0.00000 07625 83158	0.00000 15125 91966	0.00000 70348 52906
10	0.00000 00177 75559	0.00000 00200 30349	0.00000 00404 77246
12	0.00000 00004 34569	0.00000 00003 17445	0.00000 00003 63550
14	0.00000 00000 10952	0.00000 00000 05618	0.00000 00000 04163
16	0.00000 00000 00282	0.00000 00000 00107	0.00000 00000 00056
18	0.00000 00000 00007	0.00000 00000 00002	0.00000 00000 00001

k	a_{2k}	a_{4k}	a_{6k}
1	-0.12603 48571 75644	-4.19275 26396 29885	-9.41234 87210 83799
3	0.12539 89792 71593	4.17737 42734 99876	8.61234 51447 07814
5	0.00062 79118 20911	0.01534 20140 62813	0.79910 47413 99756
7	0.00000 78335 61075	0.00003 60887 12789	0.00089 76084 55303
9	0.00000 01299 93449	0.00000 02605 01145	0.00000 12207 48844
11	0.00000 00024 76288	0.00000 00028 14996	0.00000 00057 28222
13	0.00000 00000 51186	0.00000 00000 37678	0.00000 00000 43425
15	0.00000 00000 01117	0.00000 00000 00577	0.00000 00000 00430
17	0.00000 00000 00025	0.00000 00000 00010	0.00000 00000 00005
19	0.00000 00000 00001	0.00000 00000 00000	0.00000 00000 00000

Table 1 lists the nonzero coefficients a_{rk} for $1 \leq r \leq 6$. The calculations were performed as described above in 25-digit, decimal arithmetic using MAPLE, and the results then rounded to 15 decimal places. In the case $r = 1$, the coefficients were obtained by computing $a_{1k}^{(N)}$ with $N = 24$. From the behavior of $a_{1k}^{(N)}$ for N in the range $k < N \leq 30$, our values of a_{1k} appeared correct to about the 22nd decimal place before being rounded.

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