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# ON THE COMPUTATION OF BATTLE-LEMARIÉ'S WAVELETS

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ABSTRACT. We propose a matrix approach to the computation of Battle–Lemarié's wavelets. The Fourier transform of the scaling function is the product of the inverse $F(\mathbf{x})$ of a square root of a positive trigonometric polynomial and the Fourier transform of a B-spline of order $m$. The polynomial is the symbol of a bi-infinite matrix $B$ associated with a B-spline of order $2m$. We approximate this bi-infinite matrix $\mathbf{B}_{2m}$ by its finite section $A_N$, a square matrix of finite order. We use $A_N$ to compute an approximation $\mathbf{x}_N$ of $\mathbf{x}$ whose discrete Fourier transform is $F(\mathbf{x})$. We show that $\mathbf{x}_N$ converges pointwise to $\mathbf{x}$ exponentially fast. This gives a feasible method to compute the scaling function for any given tolerance. Similarly, this method can be used to compute the wavelets.

## 1. INTRODUCTION

Battle-Lemarié's wavelets [1, 3] may be constructed by using a multiresolution approximation built from polynomial splines of order $m > 0$. See, e.g., [4] or [2]. To be precise, let $V_0$ be the vector space of all functions of $L^2(\mathbf{R})$ which are $m-2$ times continuously differentiable and equal to a polynomial of degree $m-1$ on each interval $[n+m/2, n+1+m/2]$ for all $n \in \mathbf{Z}$. Define the other resolution space $V_k$ by

$$V_k := \{u(2^k t) : u \in V_0\}, \quad \forall k \in \mathbf{Z}.$$

It is known that $\{V_k\}_{k\in\mathbf{Z}}$ provide a multiresolution approximation, and there exists a unique scaling function $\varphi$ such that

$$V_k = \mathrm{span}_{L^2}\{2^{k/2}\varphi(2^k t - n) : n \in \mathbf{Z}\}$$

for all $k$, and the integer translates of $\varphi$ are orthonormal to each other. (See, e.g., [4].) Define a transfer function $H(\omega)$ by

$$H(\omega) = \frac{\hat{\varphi}(2\omega)}{\hat{\varphi}(\omega)},$$

where $\hat{\varphi}$ denotes the Fourier transform of $\varphi$. Then the wavelet $\psi$ associated with the scaling function $\varphi$ is given in terms of its Fourier transform by

$$\hat{\psi}(\omega) = e^{-j\omega/2}\overline{H}(\omega/2 + \pi)\hat{\varphi}(\omega/2).$$

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Here and throughout, $j := \sqrt{-1}$. The scaling function $\varphi$ associated with the multiresolution approximation may be given by

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{\sum_{k\in\mathbf{Z}} |\widehat{B}_m(\omega + 2k\pi)|^2}} \widehat{B}_m(\omega), \tag{1}$$

where $B_m$ is the well-known central B-spline of order $m$ whose Fourier transform is given by

$$\widehat{B}_m(\omega) = \left(\frac{\sin \omega/2}{\omega/2}\right)^m.$$

By using Poisson's summation formula, we have

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk\omega}}} \widehat{B}_m(\omega).$$

Thus, the transfer function is

$$H(\omega) = \sqrt{\frac{\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk2\omega}}{\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk\omega}}} (\cos \omega/2)^m. \tag{2}$$

Then the wavelet $\psi$ associated with $\varphi$ is given by

$$\hat{\psi}(\omega) = e^{-j\omega/2}\overline{H(\omega/2 + \pi)} \frac{1}{\sqrt{\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk\omega/2}}} \widehat{B}_m(\omega/2). \tag{3}$$

The above Fourier transforms of $\varphi$, $H$, and $\psi$ suggest that the scaling function, transfer function, and wavelet have the following representations:

$$\varphi(t) = \sum_{k\in\mathbf{Z}} \alpha_k B_m(t-k),$$

$$H(\omega) = \sum_{k\in\mathbf{Z}} \beta_k e^{-jk\omega},$$

$$\psi(t) = \sum_{k\in\mathbf{Z}} \gamma_k B_m(2t-k).$$

In this paper, we propose a matrix method to compute the $\alpha_k$'s, $\beta_k$'s, and $\gamma_k$'s. Let us use $\varphi$ to illustrate our method as follows: view $\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk\omega}$ as the symbol of a bi-infinite matrix $\mathbf{B}_{2m} = (b_{ik})_{i,k\in\mathbf{Z}}$ with $b_{i,k} = b_{0,k-i} = B_{2m}(k-i)$ for all $i, k \in \mathbf{Z}$. Similarly, $\sqrt{\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk\omega}}$ can be viewed as the symbol of another (unknown) bi-infinite matrix $\mathbf{C}_{2m}$. Then it is easy to see that

$$\mathbf{C}_{2m}^2 = \mathbf{B}_{2m}.$$

To find

$$\sum_{k\in\mathbf{Z}} \alpha_k e^{-jk\omega} = \frac{1}{\sqrt{\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk\omega}}}$$

is equivalent to solving

$$C_{2m}\mathbf{x} = \delta$$

with $\delta = (\delta_i)_{i\in\mathbf{Z}}$, $\delta_0 = 1$, and $\delta_i = 0$ for all $i \in \mathbf{Z}\backslash\{0\}$, where $\mathbf{x} = (\alpha_k)_{k\in\mathbf{Z}}$. Our numerical method is to find an approximation to $\mathbf{x}$ within a given tolerance.

Let $A_N = (b_{ik})_{-N\le i,k\le N}$ be a finite section of $\mathbf{B}_{2m}$. Note that $A_N$ is symmetric and totally positive. Thus, we can find $\widehat{P}_N$ such that

$$\widehat{P}_N^2 = A_N$$

by using, e.g., the singular value decomposition. Then we solve $\widehat{P}_N\mathbf{x}_N = \delta_N$ with $\delta_N$ a vector of $2N+1$ components which are all zeros except for the middle one, which is 1. We can show that $\mathbf{x}_N$ converges pointwise to $\mathbf{x}$ exponentially fast. Similarly, we can use this idea to compute an approximation of $\{\beta_k\}_{k\in\mathbf{Z}}$ by (2) and $\{\gamma_k\}_{k\in\mathbf{Z}}$ by (3). Therefore, the discussion mentioned above furnishes a numerical method to compute Battle-Lemarié's wavelet.

To prove the convergence of $\mathbf{x}_N$ to $\mathbf{x}$, we place ourselves in a more general setting. We study a general bi-infinite matrix $A$. (For the case of Battle-Lemarié's wavelets, $A = \mathbf{B}_{2m}$.) We look for certain conditions on $A$ such that the solution $\mathbf{x}_N$ of $\widehat{P}_N\mathbf{x}_N = \delta_N$ with $\widehat{P}_N^2 = A_N$ converges to the solution $\mathbf{x}$ of $P\mathbf{x} = \delta$ with $P^2 = A$, where $A_N$ is a finite section of $A$. This is discussed in the next section. In the last section, we show that the bi-infinite matrix $\mathbf{B}_{2m}$ satisfies the conditions on $A$ obtained in §2. This will establish our numerical method for computing Battle-Lemarié's wavelets.

## 2. Main results

Let $\mathbf{Z}$ be the set of all integers. Let $l^2 := l^2(\mathbf{Z})$ be the space of all square summable sequences with indices in $\mathbf{Z}$. That is,

$$l^2(\mathbf{Z}) = \left\{(\dots, x_{-1}, x_0, x_1, \dots)^t : \sum_{i=-\infty}^{\infty} |x_i|^2 < \infty\right\}.$$

It is known that $l^2$ is a Hilbert space. We shall use $\mathbf{x}$ to denote each vector in $l^2$ and use $A$ to denote a linear operator from $l^2$ to $l^2$. It is known that $A$ can be expressed as a bi-infinite matrix. Thus, we shall write $A = (a_{ik})_{i,k\in\mathbf{Z}}$.

In the following, we shall consider $A$ to be a banded and/or Toeplitz matrix. That is, $A$ is said to be banded if there exists a positive integer $b$ such that $a_{ik} = 0$ whenever $|i-k| > b$. The matrix $A$ is said to be Toeplitz if $a_{i+k,m+k} = a_{i,m}$ for all $i, k, m \in \mathbf{Z}$. Denote by $F(\mathbf{x})(\omega)$ the symbol of a vector $\mathbf{x} \in l^2$, i.e.,

$$F(\mathbf{x})(\omega) = \sum_{i\in\mathbf{Z}} x_i e^{-ji\omega}.$$

Denote by $F(A)(\omega)$ the symbol of a Toeplitz matrix $A = (a_{ik})_{i,k\in\mathbf{Z}}$, i.e.,

$$F(A)(\omega) = \sum_{i\in\mathbf{Z}} a_{i,0} e^{-ji\omega}.$$

Suppose that $F(A)(\omega) \neq 0$ and $\sum_{i\in\mathbf{Z}} |a_{i,0}| < \infty$. It is known from the well-known Wiener's theorem that there exists a sequence $\mathbf{x}$ such that

$$\frac{1}{F(A)(\omega)} = \sum_{k\in\mathbf{Z}} x_k e^{-jk\omega}$$

with $\sum_k |x_k| < \infty$. It is easy to see that to find this sequence $\mathbf{x}$ is equivalent to solving the linear system of bi-infinite order:

$$A\mathbf{x} = \delta,$$

where $\delta = (\ldots, \delta_{-1}, \delta_0, \delta_1, \ldots)^t$ with $\delta_0 = 1$ and $\delta_i = 0$ for all $i \in \mathbf{Z}\backslash\{0\}$.

Furthermore, if the matrix $A$ is a positive operator, then there exists a unique positive square root $P$ of $A$. That is, $P^2 = A$. The symbol representation is $F(P)(\omega) = \sqrt{F(A)(\omega)}$. To find $F(P)(\omega)$ is equivalent to finding a matrix $P$ such that $P^2 = A$.

Certainly, we cannot solve a linear system of bi-infinite order. Neither can we decompose a matrix of bi-infinite order into two matrices of bi-infinite order. However, we can do this approximatively. Let $N$ be a positive integer, and let $A_N = (a_{ik})_{-N\le i,k\le N}$ be a finite section of $A$. Let $I_{N,\infty} = (0, I_{2N+1,2N+1}, 0)$ be a matrix of $2N+1$ rows and bi-infinite columns with $I_{2N+1,2N+1}$ being the identity matrix of size $(2N+1)\times(2N+1)$ such that

$$A_N = I_{N,\infty} A I^t_{N,\infty}.$$

Denote $\delta_N = I_{N,\infty}\delta$ and $\mathbf{x}_N = I_{N,\infty}\mathbf{x}$. Then we shall solve the following linear system:

$$A_N \hat{\mathbf{x}}_N = \delta_N.$$

We claim that $\hat{\mathbf{x}}_N$ converges to $\mathbf{x}$ exponentially fast as $N$ increases to $\infty$, under certain conditions on $A$. Furthermore, we shall solve $\widehat{P}^2_N = A_N$ for $\widehat{P}_N$ by using the singular value decomposition. Once we have $\widehat{P}_N$, we shall solve

$$\widehat{P}_N \hat{\mathbf{y}}_N = \delta_N.$$

We claim that $\hat{\mathbf{y}}_N$ converges to $\mathbf{y}$ exponentially fast as $N \to \infty$, provided $A$ satisfies certain conditions.

To check the conditions on $A$, we need the following definition.

**Definition.** A matrix $A = (a_{ik})_{i,k\in\mathbf{Z}}$ is said to be of exponential decay off its diagonal if

$$|a_{ik}| \le K r^{|i-k|}$$

for some constant $K$ and $r \in (0, 1)$.

We begin with an elementary lemma.

**Lemma 1.** *Suppose that $A$ is of exponential decay off its diagonal and has a bounded inverse. Suppose that $A_N^{-1} = (\hat{a}_{ik})_{-N\le i,k\le N}$ satisfies the property that*

$$|\hat{a}_{i,k}(N)| \le K r^{|i-k|}, \quad \forall -N \le i, k \le N,$$

*for all $N > 0$. Then there exists $r_1 \in (0, 1)$ and a constant $K_1$ such that*

$$\|I_{N,\infty}\mathbf{x} - \hat{\mathbf{x}}_N\|_2 \le K_1 r_1^N,$$

*where $\mathbf{x}$ is the solution of $A\mathbf{x} = \delta$ and $\mathbf{x}_N$ is the solution of $A_N\mathbf{x}_N = \delta_N$.*

*Proof.* From the assumption of the lemma, there exist $K$ and $r \in (0, 1)$ such that $A = (a_{ik})_{i,k\in\mathbf{Z}}$ and $A_N^{-1} = (\hat{a}_{i,k}(N))_{-N\le i,k\le N}$ satisfy

$$|a_{ik}| \le K r^{|i-k|} \quad \text{and} \quad |\hat{a}_{ik}| \le K r^{|i-k|}.$$

Write

$$A I^t_{N,\infty} = \begin{bmatrix} B \\ A_N \\ C \end{bmatrix} \quad \text{and} \quad d = B A_N^{-1} \delta_N \quad \text{with } d = (\ldots, d_{-N-1}, d_{-N})^t.$$

Then we have, for each $i = -\infty, \dots, -N-1, -N$,

$$|d_i| = \left| \sum_{k=-N}^{N} a_{ik}\hat{a}_{k,0}(N) \right| \le K^2 \sum_{k=-N}^{N} r^{|i-k|} r^{|k|}$$
$$= K^2 \left( r^{-i} \sum_{k=0}^{N} r^{2k} + N r^{-i} \right) \le C\lambda^{-i}$$

for some constant $C$ and $\lambda \in (0, 1)$. Thus, $\|BA^{-1}\delta_N\|_2 \le C'\lambda^N$. Similarly, $\|CA_N^{-1}\delta_N\|_2 \le C'\lambda^N$. Hence,

$$\begin{aligned} \|I_{N,\infty}\mathbf{x} - \hat{\mathbf{x}}_N\|_2 &\le \|\mathbf{x} - I_{N,\infty}^t \hat{\mathbf{x}}_N\|_2 \le \|A^{-1}\|_2 \|\delta - A I_{N,\infty}^t A_N^{-1}\delta_N\|_2 \\ &\le \|A^{-1}\|_2 \left\| \delta - \begin{bmatrix} B \\ A_N \\ C \end{bmatrix} A_N^{-1}\delta_N \right\|_2 \\ &\le \|A^{-1}\|_2 \left\| \delta - \begin{bmatrix} BA_N^{-1} \\ I_{2N+1,2N+1} \\ CA_N^{-1} \end{bmatrix} \delta_N \right\|_2 \\ &\le \|A^{-1}\|_2 (\|BA_N^{-1}\delta_N\|_2 + \|CA_N^{-1}\delta_N\|_2) \le \|A^{-1}\|_2 2C'\lambda^N, \end{aligned}$$

hence the assertion with $K_1 = 2C'\|A^{-1}\|_2$ and $r_1 = \lambda$. This establishes the lemma. □

Next, we consider approximating the square root of a positive operator.

**Lemma 2.** *Let $P$ be the unique square root of a positive operator $A$. Suppose that $A$ is banded and $\|A - I\|_2 \le r < 1$, where $I$ is the identity operator from $l^2$ to $l^2$. Then $P = (p_{ik})_{i,k\in\mathbf{Z}}$ is of exponential decay off its diagonal.*

*Proof.* The uniqueness of $P$ and the convergence of the series

$$\sum_{i=0}^{\infty} (-1)^i \frac{(2i-3)!!}{(2i)!!} (A - I)^i$$

imply that

$$P = \sqrt{A} = \sqrt{I + (A - I)} = \sum_{i=0}^{\infty} (-1)^i \frac{(2i-3)!!}{(2i)!!} (A - I)^i.$$

The matrix $A$ is banded and so is $A - I$. If $A - I$ has bandwidth $b$, then $(A - I)^i$ is also banded with bandwidth $ib$. Thus, $|p_{ik}| \le K r^{|i-k|/b}$ for some constant $K$. This finishes the proof. □

**Lemma 3.** *Let $P$ be the unique square root of a positive operator $A$. Suppose that $A$ is banded and $\|A - I\|_2 \le r < 1$, where $I$ is the identity operator from $l^2$ to $l^2$. Then $P^{-1} = (\hat{p}_{ik})_{i,k\in\mathbf{Z}}$ is of exponential decay off its diagonal.*

*Proof.* The uniqueness of $P^{-1}$ and the convergence of the series

$$\sum_{i=0}^{\infty} (-1)^i \frac{(2i-1)!!}{(2i)!!} (A - I)^i$$

imply that

$$P^{-1} = (A)^{-1/2} = (I + (A - I))^{-1/2} = \sum_{i=0}^{\infty}(-1)^i \frac{(2i-1)!!}{(2i)!!}(A - I)^i .$$

Now we use the same argument as in the lemma above to conclude that $P^{-1}$ is of exponential decay off its diagonal. □

Let $\widehat{P}_N$ be the square root of $A_N$. That is, $\widehat{P}_N^2 = A_N$. Denote $P_N = I_{N,\infty} P I_{N,\infty}^t$. We need to estimate $P_N \widehat{P}_N - \widehat{P}_N P_N$. We have

**Lemma 4.** *Let* $R = (r_{ik})_{-N \le i, k \le N} := P_N \widehat{P}_N - \widehat{P}_N P_N$. *Then* $r_{ik} = O(r^{N/(4b)})$ *for* $k = -N/4 + 1, \ldots, N/4 - 1$ *and* $i = -N, \ldots, N$, *where* $b$ *is the bandwidth of* $A$ *and* $r$ *is as defined in Lemma* 3.

*Proof*. It is known that $P$ and $A$ commute. Let us write

$$P = \begin{bmatrix} \alpha_1 & B & \alpha_2 \\ B^t & P_N & C^t \\ \alpha_3 & C & \alpha_4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} \beta_1 & a & \beta_2 \\ a^t & A_N & c^t \\ \beta_3 & c & \beta_4 \end{bmatrix} .$$

We have $B^t a + P_N A_N + C^t c = a^t B + A_N P_N + c^t C$. Thus, $P_N A_N - A_N P_N = a^t B - B^t a + c^t C - C^t c$. Let $E = a^t B - B^t a + c^t C - C^t c$ and $I_N := I_{2N+1, 2N+1}$. We have $P_N(A_N - I_N) = (A_N - I_N)P_N + E$ and

$$P_N(A_N - I_N)^n = (A_N - I_N)^n P_N + \sum_{k=0}^{n-1}(A_N - I_N)^k E (A_N - I_N)^{n-k-1}$$

by using induction. Then, we have

$$\begin{aligned} P_N \widehat{P}_N &= \sum_{n=0}^{\infty}(-1)^n \frac{(2n-3)!!}{(2n)!!} P_N (A_N - I_N)^n \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{(2n-3)!!}{(2n)!!}(A_N - I_N)^n P_N \\ &\quad + \sum_{n=0}^{\infty}(-1)^n \frac{(2n-3)!!}{(2n)!!} \sum_{k=0}^{n-1}(A_N - I_N)^k E (A_N - I_N)^{n-k-1} \\ &= \widehat{P}_N P_N + \sum_{n=0}^{\infty}(-1)^n \frac{(2n-3)!!}{(2n)!!} \sum_{k=0}^{n-1}(A_N - I_N)^k E (A_N - I_N)^{n-k-1} . \end{aligned}$$

To estimate $R = P_N \widehat{P}_N - \widehat{P}_N P_N$ which is the summation above, we break $R$ into two parts and estimate the first by

$$\begin{aligned} &\left\| \sum_{n=N1+1}^{\infty} (-1)^n \frac{(2n-3)!!}{(2n)!!} \sum_{k=0}^{n-1}(A_N - I_N)^k E (A_N - I_N)^{n-k-1} \right\|_2 \\ &\quad \le \sum_{n=N1+1}^{\infty} \frac{(2n-3)!!\, n}{(2n)!!} \|E\|_2 \|A_N - I_N\|_2^n \le K_1 \|A_N - I_N\|_2^{N_1} . \end{aligned}$$

Thus, this part has the desired property if we choose $N1$ appropriately. Next, we note that $A_N - I_N$ is banded and its bandwidth is $b$. Thus, for $0 \le n \le N1$, $(A_N - I_N)^n$ is also banded and has bandwidth $nb \le bN1$.

Note also $E = (e_{ik})_{-N\le i,k\le N}$ has the following property:

$$e_{ik} = \begin{cases} 0 & \text{for } -N+b<k<N-b,\ -N+b<i<N-b, \\ O(r^{N-|k|}) & \text{for } -N\le i\le -N+b \text{ and } N-b\le i\le N,\ -N\le k\le N. \end{cases}$$

It follows that $(A_N - I_N)^l E$ has a similar property as $E$:

$$((A_N-I_N)^l E)_{ik} = \begin{cases} 0 & \text{for } -N+b<k<N-b, \\ & -N+kb+b<i<N-lb-b, \\ O(r^{N-|k|}) & \text{for } -N\le i\le -N+lb+b, \\ & N-lb-b\le i\le N, \text{ and } -N\le k\le N. \end{cases}$$

Choose $N1$ such that $N/(4b) \le N1 < N/(4b)+1$. Then $(A_N-I)^{N1}$ has bandwidth $bN1 < N/4+b$ and hence

$$((A_N-I)^l E(A_N-I)^{n-l-1})_{ik} = \begin{cases} O(r^{3N/4-b-|k|}) & \text{if } |k|\le N/4 \text{ and } -N\le i\le N, \\ O(1) & \text{otherwise} \end{cases}$$

for $l = 1, \ldots, N1$. Putting these two parts together, we have established that $R$ has the desired property. □

We are now ready to prove the following.

**Theorem 1.** *Suppose that $A$ is a positive operator and $\|A-I\|_2 < 1$. Suppose that $A$ is a banded matrix. Let $P$ be the unique square root of $A$ and $\mathbf{y}$ the solution of $P\mathbf{y} = \delta$. Let $\widehat{P}_N$ be a square root matrix such that $\widehat{P}_N^2 = A_N$ and $\hat{\mathbf{y}}_N$ the solution of $\widehat{P}_N\hat{\mathbf{y}}_N = \delta_N$. Then*

$$\|I_{N,\infty}\mathbf{y} - \hat{\mathbf{y}}_N\|_2 \le K\lambda^N$$

*for some $\lambda \in (0, 1)$ and a constant $K > 0$.*

*Proof.* Let $P = (p_{ik})_{i,k\in\mathbf{Z}}$ and $P_N = (p_{ik})_{-N\le i,k\le N}$. By Lemma 2, the matrix $P$ is of exponential decay off its diagonal. By Lemma 3, we know that $P_N$ is of exponential decay off its diagonal uniformly with respect to $N$ because of $\|A_N - I_{2N+1,2N+1}\|_2 < 1$, which follows from $\|A-I\|_2 < 1$. The invertibility of $A$ implies that $P$ is invertible. From $\|A-I\|_2<1$ it follows that the inverse of $P$ is bounded. Let $\tilde{\mathbf{y}}_N$ be the solution of $P_N\tilde{\mathbf{y}}_N = \delta_N$. Thus, we apply Lemma 1 to conclude that

$$\|I_{N,\infty}\mathbf{y} - \tilde{\mathbf{y}}_N\|_2 \le K_1 r^N$$

for some $r \in (0, 1)$.

We now proceed to estimate $\|\tilde{\mathbf{y}}_N - \hat{\mathbf{y}}_N\|_2$.

Note that $P^2 = A$ implies $A_N = P_N^2 + B^tB + C^tC$ or $\widehat{P}_N^2 - P_N^2 = B^tB + C^tC$. Thus, we have

$$(P_N + \widehat{P}_N)(\widehat{P}_N - P_N) = \widehat{P}_N^2 - P_N^2 + P_N\widehat{P}_N - \widehat{P}_N P_N = B^tB + C^tC + R,$$

where $R$ was defined in Lemma 4. Hence,

$$(\widehat{P}_N - P_N) = (P_N + \widehat{P}_N)^{-1}(B^tB + C^tC + R).$$

Note that the entries of $B^tB + C^tC$ have the exponential decay property: $(B^tB + C^tC)_{ik} = O(r^{N-|k|})$. By Lemma 4, we know that each entry of the middle section ($N/2$ columns) of the columns of $B^tB + C^tC + R$ has exponential decay $O(r^{N/(4b)})$. Both $P_N$ and $\widehat{P}_N$ are positive and $\|(P_N+\widehat{P}_N)^{-1}\|_2 \le \|\widehat{P}_N^{-1}\|_2$ is bounded. Recall that $P_N^{-1}$ is of exponential decay off its diagonal. We have

$$\begin{aligned}\|\tilde{\mathbf{y}}_N - \hat{\mathbf{y}}_N\|_2 &\le \|\widehat{P}_N^{-1}\|_2 \|\delta_N - \widehat{P}_N P_N^{-1}\delta_N\|_2 \\ &\le \|\widehat{P}_N^{-1}\|_2 \|(P_N - \widehat{P}_N)(P_N^{-1}\delta_N)\|_2 \\ &\le \|\widehat{P}_N^{-1}\|_2 \|(P_N + \widehat{P}_N)^{-1}\|_2 \|(B^tB + C^tC + R)P_N^{-1}\delta_N\|_2 \\ &\le K\lambda^N\end{aligned}$$

for some $\lambda \in (r, 1)$. This completes the proof. □

In the proof above, an essential step is to show that each entry of the middle section of the columns of $\widehat{P}_N - P_N$ is of exponential decay. This indeed follows from $(\widehat{P}_N - P_N) = (P_N + \widehat{P}_N)^{-1}(B^tB + C^tC + R)$, the boundedness of $(P_N + \widehat{P}_N)^{-1}$, and the fact that each entry of the middle section of the columns of $B^tB + C^tC + R$ is of exponential decay. This has its own interest. Thus, we have the following

**Theorem 2.** *Suppose that $A$ is a positive operator and $\|A - I\|_2 < 1$. Suppose that $A$ is a banded matrix. Let $P$ be the unique square root of $A$ and $P_N = I_{N,\infty}P(I_{N,\infty})^t$. Let $\widehat{P}_N$ be a square root matrix such that $\widehat{P}_N^2 = A_N$. Then*

$$\|P_N\delta_N - \widehat{P}_N\delta_N\|_2 \le K\lambda^N$$

*for some $\lambda \in (0, 1)$ and a constant $K$.*

Finally, we remark that if $\|A-I\|_2 = 1$, then each entry of the middle section of the columns of $R$ is convergent to 0 with speed $\frac{1}{N}$. The exponential decay in the above has to be replaced by

$$\|P_N\delta_N - \widehat{P}_N\delta_N\|_2 \le \frac{K}{N}.$$

## 3. Computation of Battle-Lemarié's wavelets

Fix a positive integer $m$. Let $A = \mathbf{B}_{2m}$ be the bi-infinite matrix whose symbol is $\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk\omega}$. Clearly, $A$ is a banded Toeplitz matrix. To see that $A$ is a positive operator on $l^2$, we show that $A \ge cI$ for some $c > 0$ as follows: For any $\mathbf{x} \in l^2$, we have

$$\begin{aligned}\mathbf{x}^tA\mathbf{x} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{F(\mathbf{x})(\omega)}F(A)(\omega)F(\mathbf{x})(\omega)\,d\omega \\ &= F(A)(\xi)\frac{1}{2\pi}\int_{-\pi}^{\pi} |F(\mathbf{x})(\omega)|^2\,d\omega \\ &\ge \min_{\omega} F(A)(\omega)\|\mathbf{x}\|_2^2.\end{aligned}$$

With $c = \min_\omega F(A)(\omega) > 0$, we have $A \ge cI$. Similarly, we can show that

$\|A-I\|_2 < 1$. Indeed,

$$\begin{aligned}\|(A-I)\mathbf{x}\|_2^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |F(A-I)(\omega)|^2 |F(\mathbf{x})(\omega)|^2\, d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |1-F(A)(\omega)|^2 |F(\mathbf{x})(\omega)|^2\, d\omega \\ &\le \max_{\omega} |1-F(A)(\omega)|^2 \|\mathbf{x}\|_2^2 \le \left(1-\min_{\omega} F(A)(\omega)\right)^2 \|\mathbf{x}\|_2^2 .\end{aligned}$$

Thus, we have

$$\|(A-I)\mathbf{x}\|_2 \le \left(1-\min_{\omega} F(A)(\omega)\right)\|\mathbf{x}\|_2$$

and hence, $\|A-I\|_2 < 1$. Thus, $\mathbf{B}_{2m}$ satisfies all the conditions of Theorem 1.

By (1), we have

$$\hat{\varphi}(\omega) = \frac{1}{\sqrt{\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk\omega}}}\left(\frac{\sin\omega/2}{\omega/2}\right)^m .$$

Thus, $\varphi(t) = \sum_k \alpha_k B_m(t-k)$ with $\mathbf{x} = (\alpha_k)_{k\in\mathbf{Z}}$ satisfying

$$\mathbf{C}_{2m}\mathbf{x} = \delta \quad \text{and} \quad \mathbf{C}_{2m}^2 = \mathbf{B}_{2m} .$$

Using our Theorem 1, we conclude that our numerical method is valid to compute the $\alpha_k$'s.

By (2), the transfer function is

$$H(\omega) = \frac{\sqrt{\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-j2k\omega}}}{\sqrt{\sum_{k\in\mathbf{Z}} B_{2m}(k)e^{-jk\omega}}}\cos^m(\omega/2).$$

Note that when $m$ is even, then $\cos^m(\omega/2) = (1-(e^{j\omega}+e^{-j\omega})/2)^{m/2}$, which is a finite series. However, when $m$ is odd, $\cos^m(\omega/2)$ is no longer a finite series. In order to compute $H(\omega)$, let $\mathbf{S}_m$ be the Toeplitz matrix whose symbol is $\cos^{2m}(\omega/2) = (1-(e^{j\omega}+e^{-j\omega})/2)^m$. Let $Z$ be a zero insertion operator on $l^2$ defined by

$$Z\mathbf{x} = Z(x_i)_{i\in\mathbf{Z}} = (z_i)_{i\in\mathbf{Z}} \quad \text{with } z_i = \begin{cases} x_{i/2} & \text{if } i \text{ is even}, \\ 0 & \text{if } i \text{ is odd}. \end{cases}$$

Thus, $H(\omega) = \sum_{k\in\mathbf{Z}} \beta_k e^{-jk\omega}$ with $\mathbf{x} = (\beta_k)_{k\in\mathbf{Z}}$ satisfying

$$\mathbf{x} = \mathbf{w} * \mathbf{y} * \mathbf{z},$$

where $*$ denotes the convolution operator of two vectors in $l^2$ and

$$\mathbf{y} = \mathbf{C}_m\delta, \quad \mathbf{z} = Z\mathbf{C}_m^{-1}\delta, \quad \mathbf{w} = \mathbf{T}\delta$$

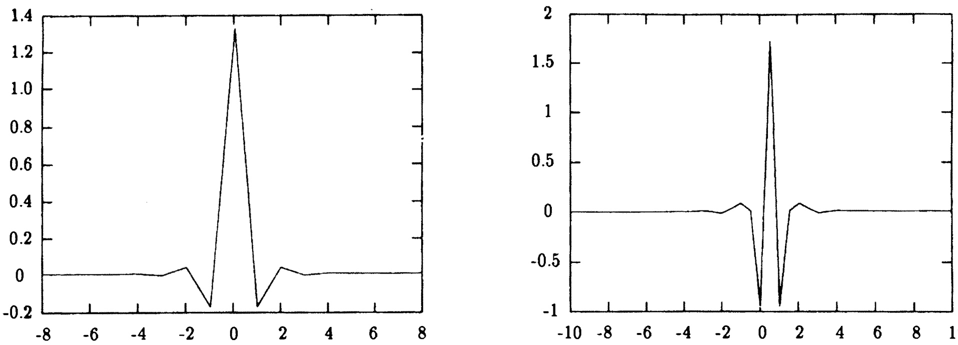
with $\mathbf{C}_m^2 = \mathbf{B}_{2m}$, $\mathbf{T}_m^2 = \mathbf{S}_m$. Using our Theorems 1 and 2, we know that our numerical method gives a good approximation to $\mathbf{y}$ and $\mathbf{z}$. For $m$ even, our numerical method produces an $\mathbf{x}_N$ which converges pointwise to $\mathbf{x}$ exponentially. When $m$ is odd, the remark after Theorem 2 has to be applied, and the $\mathbf{w}_N$ produced by this procedure does no longer converge to $\mathbf{w}$ exponentially.

By (3), the wavelet $\psi$ associated with $\varphi$ is given by

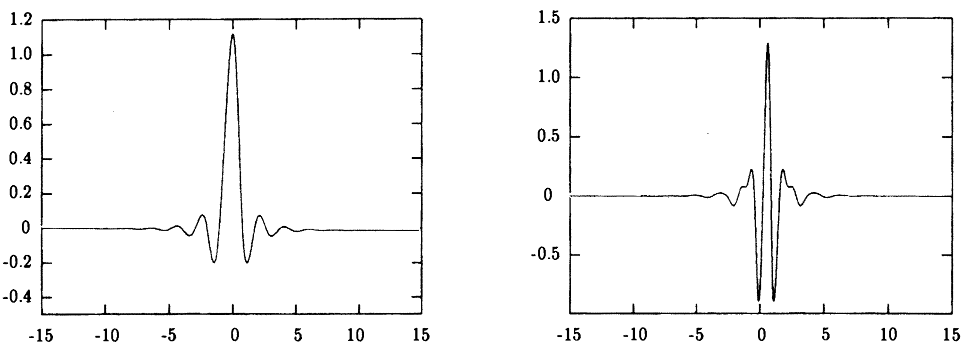
$$\hat{\psi}(2\omega) = e^{-j\omega}\overline{H(\omega+\pi)}\hat{\varphi}(\omega).$$

Once $\{\alpha_k\}_{k\in\mathbf{Z}}$ and $\{\beta_k\}_{k\in\mathbf{Z}}$ are computed, $\{\gamma_k\}_{k\in\mathbf{Z}}$ can be obtained by convolution.

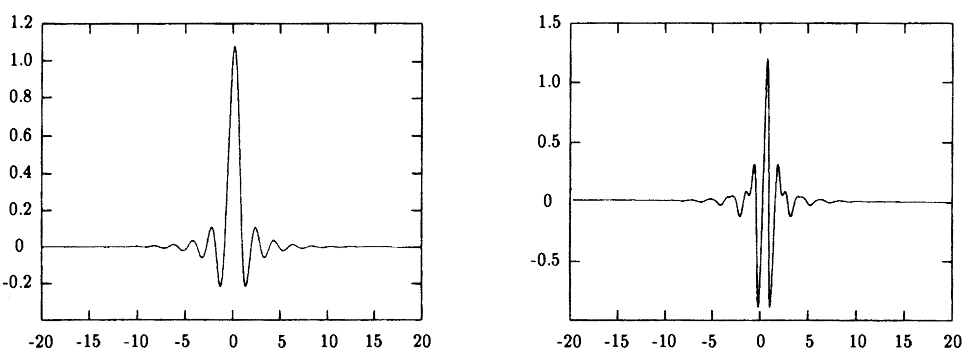
We have implemented this method to compute Battle-Lemarié's wavelets in MATLAB. The graphs of Battle-Lemarié's wavelets are shown in the following figures.



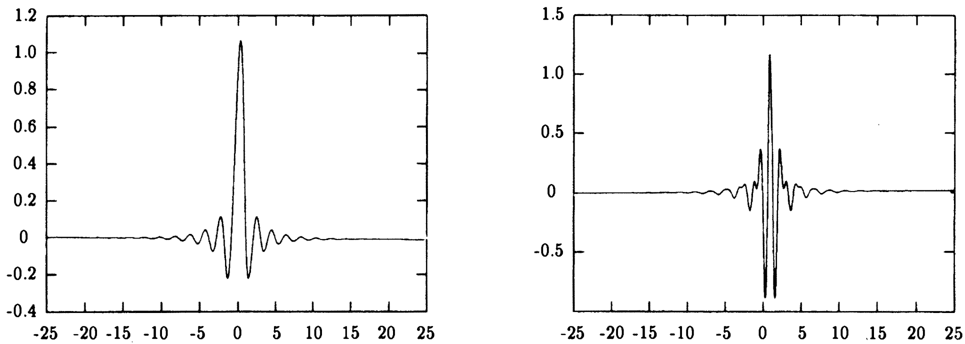

Battle-Lemarié's scaling function and wavelet of degree 1



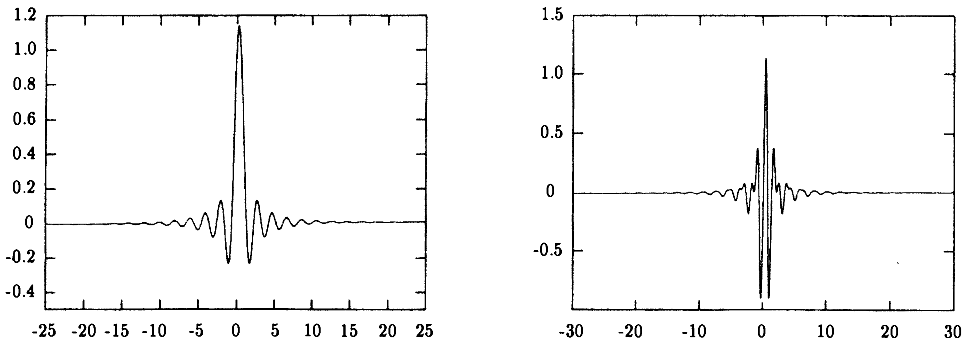

Battle-Lemarié's scaling function and wavelet of degree 3




Battle-Lemarié's scaling function and wavelet of degree 5



Battle-Lemarié's scaling function and wavelet of degree 7



Battle-Lemarié's scaling function and wavelet of degree 9

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