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IRREDUCIBLE FINITE INTEGRAL MATRIX GROUPS OF DEGREE 8 AND 10

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ABSTRACT. The lattices of eight- and ten-dimensional Euclidean space with irreducible automorphism group or, equivalently, the conjugacy classes of these groups in $GL_n(\mathbb{Z})$ for $n = 8, 10$, are classified in this paper. The number of types is 52 in the case $n = 8$, and 47 in the case $n = 10$. As a consequence of this classification one has 26, resp. 46, conjugacy classes of maximal finite irreducible subgroups of $GL_8(\mathbb{Z})$, resp. $GL_{10}(\mathbb{Z})$. In particular, each such group is absolutely irreducible, and therefore each of the maximal finite groups of degree 8 turns up in earlier lists of classifications.

INTRODUCTION

The purpose of this paper is to describe all 8- (resp. 10-)dimensional lattices in Euclidean space with irreducible automorphism groups. These groups are called the irreducible *Bravais groups* of degree 8, resp. 10, and they characterize the corresponding lattices. The method by which the Bravais groups are obtained provides some interesting information about interrelations between them: first the inclusions in other Bravais groups are determined, second it is examined whether for a set of irreducible Bravais groups there exists an irreducible subgroup of $GL_n(\mathbb{Z})$ which is $GL_n(\mathbb{Q})$-conjugate to a subgroup of each of these Bravais groups. To illustrate this last point, the simplicial complex $Br_n(\mathbb{Z})$ is introduced, which is defined similarly as the simplicial complex $M_n^F(\mathbb{Q})$ in [10].

This paper continues and extends a series of papers by W. Plesken and M. Pohst who determined the absolutely irreducible maximal finite subgroups of $GL_n(\mathbb{Z})$ for $n = 5, 6, 7, 8, 9$ in [12] and [13]. The same is done for $n = 10$ in this paper, and additionally also those irreducible Bravais groups of degree 8 and 10 are determined which are not absolutely irreducible. The results imply that for degree 8 as well as for degree 10 all maximal finite irreducible subgroups of $GL_n(\mathbb{Z})$ are absolutely irreducible (which is not true in general, cf. [10]). Hence, the list of the absolutely irreducible maximal finite subgroups of $GL_8(\mathbb{Z})$ in [13] is a complete list of the maximal finite irreducible subgroups.

On the microfiche appendix, generators for the irreducible Bravais groups of degree 8 and 10 and Gram matrices for the quadratic forms of their lattices are given (except for the absolutely irreducible groups of degree 8 which were already described in [13]).

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The extensive calculations were carried out on a MIPS C2050 and on a HP 9000/800, partially using the computer algebra systems GAP ([6]) and CAYLEY ([4]).

1. General methods and definitions

1.1. **Terminology and notation.** Two subgroups G and G' of $\mathrm{GL}_n(\mathbb{Z})$ are called $\mathbb{Z}$- (resp. $\mathbb{Q}$-)equivalent if they are conjugate by an element of $\mathrm{GL}_n(\mathbb{Z})$, resp. $\mathrm{GL}_n(\mathbb{Q})$; the conjugacy classes are called $\mathbb{Z}$- (resp. $\mathbb{Q}$-)classes.

Definition 1.1 (cf. [1]). (i) Let $G \leq \mathrm{GL}_n(\mathbb{Z})$; then $\mathcal{F}(G) := \{F \in \mathbb{R}^{n\times n}_{\text{symm}} \mid gFg^{\text{tr}} = F \text{ for all } g \in G\}$ is called the *space of forms of* G.

(ii) Let $\mathcal{F} \subseteq \mathbb{R}^{n\times n}_{\text{symm}}$ be a subset of the symmetric $n \times n$-matrices; then $\mathcal{B}(\mathcal{F}) := \{g \in \mathrm{GL}_n(\mathbb{Z}) \mid gFg^{\text{tr}} = F \text{ for all } F \in \mathcal{F}\}$ is called the *Bravais group of* $\mathcal{F}$.

(iii) $\mathcal{B}(G) := \mathcal{B}(\mathcal{F}(G))$ is called the *Bravais group of* G.

For $G \leq \mathrm{GL}_n(\mathbb{Z})$ the space of forms of G is clearly a vector space over $\mathbb{R}$ which has dimension ≥ 1 if G is finite. This dimension can easily be computed in terms of the natural character of G. Groups with a space of forms of dimension 1 are called *uniform*.

Definition 1.2 (cf. [11]). A finite group $G \leq \mathrm{GL}_n(\mathbb{Z})$ is called *Bravais-minimal*, if $\dim_{\mathbb{R}} \mathcal{F}(H) > \dim_{\mathbb{R}} \mathcal{F}(G)$ for each $H < G$.

The property of being Bravais-minimal is clearly a property of $\mathbb{Q}$-classes because it only depends on the character of the natural representation of a group.

As notation for abstract groups, let the following be fixed: $G{:}\,H$, resp. $G.H$, is a split, resp. nonsplit, extension of G by H, $G \curlyvee H$ is a central product of G and H, and $G \curlywedge^F H$, resp. $G \between^F H$, is a subdirect, resp. subcentral, product of G and H amalgamated over the common factor F (cf. [8]).

1.2. **The method of computing irreducible Bravais groups.** The method of computing all irreducible Bravais groups of $\mathrm{GL}_n(\mathbb{Z})$ up to $\mathbb{Z}$-equivalence is divided into four steps:

(i) construct the irreducible Bravais-minimal subgroups of $\mathrm{GL}_n(\mathbb{Z})$ up to $\mathbb{Q}$-equivalence;

(ii) determine the $\mathbb{Z}$-classes of these Bravais-minimal groups;

(iii) for each $\mathbb{Z}$-class calculate the Bravais group of a representative of that class;

(iv) check whether the Bravais group so found is $\mathbb{Z}$-equivalent to one which was already obtained.

ad (i) The first step will be treated separately in §§2 and 3 of this paper since it strongly depends on the information which is already known for the dimension in question.

ad (ii) The $\mathbb{Z}$-classes in the $\mathbb{Q}$-class of a group $G < \mathrm{GL}_n(\mathbb{Z})$ can be computed by means of the *centering algorithm* described in [11]. This algorithm determines a set of representatives for the genera of $\mathbb{Z}G$-lattices. The isomorphism classes are obtained by multiplying these representatives with representatives for the ideal classes of the centralizing algebra C of G in $\mathbb{Z}^{n\times n}$ (cf. [15]).

ad (iii) The Bravais groups of the Bravais-minimal groups are computed by an extension of an algorithm described in [14], where an automorphism is constructed as a basis transformation which fixes the Gram matrices in the space of forms. A new implementation of the algorithm by the author constructs the full group of automorphisms by descending the Sims-chain of point stabilizers of the vectors in the standard basis.

ad (iv) The problem of deciding whether two Bravais groups are $\mathbb{Z}$-equivalent is attacked by searching for isometries for the spaces of forms. This is trivial for uniform Bravais groups. For Bravais groups which are not uniform the problem can be solved by using canonical bases for the spaces of forms. A conjugating element can then be found as a simultaneous isometry of the forms in the bases. The following lemma describes an identification of the space of forms of an irreducible group $G \leq \mathrm{GL}_n(\mathbb{Z})$ with a real subfield of the centralizing algebra C of G in $\mathbb{Q}^{n\times n}$ which leads to a canonical basis in many cases.

Lemma 1.3. *Let $G \leq \mathrm{GL}_n(\mathbb{Z})$ be finite, irreducible, $F_0 \in \mathcal{F}(G)$ positive definite, $C := C_{\mathrm{GL}_n(\mathbb{Q})}(G)$, and K a real subfield of $Z(C)$. Define $\tau\colon C \to \mathbb{Q}^{n\times n}\colon X \mapsto XF_0$ and $\kappa\colon \mathcal{F}(G) \to \mathbb{Q}^{n\times n}\colon F \mapsto FF_0^{-1}$.*

Then $K\tau \subseteq \mathcal{F}(G)$ and $\mathcal{F}(G)\kappa \subseteq C$. Furthermore, $\tau_{|K}$ induces an isomorphism of vector spaces from the additive group of K onto a subspace of $\mathcal{F}(G)$ which maps totally positive elements of K to positive definite forms of $\mathcal{F}(G)$.

Proof. Let $F \in \mathcal{F}(G)$, $g \in G$; then $gFF_0^{-1} = Fg^{-\mathrm{tr}}F_0^{-1} = FF_0^{-1}g$, and hence $F\kappa \in C$. For $X \in C$, clearly XF_0 is invariant under G. Now let $X \in K$; then $X \in \mathbb{Q}G$ because C has the double centralizer property (cf. [15]), and so $Z(C) = Z(\mathbb{Q}G)$. Hence, X may be written as $X = \sum_{g\in G} a_g g$. This yields $XF_0X^{\mathrm{tr}} = (\sum_{g\in G} a_g g)F_0(\sum_{g\in G} a_g g^{\mathrm{tr}}) = (\sum_{g\in G} a_g g)(\sum_{g\in G} a_g g^{-1})F_0$. But K is real, and since $g \mapsto g^{-1}$ is induced by complex conjugation, one has $\sum_{g\in G} a_g g = \sum_{g\in G} a_g g^{-1}$, hence $XF_0X^{\mathrm{tr}} = X^2F_0$, which is equivalent to $(XF_0)^{\mathrm{tr}} = XF_0$.

The injectivity of $\tau_{|K}$ is clear. The last claim follows from the fact that the conditions for X to be totally positive and for XF_0 to be positive definite coincide up to a scalar factor (which is the determinant of F_0). □

Remark 1. The condition that the real subfield is central is necessary to be sure that for a primitive element X of the field the matrix XF_0 is symmetric, but for a suitable choice of X and F_0 this may also be true even if the field is not central. Then the other statements of the lemma clearly hold.

The canonical bases are now obtained as the images under τ of an integral basis of totally positive elements of the maximal real subfield of C. The isometries are constructed by an algorithm which is a slightly modified version of the algorithm for finding automorphisms.

Remark 2. Lemma 1.3 can also be used to determine inclusions between Bravais groups. In this context, by inclusion always inclusion of a $\mathrm{GL}_n(\mathbb{Z})$-conjugate is meant. Clearly, inclusion of Bravais groups is equivalent to opposite inclusion of their spaces of forms, and the latter can be analyzed using the identifications with a real field.

1.3. **The simplicial complex of the irreducible Bravais groups.** One classical aspect of looking at Bravais groups is to use them to classify the crystal families (cf. [1]). These crystal families correspond to the connected components of the simplicial complex $\mathrm{Br}_n(\mathbb{Z})$ which is defined as follows.

Definition 1.4. The simplicial complex $\mathrm{Br}_n(\mathbb{Z})$ of the irreducible Bravais groups of degree n has the $\mathbb{Z}$-classes of irreducible Bravais groups as vertices. The

$s+1$ vertices $P_0, \dots, P_s$ represented by the Bravais groups $G_0, \dots, G_s$ form an *s-simplex* if there exists a group $H \le \mathrm{GL}_n(\mathbb{Z})$ with the properties:

(i) for all $0 \le i \le s$ there is a subgroup $H_i \le G_i$ in the $\mathbb{Q}$-class of H;

(ii) for all $0 \le i \le s$ one has $\dim_{\mathbb{R}} \mathcal{F}(H) = \dim_{\mathbb{R}} \mathcal{F}(G_i)$.

The method of constructing the Bravais-minimal subgroups of $\mathrm{GL}_n(\mathbb{Z})$ gives all the information to determine this simplicial complex. It will be visualized as follows: the maximal simplices are shown as n-gons which are glued together along dotted lines. With each of the maximal simplices the isomorphism type of an irreducible Bravais-minimal group H is given such that for each vertex of the simplex there exists a group $\widetilde{H} \le \mathrm{GL}_n(\mathbb{Z})$ in the $\mathbb{Q}$-class of H such that $\mathcal{B}(\widetilde{H})$ represents that vertex.

2. The irreducible Bravais groups of degree 8

Theorem 2.1. *In* $\mathrm{GL}_8(\mathbb{Z})$ *there are*

(i) *no maximal finite irreducible subgroups which are not absolutely irreducible*;

(ii) 26 *$\mathbb{Z}$-classes of uniform irreducible Bravais groups falling into* 16 *$\mathbb{Q}$-classes* (*cf.* [13]);

(iii) 18 *$\mathbb{Z}$-classes of irreducible Bravais groups with* 2*-dimensional space of forms falling into* 15 *$\mathbb{Q}$-classes*;

(iv) 4 *$\mathbb{Z}$-classes of irreducible Bravais groups with* 3*-dimensional space of forms falling into* 3 *$\mathbb{Q}$-classes*;

(v) 4 *$\mathbb{Z}$-classes of irreducible Bravais groups with* 4*-dimensional space of forms falling into* 4 *$\mathbb{Q}$-classes.*

Table 1 gives some specific information about the irreducible Bravais groups of degree 8 and about their interrelations. The uniform groups are omitted in the table since they were already described in [13]; the notation $F_1, \dots, F_{26}$ for their invariant forms is adapted from there. The first column of the table gives the name and the second the isomorphism type of the irreducible Bravais group. In the third column the forms of the uniform Bravais groups in which it is contained can be found and in the fourth column the irreducible Bravais groups with 2-dimensional spaces of forms in which it is properly contained (in this context inclusion always means inclusion of a $\mathrm{GL}_n(\mathbb{Z})$-conjugate). Finally, the last column gives the centralizer of the group in $\mathbb{Z}^{8\times 8}$. Here, $\theta_n := \zeta_n + \zeta_n^{-1}$, where ζ_n is a primitive nth root of unity, $Q_{2,3}$ denotes a maximal order of an indefinite quaternion algebra over $\mathbb{Q}$ ramified only at 2 and 3, and $\Lambda_{2,3}$ is a suborder of $Q_{2,3}$ of index 2. The horizontal lines separate the $\mathbb{Q}$-classes of the Bravais groups.

By inspecting the result one notices a nice characterization of the irreducible Bravais groups of degree 8.

Corollary 2.2. *The irreducible Bravais groups in* $\mathrm{GL}_8(\mathbb{Z})$ *are uniquely determined by their centralizer in* $\mathbb{Z}^{n\times n}$ *and by the $\mathbb{Z}$-classes of the maximal finite irreducible subgroups of* $\mathrm{GL}_8(\mathbb{Z})$ *they are contained in.*

2.1. **Constructing the Bravais-minimal groups of degree 8.** Let $G \le \mathrm{GL}_8(\mathbb{Z})$ be finite, irreducible; then the absolutely irreducible constituents of the natural representation of G all have the same degree d which is a divisor of 8. If $d = 1$, the group G has to be cyclic of order m with $\varphi(m) = 8$ (where φ denotes the Euler φ-function); hence $m \in \{15, 30, 16, 20, 24\}$. Clearly,

TABLE 1

name	isomorphism type	1-dim.	2-dim.	centr.
B_1	$D_{16} \wr C_2$	F_1, F_3		$\mathbb{Z}[\sqrt{2}]$
B_2		F_2, F_4		$\mathbb{Z}[\sqrt{2}]$
B_3	$(\tilde{S}_4 \mathsf{Y} \tilde{S}_4) : C_2$	F_3, F_5		$\mathbb{Z}[\sqrt{2}]$
B_4	$D_{16} \mathsf{Y} D_{12}$	F_6, F_7		$\mathbb{Z}[\sqrt{2}]$
B_5	$D_{24} \wr C_2$	F_3, F_7		$\mathbb{Z}[\sqrt{3}]$
B_6	$D_{24} \mathsf{Y} D_{12}$	F_6, F_9		$\mathbb{Z}[\sqrt{3}]$
B_7	$D_{24} \mathsf{Y} \tilde{S}_4$	F_5, F_6		$\mathbb{Z}[\sqrt{3}]$
B_8	$D_{24} \mathsf{Y}\, SL_2(3)$	F_3, F_{22}		$\mathbb{Z}[\sqrt{3}]$
B_9	$D_8 \diamond^{C_2} \mathrm{GL}_2(3)$	F_1, F_5, F_6		$\mathbb{Z}[\sqrt{3}]$
B_{10}		F_2, F_{20}		$\mathbb{Z}[\sqrt{3}]$
B_{11}		F_4, F_{21}		$\mathbb{Z}[\sqrt{3}]$
B_{12}	$\tilde{S}_4 \diamond^{C_2} Q_{24}$	F_3, F_6, F_{15}		$\mathbb{Z}[\sqrt{6}]$
B_{13}	$D_{16} \diamond^{C_2} D_{24}$	F_5, F_7		$\mathbb{Z}[\sqrt{6}]$
B_{14}	$(\mathrm{SL}_2(5) \mathsf{Y} \mathrm{SL}_2(5)) : C_2$	F_5, F_{15}		$\mathbb{Z}[\theta_5]$
B_{15}	$D_{20} \wr C_2$	F_{14}, F_{16}		$\mathbb{Z}[\theta_5]$
B_{16}	$C_2 \times (D_{10} \wr C_2)$	F_{15}, F_{17}		$\mathbb{Z}[\theta_5]$
B_{17}	$D_{20} \mathsf{Y} D_{12}$	F_{18}, F_{19}		$\mathbb{Z}[\theta_5]$
B_{18}	$(C_{10} \times C_2) : C_4$	F_{14}, F_{16}		$\mathbb{Z}[\sqrt{5}]$
B_{19}	$C_{12}.V_4$	F_5, F_6, F_7	B_4, B_7, B_{13}	$Q_{2,3}$
B_{20}		$F_3, F_6, F_7, F_{15}, F_{22}$	B_4, F_5, B_8, B_{12}	$\Lambda_{2,3}$
B_{21}	$(C_{12}.V_4) : C_3$	F_3, F_5, F_6, F_{15}	B_3, B_7, B_{12}, B_{14}	$Q_{2,3}$
B_{22}	$C_{12}.V_4$	F_3, F_5, F_7	B_3, B_5, B_{13}	$\Lambda_{2,3}$
D_{60}	D_{60}	$F_5, F_{15}, F_{18}, F_{19}$	B_{14}, B_{17}	$\mathbb{Z}[\theta_{15}]$
D_{32}	D_{32}	F_1, F_2, F_3, F_4	B_1, B_2	$\mathbb{Z}[\theta_{16}]$
D_{40}	D_{40}	$F_5, F_{14}, F_{15}, F_{16}$	B_{14}, B_{15}	$\mathbb{Z}[\theta_{20}]$
D_{48}	D_{48}	$F_3, F_5, F_6, F_7, F_{15}$	$B_3, B_4, B_5, B_7, B_{12}, B_{13}$	$\mathbb{Z}[\theta_{24}]$

the group C_{30} is not Bravais-minimal. If $d = 8$, the group G is absolutely irreducible; therefore $\mathcal{B}(G)$ must be one of the absolutely irreducible maximal finite subgroups of $\mathrm{GL}_8(\mathbb{Z})$, which were determined in [13].

The cases $d = 2$ and $d = 4$ are treated, using [2], where the finite subgroups of $\mathrm{PSL}_n(\mathbb{C})$ are classified for $n = 2, 3, 4$. The following definition and lemma describe how finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ can be reconstructed from their images in $\mathrm{PSL}_n(\mathbb{C})$.

Definition 2.3. Let $H \le \mathrm{PSL}_n(\mathbb{C})$ be finite, irreducible, S the group of scalar matrices in $\mathrm{GL}_n(\mathbb{C})$.

(i) A finite group $\tilde{H} \le \mathrm{GL}_n(\mathbb{C})$ is called a *covering group of the projective group* H if $\langle \tilde{H}, S \rangle$ is the full preimage of H in $\mathrm{GL}_n(\mathbb{C})$ and $\tilde{H}$ is minimal (with respect to order) with this property. (Note: $\tilde{H}$ is not uniquely determined by these properties.)

(ii) $A := \tilde{H} \cap S$ then is called a *multiplier of the projective group* H. (Note: A is an epimorphic image of the multiplier of the abstract group H.)

Lemma 2.4. *Let* $H \le \mathrm{PSL}_n(\mathbb{C})$ *be finite, irreducible,* $\tilde{H} \le \mathrm{GL}_n(\mathbb{C})$ *a covering group of the projective group* H. *Then the finite preimages* $G \le \mathrm{GL}_n(\mathbb{C})$ *of* H *are obtained as central, subdirect, or subcentral products of a finite cyclic group* C_m *with* $\tilde{H}$.

Proof. Denote by U the group of roots of unity in $\mathbb{C}$; then the full preimage of H in $\mathrm{GL}_n(\mathbb{C})$ is $\widetilde{H} \curlyvee U$, where the identified subgroup is $Z(\widetilde{H})$. Let π_1, π_2 be the projections of $\widetilde{H} \curlyvee U$ onto $\widetilde{H}$, resp. U. Since H is irreducible, the center of H is trivial; hence one has $A = Z(\widetilde{H})$. Let G be a finite preimage of H in $\mathrm{GL}_n(\mathbb{C})$. The minimality of $\widetilde{H}$ implies $G\pi_1 = \widetilde{H}$, and one has $G\pi_2 = C_m$ for a finite cyclic group C_m. Now surjectivity of the projections implies that G is a central, subdirect, or subcentral product of $\widetilde{H}$ with C_m, where in the cases concerned the identified subgroup is $Z(\widetilde{H})$. (For a detailed description of the construction of central, subdirect, and subcentral products, see e.g. [8].) □

The next step is to decide whether an irreducible group $\widetilde{G} \le \mathrm{GL}_d(\mathbb{C})$ is equivalent to an absolutely irreducible constituent of a rationally irreducible group $G \le \mathrm{GL}_n(\mathbb{Q})$. Denote by k the degree over $\mathbb{Q}$ of the character field of the natural representation of $\widetilde{G}$ and by s its rational Schur index. Then one must have $n = d \cdot k \cdot s$. The value of k is clear from the character table of $\widetilde{G}$, and for $n = 8$ it is easy to see that s has to be 1 or 2. Since the finite subgroups of $\mathrm{GL}_4(\mathbb{Q})$ are well known from [1], the case $d \cdot k = 4$ is easy to handle. For $d \cdot k = 8$ it is in most cases easy to construct an irreducible group $G \le \mathrm{GL}_8(\mathbb{Z})$ with constituent isomorphic to $\widetilde{G}$. If not, one constructs a group $G \le \mathrm{GL}_{16}(\mathbb{Z})$. By means of the centering algorithm it is easy to check whether the natural lattice $\mathbb{Z}^{1\times 16}$ is an irreducible $\mathbb{Z}G$-lattice. If this is the case, one has $s = 2$; otherwise $s = 1$.

For $d = 2$ it follows from [2] that each of the covering groups that have to be considered is isomorphic either to a dihedral group D_{2m} of order $2m$, where $\varphi(m) \mid 8$, or to one of the groups $\mathrm{SL}_2(3)$, $\widetilde{S}_4$, $\mathrm{SL}_2(5)$ (where $\widetilde{S}_4$ denotes the double cover of S_4). As Bravais-minimal irreducible subgroups of $\mathrm{GL}_8(\mathbb{Z})$ one obtains groups isomorphic to: $C_5 \times D_6$, $C_8 \times D_6$, $C_{12} \times D_6$, $C_{20} \curlywedge D_6$, $C_{24} \curlywedge D_6$, $C_5 \times D_8$, $C_8 \curlyvee D_8$, $C_{12} \curlyvee D_8$, $C_{16} \diamondsuit^{C_2} D_8$, $C_{20} \diamondsuit^{C_2} D_8$, $C_{24} \diamondsuit^{C_2} D_8$, $C_3 \times D_{10}$, $C_5 \times D_{10}$, $C_4 \curlywedge D_{10}$, $C_3 \times D_{16}$, $C_4 \diamondsuit^{C_2} D_{16}$, $C_8 \diamondsuit^{C_2} D_{16}$, $C_{12} \diamondsuit^{C_2} D_{16}$, $C_3 \times D_{24}$, $C_4 \diamondsuit^{C_2} D_{24}$, $C_8 \diamondsuit^{C_2} D_{24}$, $C_4 \curlyvee \mathrm{SL}_2(3)$.

For $d = 4$ it follows from [2] that the covering groups are of one of the following three types:

(i) imprimitive groups with blocks of imprimitivity of length 1: a group of this type is obtained as an extension of an abelian group by a transitive subgroup of S_4;

(ii) imprimitive groups with blocks of imprimitivity of length 2: a group of this type is obtained as an extension by C_2 of a group which is a subdirect product of a subgroup of $\mathrm{GL}_2(\mathbb{C})$ with itself;

(iii) primitive groups.

Bravais-minimal groups which are not uniform result only from groups of type (i). One obtains groups isomorphic to: $(C_8 \times C_4) : C_4$, $(C_{12} \times C_6) : C_4$, $(C_{10} \times C_2) : C_4$, $(C_5 \times C_5) : C_4$, $(C_6 \times C_6).D_8$ (2-dim. space of forms) and two nonsplit extensions of C_{12} by V_4 (3-dim.). For the uniform groups one checks that each of them fixes a form isometric to one of the forms from [13].

A more detailed description of the determination of the Bravais-minimal groups can be found in [17].

2.2. **Determining the class number of the centralizer.** It was pointed out in §1.2 that one needs representatives for the ideal classes of the centralizer in $\mathbb{Z}^{n\times n}$

of a group $G \leq \mathrm{GL}_n(\mathbb{Z})$ to get a full set of representatives for the isomorphism classes of $\mathbb{Z}G$-sublattices of $\mathbb{Z}^{1\times n}$. For the Bravais-minimal groups of degree 8 the centralizers with class number 1 are: $\mathbb{Z}[\zeta_3]$, $\mathbb{Z}[\zeta_4]$, $\mathbb{Z}[\zeta_5]$, $\mathbb{Z}[\zeta_8]$, $\mathbb{Z}[\zeta_{12}]$, $\mathbb{Z}[\zeta_{15}]$, $\mathbb{Z}[\zeta_{16}]$, $\mathbb{Z}[\zeta_{20}]$, $\mathbb{Z}[\zeta_{24}]$, (see [18]), $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{3}]$, $\mathbb{Z}[\theta_5]$, $\mathbb{Z}[\sqrt{-2}]$ (see [3]), $\mathbb{Z}[\zeta_3, \sqrt{2}]$, $\mathbb{Z}[\zeta_3, \sqrt{-2}]$, $\mathbb{Z}[\zeta_3, \theta_5]$ (see [7]), a maximal order of an indefinite quaternion algebra over $\mathbb{Q}$ ramified only at 2 and 3 (see [15]), maximal orders of totally definite quaternion algebras C over $\mathbb{Q}$ ramified at ∞ and 2, at ∞ and 3, or at ∞ and 5, and over $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{5})$ ramified only at the infinite primes (see [5]; for the calculation of $\zeta_K(2)$ of the Dedekind ζ-function for $K := Z(C)$ see [9]).

The only cases where the class number is not 1 are: $\mathbb{Z}[\sqrt{-5}]$, $\mathbb{Z}[\sqrt{-15}]$, $\mathbb{Z}[\zeta_4, \sqrt{3}\zeta_8]$, and a maximal order of the totally definite quaternion algebra over $\mathbb{Q}(\sqrt{3})$ which is ramified only at the infinite primes. In these four cases the class number is 2 (by the same references as above), and by looking at the norm functions one sees that in each of these cases the ideal generated by 2 is contained in a nonprincipal ideal. This shows that a lattice which is multiplied by a nonprincipal ideal will be found by the centering algorithm when it is run for the prime $p = 2$.

2.3. **Inclusions.** In a first step the following lemma shows in which of the uniform Bravais groups a given irreducible Bravais group is included. As before, $F_1, \dots, F_{26}$ denote the forms of the uniform groups as in [13].

Theorem 2.5. *Let $G \leq \mathrm{GL}_8(\mathbb{Z})$ be irreducible, K the maximal real subfield of $Z(C_{\mathrm{GL}_8(\mathbb{Q})}(G))$. Suppose $[K : \mathbb{Q}] = \dim_{\mathbb{R}}(\mathcal{F}(G))$, and let R be the ring of integral elements in K. Let $F_0 \in \mathcal{F}(G)$ be isomorphic to one of the F_i. Let $\{x_1R, \dots, x_lR\}$ be the set of principal ideals I of R with $I \supset pR$ for a $p \in \{2, 3, 5, 7\}$, and let $\{u_1, \dots, u_s\}$ be a set of representatives of R^* modulo $(R^*)^2$. Then the set $\{(u_k x_{j_1} \cdots x_{j_r})F_0 \mid u_k x_{j_1} \cdots x_{j_r}$ totally positive, $1 \leq k \leq s$, $0 \leq r \leq l$, $1 \leq j_1 \leq \cdots \leq j_r \leq l$, $x_i \neq x_j$ for $i \neq j\}$ contains a set of representatives for the isometry classes of forms in $\mathcal{F}(G)$ which are isometric to one of the F_i.*

Proof. Clearly, only forms corresponding to totally positive elements which lie in $R - pR$ for every prime number p need to be considered. Furthermore, the only primes dividing the determinant of one of the F_i are 2, 3, 5, 7; hence a form $F \in \mathcal{F}(G)$ which is isometric to one of the F_i corresponds to a totally positive element x that generates an ideal which is a product of the ideals x_jR. Thus, $x = ux_{j_1} \cdots x_{j_r}$ with $u \in R^*$. But from the proof of Lemma 1.3 one sees that for $u \in R^*$ and $F \in \mathcal{F}(G)$ one has $u^2F = uFu^{\mathrm{tr}} \sim F$; hence, to get representatives for the isometry classes, u may be replaced by one of the u_k. □

This lemma cannot be directly applied to the case where the centralizer is an indefinite quaternion algebra ramified at 2 and 3. In this case, forms can be found by identifying 2-dimensional subspaces of the space of forms with noncentral real subfields of the algebra (which are $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{5})$, and $\mathbb{Q}(\sqrt{6})$) using the above lemma. By looking at the norm form on the space of forms it is possible to exclude the remaining F_i to be isometric to a form in the space of forms.

The inclusions of 2-dimensional spaces of forms in 3- or 4-dimensional ones are now easily determined by using the information which of the forms of the uniform groups are contained in them.

It is clear that none of the irreducible Bravais groups with 3-dimensional space of forms can contain one of the groups with 4-dimensional space of forms.

2.4. The simplicial complex $\mathrm{Br}_8(\mathbb{Z})$. Figure 1 shows the simplicial complex $\mathrm{Br}_8(\mathbb{Z})$. The vertices for the uniform groups are denoted by the names of their invariant forms, the vertices for the other groups by the names introduced in Table 1.

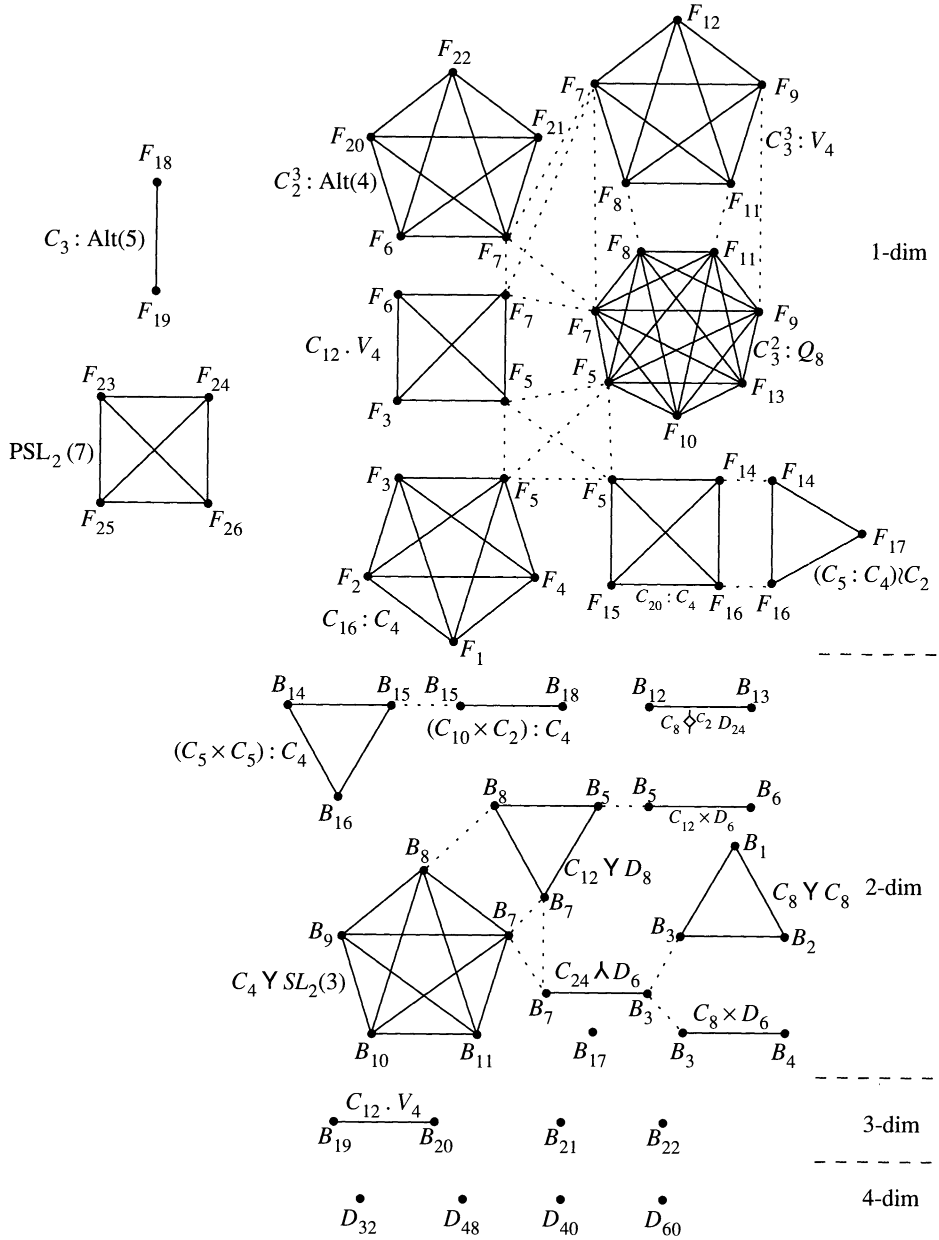

FIGURE 1

3. The irreducible Bravais groups of degree 10

3.1. **The irreducible Bravais groups which are not uniform.** The first step in finding the irreducible Bravais groups of degree 10 is to determine the groups which are not uniform.

Theorem 3.1. *There is up to* $\mathbb{Z}$*-equivalence only one irreducible Bravais group of degree* 10 *which is not uniform. This group is isomorphic to the dihedral group* D_{44}.

Proof. In [10, II.16] the finite irreducible, but not absolutely irreducible subgroups of $GL_{10}(\mathbb{Q})$ are determined. The only groups that have a space of forms of dimension larger than 1 are isomorphic either to one of the cyclic groups C_{11} or C_{22}, or to one of the corresponding dihedral groups D_{22} or D_{44}. The only Bravais-minimal group among these is C_{11}. In the $\mathbb{Q}$-class of this group there is only one $\mathbb{Z}$-class and the Bravais group is isomorphic to D_{44}. □

3.2. **The maximal finite irreducible subgroups of** $GL_{10}(\mathbb{Z})$.

Theorem 3.2. *There are* 46 $\mathbb{Z}$*-classes of maximal finite irreducible subgroups of* $GL_{10}(\mathbb{Z})$. *They fall into* 21 $\mathbb{Q}$*-classes. All of them are absolutely irreducible. (For further information about the groups, see Table* 2 *on next page.)*

Proof. The $\mathbb{Z}$-classes of the maximal finite irreducible subgroups of $GL_{10}(\mathbb{Z})$ will be determined in §§3.3.1–3.3.5. The determination of the $\mathbb{Q}$-classes follows from the following two criteria: Let $G, G' \leq GL_n(\mathbb{Z})$ be two uniform irreducible Bravais groups, with invariant forms F, F' and corresponding lattices L, L'.

(i) If the dual form $F^{\#}$ of F is isometric to F', then G and G' are $\mathbb{Q}$-equivalent (cf. [12]).

(ii) If L is an odd lattice and L' is $\mathbb{Z}$-equivalent to the even sublattice of L, then G is $\mathbb{Q}$-equivalent to a subgroup of G'. Hence, G and G' are $\mathbb{Q}$-equivalent if they have the same order. The fact that all the groups are absolutely irreducible is easily checked, e.g., by calculating the centralizer in $GL_{10}(\mathbb{Q})$. □

From the analysis of the result one gets some further interesting information, for example the following corollary.

Corollary 3.3. *Each of the maximal finite irreducible subgroups of* $GL_{10}(\mathbb{Z})$ *is contained in a maximal finite irreducible subgroup of* $GL_{10}(\mathbb{Q})$, *which is unique up to* $GL_{10}(\mathbb{Q})$*-conjugation.*

The statement of this corollary is true for all dimensions less than or equal to 10 (cf. [12, 13]), but it is not known whether it is true in general.

Table 2 summarizes the result about the uniform irreducible Bravais groups of degree 10 which are exactly the maximal finite irreducible subgroups of $GL_{10}(\mathbb{Z})$. The first column gives the notation of the invariant form (as used in the subsequent sections) and possibly a characterization of it by root systems, resp. by its dual form (denoted by $F^{\#}$); the second gives the maximal finite irreducible subgroups of $GL_{10}(\mathbb{Q})$ (cf. [10]) in which the Bravais group is contained, and which turns out to be unique. In the third column one finds the isomorphism type of the Bravais group. Column four gives the elementary

TABLE 2

form	cont. in	Aut(F)	elt. div.	min.	# min. vect.
$F_1 \cong B_{10} = I_{10}$	Aut(B_{10})	$C_2 \wr S_{10}$	1^{10}	1	$2 \cdot 10$
F_2			$1^8, 2^2$	2	$2 \cdot 90$
$F_3 \cong F_2^{\#}$			$1^2, 2^8$	2	$2 \cdot 10$
F_4	Aut(B_{10})	$C_2^9 : S_{10}$	$1, 2^8, 4$	4	$2 \cdot 90$
F_5	Aut(B_{10})	$C_2^{10} : S_6$	$1^4, 2^2, 4^4$	4	$2 \cdot 130$
F_6	Aut(B_{10})	$C_2^{10} : (S_5 \wr C_2)$	$1^8, 4^2$	2	$2 \cdot 40$
$F_7 \cong F_6^{\#}$			$1^2, 4^8$	4	$2 \cdot 10$
F_8	Aut(B_{10})	$C_2^{10} : S_5$	$1^6, 4^4$	3	$2 \cdot 40$
$F_9 \cong F_8^{\#}$			$1^4, 4^6$	4	$2 \cdot 50$
F_{10}	Aut(B_{10})	$C_2^6 : S_5$	$1^4, 4^4, 8^2$	4	$2 \cdot 20$
$F_{11} \cong F_{10}^{\#}$			$1^2, 2^4, 8^4$	5	$2 \cdot 32$
F_{12}	Aut(B_{10})	$C_2^5 : S_6$	$1^4, 2, 4^4, 8$	4	$2 \cdot 60$
$F_{13} \cong F_{12}^{\#}$			$1, 2^4, 4, 8^4$	5	$2 \cdot 16$
F_{14}	Aut(A_5^2)	$(C_2 \times S_6) \wr C_2$	$1^8, 3^2$	2	$2 \cdot 30$
$F_{15} \cong F_{14}^{\#}$			$1^2, 3^8$	4	$2 \cdot 30$
$F_{16} \cong A_5^2$			$1^8, 6^2$	2	$2 \cdot 30$
$F_{17} \cong F_{16}^{\#}$			$1^2, 6^8$	5	$2 \cdot 12$
F_{18}			$1^2, 2^6, 6^2$	3	$2 \cdot 20$
$F_{19} \cong F_{18}^{\#}$			$1^2, 3^6, 6^2$	4	$2 \cdot 30$
F_{20}	Aut(A_5^2)	$C_2 \times (S_6 \wr C_2)$	$1, 3^8, 9$	5	$2 \cdot 36$
F_{21}			$1, 3^7, 6, 18$	6	$2 \cdot 30$
$F_{22} \cong F_{21}^{\#}$			$1, 3, 6^7, 18$	9	$2 \cdot 20$
F_{23}	Aut(A_5^2)	$(V_4 \times \mathrm{Alt}(5)) : C_2$	$1^4, 2^2, 4^2, 12^2$	4	$2 \cdot 30$
$F_{24} \cong F_{23}^{\#}$			$1^2, 3^2, 6^2, 12^4$	8	$2 \cdot 30$
$F_{25} \cong A_2^5$	Aut(A_2^5)	$(C_2 \times S_3) \wr S_5$	$1^5, 3^5$	2	$2 \cdot 15$
F_{26}	Aut(A_2^5)	$C_2 \times (S_3 \wr S_5)$	$1^4, 3^5, 9$	4	$2 \cdot 90$
$F_{27} \cong F_{26}^{\#}$			$1, 3^5, 9^4$	6	$2 \cdot 15$
F_{28}	Aut(A_2^5)	$C_2 \times (C_3^4 : C_2) : S_5$	$1, 3^4, 9^4, 27$	10	$2 \cdot 81$
F_{29}	Aut(A_2^5)	$S_3 \times (C_2 \wr S_5)$	$1^5, 3^3, 12^2$	4	$2 \cdot 60$
$F_{30} \cong F_{29}^{\#}$			$1^2, 4^3, 12^5$	8	$2 \cdot 15$
F_{31}	$G_3(10)$	$(C_6 \times \mathrm{SU}_4(2)) : C_2$	$1^5, 3^3, 6^2$	4	$2 \cdot 135$
$F_{32} \cong F_{31}^{\#}$			$1^2, 2^3, 6^5$	6	$2 \cdot 120$
F_{33}	$G_3(10)$	$C_2 \times \mathrm{SU}_4(2) : C_2$	$1^5, 3^5$	3	$2 \cdot 40$
$F_{34} \cong A_5 \otimes A_2$	$G_3(10)$	$D_{12} \times S_6$	$1^4, 3^4, 6, 18$	4	$2 \cdot 45$
$F_{35} \cong F_{34}^{\#}$			$1, 3, 6^4, 18^4$	10	$2 \cdot 18$
F_{36}			$1^2, 2^2, 6^5, 18$	6	$2 \cdot 30$
$F_{37} \cong F_{36}^{\#}$			$1, 3^5, 9^2, 18^2$	8	$2 \cdot 45$
F_{38}	$G_4(10)$	$C_2 \times S_6$	$1^6, 6^4$	3	$2 \cdot 20$
$F_{39} \cong F_{38}^{\#}$			$1^4, 6^6$	4	$2 \cdot 15$
F_{40}			$1^4, 2^2, 6^4$	4	$2 \cdot 45$
$F_{41} \cong F_{40}^{\#}$			$1^4, 3^2, 6^4$	4	$2 \cdot 15$
$F_{42} \cong A_{10}$	Aut(A_{10})	$C_2 \times S_{11}$	$1^9, 11$	2	$2 \cdot 55$
$F_{43} \cong F_{42}^{\#}$			$1, 11^9$	10	$2 \cdot 11$
F_{44}	$G_1(10)$	$C_2 \times \mathrm{PGL}_2(11)$	$1^7, 11^3$	4	$2 \cdot 110$
$F_{45} \cong F_{44}^{\#}$			$1^3, 11^7$	10	$2 \cdot 66$
F_{46}	$G_2(10)$	$C_2 \times \mathrm{PGL}_2(11)$	$1^5, 11^5$	6	$2 \cdot 55$

divisors of the form, column five contains the length of the minimal vectors, and the last column their number. The $\mathbb{Q}$-classes of the maximal groups are separated by horizontal lines. Representatives for the maximal finite subgroups of $\mathrm{GL}_{10}(\mathbb{Q})$ are the automorphism groups of the forms F_1, F_{14}, F_{25}, F_{31}, F_{38}, F_{42}, F_{44}, and F_{46}.

3.3. **Constructing the uniform Bravais-minimal groups of degree 10.** The method of finding the uniform Bravais-minimal subgroups of $\mathrm{GL}_{10}(\mathbb{Z})$ is structured by means of the simplicial complex $M_{10}^F(\mathbb{Q})$ determined in [10]. This has four maximal simplices consisting of the $\mathbb{Q}$-classes of the following maximal finite irreducible subgroups of $\mathrm{GL}_{10}(\mathbb{Q})$:

(i) $\mathrm{Aut}(A_{10}) \cong C_2 \times S_{11}$, $G_1(10) \cong C_2 \times \mathrm{PGL}_2(11)$, $G_2(10) \cong C_2 \times \mathrm{PGL}_2(11)$;
(ii) $\mathrm{Aut}(A_2^5) \cong (C_2 \times S_3) \wr S_5$, $G_3(10) \cong (C_6 \times \mathrm{SU}_4(2)) : C_2$;
(iii) $G_4(10) \cong C_2 \times S_6$;
(iv) $\mathrm{Aut}(B_{10}) \cong C_2 \wr S_{10}$, $\mathrm{Aut}(A_5^2) \cong (C_2 \times S_6) \wr C_2$.

3.3.1. Subgroups of $\mathrm{Aut}(A_{10})$, $G_1(10)$, $G_2(10)$.

Lemma 3.4. *The groups* $\mathrm{Aut}(A_{10})$, $G_1(10)$, *and* $G_2(10)$ *have up to* $\mathbb{Q}$*-equivalence only one minimal irreducible uniform subgroup. This group is isomorphic to* $C_{11} : C_5$.

Proof. It is clear that neither $C_2 \times S_{11}$ nor $C_2 \times \mathrm{PGL}_2(11)$ contain an irreducible subgroup of order not divisible by 11. Hence, for each subgroup G which is minimal irreducible and uniform one has $11 \mid |G|$. From [10, II.20 (ii)] it follows that G contains a subgroup isomorphic to $C_{11} : C_5$. □

The $\mathbb{Q}$-class of this Bravais-minimal group splits up into five $\mathbb{Z}$-classes with forms F_{42}, F_{43}, F_{44}, F_{45}, F_{46}.

3.3.2. Subgroups of $\mathrm{Aut}(A_2^5)$, $G_3(10)$.

Lemma 3.5. *The groups* $\mathrm{Aut}(A_2^5)$ *and* $G_3(10)$ *have up to* $\mathbb{Q}$*-equivalence four minimal irreducible uniform subgroups, which are isomorphic to* $C_3 \times \mathrm{Alt}(5)$, $C_2^4 : \mathrm{Alt}(5)$, $C_3^4 : C_5$, *and* $(C_3 \times C_2^4) : C_5$.

Proof. (i) One has $\mathrm{Aut}(A_2^5) \cong (C_2 \times S_3)^5 : S_5$; hence for finding minimal irreducible uniform groups, one has to look at extensions of subgroups of $(C_2 \times S_3)^5$ by subgroups of S_5 with order divisible by 5. Clearly, it suffices to look at extensions of subgroups of C_6^5 on which at least C_5 acts. Furthermore, irreducible extensions of groups which contain a C_2 on which C_5 acts trivially are not minimal because one can omit this C_2. One so obtains as Bravais-minimal groups the four groups given in the claim.

(ii) $G_3(10) \cong (C_6 \times \mathrm{SU}_4(2)) : C_2$, where the C_2 acts nontrivially on both factors. In this case the minimal irreducible uniform subgroups can be obtained by descending chains of maximal subgroups on which the irreducible rational character of degree 10 of $G_3(10)$ stays irreducible. Clearly, the central C_2 can be omitted. One obtains three minimal groups which are isomorphic to $C_3 \times (C_2^4 : C_5)$, $C_3 \times \mathrm{Alt}(5)$, and $C_2^4 : \mathrm{Alt}(5)$, but these are $\mathbb{Q}$-equivalent to the ones obtained in part (i). □

The $\mathbb{Q}$-classes of these Bravais-minimal groups split up into $\mathbb{Z}$-classes as follows:

$C_2^4 : \mathrm{Alt}(5)$: four classes, forms F_{25}, F_{31}, F_{32}, F_{33};
$C_3 \times \mathrm{Alt}(5)$: nine classes, forms F_{25}, F_{26}, F_{27}, F_{31}, F_{32}, F_{34}, F_{35}, F_{36}, F_{37};
$C_3^4 : C_5$: four classes, forms F_{25}, F_{26}, F_{27}, F_{28};
$(C_2^4 \times C_3) : C_5$: five classes, forms F_{25}, F_{29}, F_{30}, F_{31}, F_{32}.

3.3.3. Subgroups of $G_4(10)$.

Lemma 3.6. *The group $G_4(10)$ contains up to $\mathbb{Q}$-equivalence only one minimal irreducible uniform subgroup. This group is isomorphic to the alternating group* $\mathrm{Alt}(6)$.

Proof. The alternating group $\mathrm{Alt}(6)$ is the only proper subgroup of S_6 with order divisible by 5 and an irreducible rational representation of degree 10. □

The $\mathbb{Q}$-class of this Bravais-minimal group splits up into four $\mathbb{Z}$-classes with forms F_{38}, F_{39}, F_{40}, F_{41}.

For the purpose of clarity the minimal irreducible subgroups of $\mathrm{Aut}(B_{10})$ and $\mathrm{Aut}(A_5^2)$ will be determined separately. As before, $G \curlywedge^F H$ denotes a subdirect product of G and H amalgamated over the common factor group F.

3.3.4. Subgroups of $\mathrm{Aut}(A_5^2)$.

Lemma 3.7. *There are up to $\mathbb{Q}$-equivalence nine minimal irreducible uniform subgroups of* $\mathrm{Aut}(A_5^2)$. *They are isomorphic to* $\mathrm{Alt}(5) \times C_4$, $(C_2 \times \mathrm{Alt}(5)).C_2$, M_{10}, $\mathrm{PGL}_2(9)$, $V_4 \curlywedge^{C_2} M_{10}$, $V_4 \curlywedge^{C_2} \mathrm{PGL}_2(9)$, $\mathrm{Alt}(5) \wr C_2$, $(S_5 \curlywedge^{C_2} S_5).C_2$, *and* $(C_2 \times S_5 \curlywedge^{V_4} C_2 \times S_5).C_2$.

Proof. Each of the irreducible subgroups of $\mathrm{Aut}(A_5^2)$ has a reducible subgroup of index 2. The image of the projection onto one of the irreducible constituents must be an irreducible subgroup of $\mathrm{GL}_5(\mathbb{Z})$ contained in $\mathrm{Aut}(A_5) \cong C_2 \times S_6$. From [12] one sees that the minimal irreducible subgroups of $\mathrm{Aut}(A_5)$ are isomorphic to $\mathrm{Alt}(5)$; hence the image of the projection must be isomorphic to $\mathrm{Alt}(5)$, S_5, $\mathrm{Alt}(6)$, S_6, or to a direct product of one of these with C_2. One now obtains the irreducible subgroups of $\mathrm{Aut}(A_5^2)$ as extensions of a subdirect product of one of these groups with itself by C_2.

As Bravais-minimal groups one gets: from $\mathrm{Alt}(5)$ the group $\mathrm{Alt}(5) \wr C_2$, from $C_2 \times \mathrm{Alt}(5)$ the groups $\mathrm{Alt}(5) \times C_4$ and $(C_2 \times \mathrm{Alt}(5)).C_2$, from S_5 the group $(S_5 \curlywedge^{C_2} S_5).C_2$, from $C_2 \times S_5$ the group $(C_2 \times S_5 \curlywedge^{V_4} C_2 \times S_5).C_2$, from $\mathrm{Alt}(6)$ the groups M_{10} and $\mathrm{PGL}_2(9)$, from $C_2 \times \mathrm{Alt}(6)$ the groups $V_4 \curlywedge^{C_2} M_{10}$ and $V_4 \curlywedge^{C_2} \mathrm{PGL}_2(9)$, no group from S_6 and $C_2 \times S_6$. □

The $\mathbb{Q}$-classes of these Bravais-minimal groups split up into $\mathbb{Z}$-classes as follows:

$\mathrm{Alt}(5) \times C_4$, $(V_4 \curlywedge^{C_2} M_{10}).C_2$, $(C_2 \times S_5 \curlywedge^{V_4} C_2 \times S_5).C_2$: six classes, forms F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19};

$(C_2 \times \mathrm{Alt}(5)).C_2$: eight classes, forms F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19}, F_{23}, F_{24};

$\mathrm{PGL}_2(9)$, $\mathrm{Alt}(5) \wr C_2$, $(S_5 \curlywedge^{C_2} S_5).C_2$; nine classes, forms F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19}, F_{20}, F_{21}, F_{22};

$V_4 \curlywedge^{C_2} \mathrm{PGL}_2(9)$: nine classes, forms F_1, F_2, F_3, F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19};

M_{10}: twelve classes, forms F_1, F_2, F_3, F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19}, F_{20}, F_{21}, F_{22}.

3.3.5. Subgroups of $\mathrm{Aut}(B_{10})$,

Lemma 3.8. *Let $G \leq \mathrm{Aut}(B_{10})$, denote by D the diagonal matrices and by S the permutation matrices in $\mathrm{Aut}(B_{10})$, let P be the projection of G into S, and $N := D \cap G$. Then G is the full preimage in $D : P$ of a complement of D/N in $(D : P)/N$.*

Proof. Clearly, $G/N \cap D/N = \{1\}$, and since $D \cdot G = D : P$ the claim follows. □

The use of this lemma is that the irreducible subgroups of $\mathrm{Aut}(B_{10})$ can now be obtained by calculating for the transitive subgroups P of S_{10} and the subgroups N of C_2^{10} on which they act the first cohomology group $H^1(P, C_2^{10}/N)$, which stands in bijection to the complements of C_2^{10}/N in $(C_2 \wr P)/N$ and which is much easier to calculate than $H^2(P, N)$.

As $\mathrm{Aut}(B_{10}) \cong C_2 \wr S_{10}$, the determination of the minimal irreducible uniform subgroups of $\mathrm{Aut}(B_{10})$ will be divided into three parts depending on whether the image under the projection into S_{10} is primitive, is a subgroup of $S_5 \wr C_2$, or is a subgroup of $C_2 \wr S_5$.

Lemma 3.9. *There are up to $\mathbb{Q}$-equivalence four minimal irreducible uniform subgroups of $\mathrm{Aut}(B_{10})$ such that their projection into S_{10} is primitive. Two of them are isomorphic to $C_2^4 : \mathrm{Alt}(5)$ one to $C_2^5 : \mathrm{Alt}(5)$, and one to M_{10}.*

Proof. The primitive subgroups of S_{10} are $\mathrm{Alt}(5)$, S_5, $\mathrm{Alt}(6)$, S_6, $\mathrm{PGL}_2(9)$, M_{10}, $P\Gamma L_2(9)$, $\mathrm{Alt}(10)$, and S_{10} (cf. [16]). Using Lemma 3.8, one obtains as Bravais-minimal groups the groups given in the claim. □

The $\mathbb{Q}$-classes of these Bravais-minimal groups split up into $\mathbb{Z}$-classes as follows:

$C_2^4 : \mathrm{Alt}(5)(a)$: seven classes, forms F_1, F_2, F_3, F_4, F_5, F_8, F_9;

$C_2^4 : \mathrm{Alt}(5)(b)$: nine classes, forms F_1, F_2, F_3, F_4, F_5, F_8, F_9, F_{12}, F_{13};

$C_2^5 : \mathrm{Alt}(5)$: eleven classes, forms F_1, F_2, F_3, F_4, F_5, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13};

M_{10}: twelve classes, forms F_1, F_2, F_3, F_{14}, F_{15}, F_{16}, F_{17}, F_{18}, F_{19}, F_{20}, F_{21}, F_{22}.

Lemma 3.10. *There are up to $\mathbb{Q}$-equivalence nine minimal irreducible uniform subgroups of $\mathrm{Aut}(B_{10})$ such that their projection into S_{10} is a subgroup of $S_5 \wr C_2$. One of them is isomorphic to $C_2^5.C_{10}$, one to $C_2^8 : C_{10}$, one to $C_2^5.D_{10}$, one to $C_2^8 : D_{10}$, three to $C_2^4 : (C_5 : C_4)$, and two to $C_2^8 : (C_5 : C_4 \curlywedge^{C_4} C_5 : C_4).C_2$.*

Proof. The subgroups of $S_5 \wr C_2$ are constructed as extensions by C_2 of a subgroup of S_5 amalgamated with itself over some factor. To get an irreducible group of degree 10, the subgroup of S_5 must have order divisible by 5. Thus, one obtains the following possible isomorphism types for the subgroups of $S_5 \wr C_2$: $S_5 \wr C_2$, $(S_5 \curlywedge^{C_2} S_5).C_2$ (two possibilities), $S_5 \times C_2$,

$\mathrm{Alt}(5) \wr C_2$, S_5, $\mathrm{Alt}(5) \times C_2$, $(C_5 : C_4) \wr C_2$, $(C_5 : C_4 \curlywedge^{C_2} C_5 : C_4).C_2$ (two possibilities), $(C_5 : C_4 \curlywedge^{C_4} C_5 : C_4).C_2$ (four possibilities), $(C_5 : C_4) \times C_2$, $D_{10} \wr C_2$, $(D_{10} \curlywedge^{C_2} D_{10}).C_2$ (two possibilities), $C_5 : C_4$, $D_{10} \times C_2$, $C_5 \wr C_2$, D_{10}, C_{10}. From these groups one obtains the Bravais-minimal groups using Lemma 3.8. □

The $\mathbb{Q}$-classes of these Bravais-minimal groups split up into $\mathbb{Z}$-classes as follows:

$C_2^5.C_{10}$, $C_2^5.D_{10}$, $C_2^8 : (C_5 : C_4 \curlywedge^{C_4} C_5 : C_4).C_2(a)$: five classes, forms F_1, F_2, F_3, F_6, F_7;

$C_2^8 : C_{10}$, $C_2^8 : D_{10}$, $C_2^8 : (C_5 : C_4 \curlywedge^{C_4} C_5 : C_4).C_2(b)$: six classes, forms F_1, F_2, F_3, F_4, F_6, F_7;

$C_2^4 : (C_5 : C_4)(a)$: eight classes, forms F_1, F_2, F_3, F_5, F_6, F_7, F_8, F_9;

$C_2^4 : (C_5 : C_4)(b)$: ten classes, forms F_1, F_2, F_3, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11};

$C_2^4 : (C_5 : C_4)(c)$: thirteen classes, forms F_1, F_2, F_3, F_5, F_6, F_7, F_8, F_9, F_{10}, F_{11}, F_{12}, F_{13}.

Lemma 3.11. *There are up to $\mathbb{Q}$-equivalence four minimal irreducible uniform subgroups of* $\mathrm{Aut}(B_{10})$ *such that their projection into S_{10} is a subgroup of $C_2 \wr S_5$ and not a subgroup of $S_5 \wr C_2$. Two of them are isomorphic to $C_2^5.(C_2^4 : C_5)$ and two to $C_2^4.(C_2^4 : C_5)$.*

Proof. The subgroups of $C_2 \wr S_5$ can be found as extensions of a subgroup of C_2^5 by a subgroup of S_5 with order divisible by 5. As transitive permutation groups of degree 10, one obtains the following possible isomorphism types: D_{10}, $C_5 : C_4$, S_5, $C_2^i : G$ for $i \in \{1, 4, 5\}$ and $G \in \{C_5, D_{10}, C_5 : C_4, \mathrm{Alt}(5), S_5\}$. Now the groups isomorphic to $C_2^i : G$ for $i \in \{0, 1\}$ have also a projection into $S_5 \wr C_2$; hence they were already considered in Lemma 3.10. Furthermore, the groups isomorphic to $C_2^5 : G$ need not be considered for if a group of the form $C_2^k.C_2^5 : G$ is rationally irreducible, then so is $C_2^k.C_2^4 : G$. The remaining groups yield the Bravais-minimal groups given in the claim via Lemma 3.8. □

The $\mathbb{Q}$-class of each of the Bravais-minimal groups determined in the above lemma splits up into four $\mathbb{Z}$-classes with forms F_1, F_2, F_3 and F_4.

3.4. Inclusions.

Lemma 3.12. *The automorphism groups of the forms F_{42}, F_{43}, F_{44}, F_{45}, and F_{46} are the only maximal finite irreducible subgroups of* $\mathrm{GL}_{10}(\mathbb{Z})$ *which contain the irreducible Bravais group isomorphic to D_{44}.*

Proof. It is clear that these forms are the only candidates because the order of the automorphism group has to be divisible by 11. On the other hand, the groups $C_2 \times S_{11}$ and $C_2 \times \mathrm{PGL}_2(11)$ both contain a subgroup isomorphic to D_{44}, which clearly is rationally irreducible. □

3.5. **The simplicial complex** $\mathrm{Br}_{10}(\mathbb{Z})$. Figure 2 shows the simplicial complex $\mathrm{Br}_{10}(\mathbb{Z})$. The vertices are denoted by the names of the invariant quadratic forms as introduced in Table 1 of §3.2.

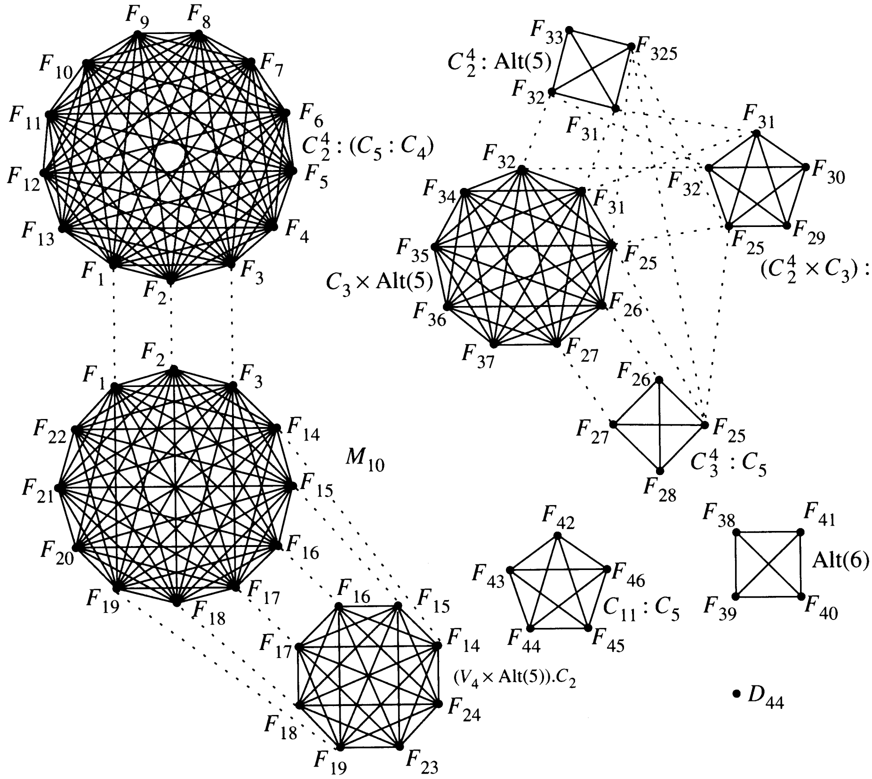

FIGURE 2

APPENDIX

The appendix on the microfiche supplement contains a complete list of representatives for the $\mathbb{Z}$-classes of the irreducible Bravais groups of degree 8 and 10 with the exception of the uniform Bravais groups of degree 8, which were already described in [13]. For each group G the following information is given:

- the isomorphism type of G,
- the order of G,
- the dimension of the space of forms of G (unless it is 1),
- the Gram matrices of an integral basis for the space of forms of G,
- a minimal set of generators of G.

Additionally, for the quadratic forms of the uniform Bravais groups of degree 10 the elementary divisors of the Gram matrix and the number of vectors of shortest length are given.

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