MATHEMATICS OF COMPUTATION
VOLUME 63, NUMBER 208
OCTOBER 1994, PAGES S7–S10

Supplement to

ON THE ACCURATE LONG-TIME SOLUTION OF THE WAVE EQUATION IN EXTERIOR DOMAINS: ASYMPTOTIC EXPANSIONS AND CORRECTED BOUNDARY CONDITIONS

THOMAS HAGSTROM, S. I. HARIHARAN, AND R. C. MACCAMY

APPENDIX A. CONSTRUCTION AND ANALYSIS OF H

We seek a function, $\mathcal{H}(\tau)$, whose Laplace transform, $\hat{\mathcal{H}}(s) = (s\ln s)^{-1} + O(1)$ as $s \to 0$, has no other singularities for $\Re s \geq 0$ and decays as fast as $(s\ln s)^{-1}$ as $s \to \infty$. We make use of the following formula, which may be found in most tables of Laplace transforms or verified by direct computation

$$\mathcal{L}\left(\int_0^\infty \frac{\tau^{u-1+\alpha}}{\Gamma(u+\alpha)}du\right) = \frac{1}{s^\alpha \ln s}, \quad \alpha \geq 0. \tag{1.0.153}$$

Let

$$F(\tau) = \int_0^1 \frac{\tau^{u-1}}{\Gamma(u)}du, \tag{1.0.154}$$

so that

$$\hat{F}(s) = \frac{1}{\ln s} - \frac{1}{s\ln s}. \tag{1.0.155}$$

Define $\mathcal{H}(\tau)$ to be the unique decaying solution of

$$\frac{d\mathcal{H}}{d\tau} - \mathcal{H} = F. \tag{1.0.156}$$

It may be represented in integral form as

$$\mathcal{H}(\tau) = -\int_\tau^\infty e^{\tau-p}\left(\int_0^1 \frac{p^{u-1}}{\Gamma(u)}du\right)dp. \tag{1.0.157}$$

To compute the large τ asymptotics of $\mathcal{H}$, we first compute the large p asymptotics of $F(p)$. By Watson's lemma we find

$$F(p) = \frac{1}{\ln p} - \frac{\gamma}{(\ln p)^2} + O((\ln p)^{-3}). \tag{1.0.158}$$

Substituting this into the expression for $\mathcal{H}$ and integrating by parts, we finally have

$$\mathcal{H}(\tau) = -\frac{1}{\ln \tau} + \frac{\gamma}{(\ln\tau)^2} + O((\ln\tau)^{-3}), \quad \tau \to \infty. \tag{1.0.159}$$

APPENDIX B. GENERALIZATIONS TO NONCIRCULAR BOUNDARIES

In this section we consider general artificial boundaries, concentrating on the construction of the low-frequency expansion of the exact condition. Our purpose is twofold. First, we want to show that the lower bounds on error decay for local in time constant-coefficient conditions cannot be improved by changing the artificial boundary shape. Second, we want to indicate how to construct uniformly accurate conditions for such boundaries. (This may be useful for scattering by bodies with large aspect ratio.) Similar ideas may be applied in three space dimensions. The construction of the low-frequency expansion is discussed in more detail in [18].

To derive an expression for the exact condition, we study the Dirichlet problem exterior to the *artificial boundary*, Γ, which we will suppose is a level set of some smooth function $f(x,y)$. After Laplace transformation we have

$$\nabla^2 \hat{u} - s^2 \hat{u} = 0, \tag{2.0.160}$$

exterior to Γ. By Duhamel's principle we have

$$\hat{u}(\vec{x};s) = \int_0^\infty e^{-st} \hat{U}(\vec{x},s;t)dt, \tag{2.0.161}$$

where $\hat{U}$ satisfies (2.0.160) and the boundary condition at Γ,

$$\hat{U}(\vec{x},s;t) = u(\vec{x},t). \tag{2.0.162}$$

We now solve for $\hat{U}$, using integral equation techniques:

$$\hat{U}(\vec{x},s;t) = \int_\Gamma \sigma(\vec{y},s;t) K_0(s|\vec{x}-\vec{y}|) dS_y, \tag{2.0.163}$$

leading to the singular integral equation for σ

$$\int_\Gamma \sigma(\vec{y},s;t) K_0(s|\vec{x}-\vec{y}|) dS_y = u(\vec{x},t), \quad \vec{x} \in \Gamma. \tag{2.0.164}$$

We recall the expansion of K_0 for small argument [1]:

$$K_0(s|\vec{x}-\vec{y}|) = -\ln|\vec{x}-\vec{y}| - \beta s + O(s^2 \ln s), \quad s \to 0, \tag{2.0.165}$$

where β is given by

$$\beta(s) = \ln s - \ln 2 + \gamma. \tag{2.0.166}$$

From [18], the small-s asymptotics of $\hat{U}$ may be computed by substituting the small-s asymptotics of K_0 into the integral equation. We then have

Theorem 10 (Hariharan and MacCamy [18].**).** *Let $f_0(\vec{y})$ and $f_1(\vec{y};t)$ be solutions of the integral equations*

$$\int_\Gamma f_0(\vec{y}) \ln|\vec{x}-\vec{y}| dS_y + c_0 = 0, \quad \vec{x} \in \Gamma, \tag{2.0.167}$$

$$\int_\Gamma f_0(\vec{y}) dS_y = 1, \tag{2.0.168}$$

$$\int_\Gamma f_1(\vec{y};t) \ln|\vec{x}-\vec{y}| dS_y + c_1 = -u(\vec{x},t), \quad \vec{x} \in \Gamma, \tag{2.0.169}$$

$$\int_\Gamma f_1(\vec{y};t) dS_y = 0. \tag{2.0.170}$$

Here, c_0 and c_1 are constants. Let

$$W(\vec{x},s;t) = -\int_\Gamma \sigma_0(\vec{y},s;t) \ln|\vec{x}-\vec{y}| dS_y - \beta(s) \int_\Gamma \sigma_0(\vec{y},s;t) dS_y \tag{2.0.171}$$

in the exterior domain, where

$$\sigma_0(\vec{y},s;t) = f_1(\vec{y};t) + \frac{c_1}{\beta(s)-c_0} f_0(\vec{y}). \tag{2.0.172}$$

Then

$$\hat{U} = W + O(s^2 \ln^2 s), \quad \frac{\partial \hat{U}}{\partial n} = \frac{\partial W}{\partial n} + O(s^2 \ln^2 s), \quad s \to 0. \tag{2.0.173}$$

To turn this into a representation of the exact boundary operator, we simply compute the normal derivative of $\hat{u}$,

$$\frac{\partial \hat{u}}{\partial n} \approx \int_0^\infty e^{-st} \frac{\partial W}{\partial n} dt. \tag{2.0.174}$$

We have

$$\frac{\partial W}{\partial n} = -M\sigma_0 = -Mf_1 - \frac{c_1}{\beta(s)-c_0} M f_0, \tag{2.0.175}$$

where the Neumann operator M is defined by

$$Mf = \pi f + \int_\Gamma f(\vec{y}) \frac{\partial}{\partial n_x} \ln|\vec{x}-\vec{y}| dS_y. \tag{2.0.176}$$

Now we first note that f_1 corresponds to a bounded solution to Laplace's equation exterior to Γ with Dirichlet data $-u(\vec{x},t)$. Therefore,

$$-\int_0^\infty e^{-st} M f_1 dt = K\hat{u}, \tag{2.0.177}$$

where K is the Dirichlet-to-Neumann map for bounded solutions of Laplace's equation exterior to Γ. (It corresponds to the terms $-\frac{n}{R}$ for the circular boundary.) The only dependence of the second term on u is through c_1:

$$c_1(t) = -Lu(\vec{x},t), \tag{2.0.178}$$

where L is some linear functional. This yields

$$-\frac{c_1}{\beta(s)-c_0} M f_0 = \frac{Q(\vec{x})}{\beta(s)-c_0} L\hat{u}, \tag{2.0.179}$$

where the function Q, the constant c_0, and the operator L depend only on the boundary and could be explicitly constructed. In transform space we therefore have for $\vec{x} \in \Gamma$

$$\frac{\partial \hat{u}}{\partial n} = K\hat{u} + \frac{Q(\vec{x})}{\beta(s)-c_0} L\hat{u} + O(s^2 \ln^2 s), \quad s \to 0. \tag{2.0.180}$$

Noting the behavior of β on s, we again see a term which cannot be well approximated by a rational function. This means that the slow decay rates for the error which hold for constant-coefficient conditions with circular boundaries will be present for arbitrary boundaries.

Equation (2.0.180) can also be used to compute variable-coefficient conditions. Again assuming that u approaches a steady state, we have

$$\frac{\partial u}{\partial n} \approx Ku + \mathcal{H}(2e^{(c_0-\gamma)}t)Q(\vec{x})Lu, \quad t \to \infty. \tag{2.0.181}$$

As above, the function $\mathcal{H}$ can be approximated by a simpler function, $G(t)$, which has the same asymptotic expansion up to some number of terms. For example, we could take $G(t) = 1/(c_0 + \ln 2(t+D))$.

For short times, on the other hand, we may use the geometrical optics expansion, which corresponds to studying the behavior of the transformed problem for s large. We begin with the assumption that we have chosen the boundary such that wave propagation is nearly normal to it. Asymptotically, this would correspond to circular boundaries and the Friedlander expansion. But the goal of using a noncircular boundary would most likely be to avoid computing so far from a high aspect ratio body as required by that expansion.

We further assume that $f(x,y)$ is normalized so that $|\nabla f| = 1$ and is increasing in the outward normal direction. Then, by computing an expansion of the form

$$\hat{u} \sim e^{-sf(x,y)}(g_0(x,y) + \frac{1}{s}g_1(x,y) + \cdots), \tag{2.0.182}$$

we can compute an expansion for the normal derivative of u. Retaining as many terms as we did in the case of a circular boundary, we obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial n} + \frac{\partial u}{\partial t} + \frac{\kappa(x,y)}{2}u\right) - \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} - \frac{\gamma^2(x,y)}{8}u \approx 0. \tag{2.0.183}$$

Here, $\frac{\partial}{\partial \tau}$ denotes the tangential derivative and κ, the curvature, and γ^2 are given by

$$\kappa = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}; \quad \gamma^2 = \left(\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right)^2 + 4\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2. \tag{2.0.184}$$

The uniform condition is then given by

$$\left(\frac{\partial}{\partial t} + \mathcal{D}(t)\right)\left(\frac{\partial u}{\partial n} + \frac{\partial u}{\partial t} + \frac{\kappa(x,y)}{2}u\right) - \frac{1}{2}\frac{\partial^2 u}{\partial \tau^2} - \frac{\gamma^2(x,y)}{8}u = 0, \tag{2.0.185}$$

where $\mathcal{D}$ is determined by

$$\mathcal{D} = \frac{1}{8}\left(4\frac{\partial^2}{\partial \tau^2} + \gamma^2\right)\left(K + G(t)Q(\vec{x})L + \frac{\kappa}{2}\right)^{-1}. \tag{2.0.186}$$

(Note that the time-dependent piece represents a rank-one correction, so only $K + \kappa/2$ need be inverted.)

Appendix C. Uniform Asymptotic Conditions in 3 Dimensions

In this section we construct conditions analagous to those constructed in§3 for the easier case of three space dimensions. Now we take a spherical artificial boundary and expand our approximate solution, v, in terms of spherical harmonics, $Y_n(\theta,\phi)$: $v = \sum_{n=0}^{\infty} v_n(r,t)Y_n$. For v_n we impose a condition similar to (3.0.42), but with constant coefficients:

$$\mathcal{K}_n v_n \equiv \left(\frac{\partial}{\partial t} + \delta_n\right)\frac{\partial v_n}{\partial r} + \frac{\partial^2 v_n}{\partial t^2} + \left(\frac{1}{R} + \delta_n\right)\frac{\partial v_n}{\partial t} + \delta_n G_n v_n = 0. \tag{3.0.187}$$

This may be represented in transform space as

$$\frac{\partial \hat{v}_n}{\partial r} + b_n(s)\hat{v}_n = 0, \tag{3.0.188}$$

where

$$b_n(s) = \frac{s^2 + s\left(\frac{1}{R} + \delta_n\right) + \delta_n G_n}{s + \delta_n}. \tag{3.0.189}$$

To determine the parameters δ_n, G_n, we study the behavior of the exact condition, represented by $b_n^e(s)$, for large and small s. We have

$$b_n^e(s) = -\frac{sQ'(Rs)}{Q(Rs)}, \tag{3.0.190}$$

$$Q(z) = \frac{1}{\sqrt{z}}K_{n+\frac{1}{2}}(z) = \sqrt{\frac{\pi}{2}}\frac{e^{-z}}{z}\sum_{k=0}^{n}\frac{(n+k)!}{k!(n-k)!}(2z)^{-k}. \tag{3.0.191}$$

By direct calculation we find

$$b_n^e(s) = s + \frac{1}{R} + \frac{n(n+1)}{2R^2 s} + O(R^{-3}s^{-2}), \quad R, s \to \infty, \tag{3.0.192}$$

$$b_n^e(s) = \frac{n+1}{R} + O(s), \quad s \to 0. \tag{3.0.193}$$

Choosing

$$\delta_n = \frac{n+1}{2R}, \quad G_n = \frac{n+1}{R}, \tag{3.0.194}$$

we have that b_n agrees with b_n^e to all terms listed above.

Using the fact that

$$\frac{1}{\sin^2\theta}\frac{\partial^2 Y_n}{\partial\phi^2}+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial Y_n}{\theta})\equiv L_s Y_n=-n(n+1)Y_n, \tag{3.0.195}$$

we obtain a spatially nonlocal boundary condition in a fairly simple form:

$$\left(\frac{\partial}{\partial t}+\mathcal{D}\right)\left(\frac{\partial v}{\partial r}+\frac{\partial v}{\partial t}+\frac{v}{R}\right)-\frac{1}{2R^2}L_s v=0, \tag{3.0.196}$$

where the spatially nonlocal operator $\mathcal{D}$ is defined by

$$\mathcal{D}w(\theta,\phi)=\sum_{n=0}^{\infty}\delta_n w_n Y_n(\theta,\phi),\quad w(\theta,\phi)=\sum_{n=0}^{\infty}w_n Y_n(\theta,\phi). \tag{3.0.197}$$

Use of this operator will result in approximations which are accurate for short to moderate times and exact at steady state. Given the exponential convergence to steady state for three-dimensional problems, we expect the approximation to be uniformly accurate. However, the application of the nonlocal operator $\mathcal{D}$ will be significantly more expensive than in two dimensions, though still a small cost in comparison with the full solution process. (See again [21].) It is therefore useful to consider simpler approximations. Again, the steady state error behavior for general approximations is easy to obtain. For the nth harmonic the maximum error scales like $R^{-(n+1)}$ and decays like $(\frac{r}{R})^n$. Therefore, the local operator defined by replacing $\mathcal{D}$ with δ_0 will have a steady state error which is $O(R^{-2})$ at the boundary and decays linearly into the interior. Replacing $\mathcal{D}$ with the simpler nonlocal operator $\mathcal{D}_1$ defined by

$$\mathcal{D}_1 w(\theta,\phi)=\delta_0\bar{w}+\delta_1(w-\bar{w}),\quad \bar{w}=\frac{1}{4\pi}\int_{-\pi}^{\pi}\int_0^{\pi}w(\theta,\phi)\sin\theta d\theta d\phi, \tag{3.0.198}$$

yields an $O(R^{-3})$ steady-state error decaying quadratically.

Appendix D. Differencing of the Boundary Conditions

For brevity let $v^{i,j}=v_n(R-i\Delta r,t+j\Delta t)$. Solution values off the grid are obtained by linear interpolation.
Differencing of (2.1.16):

$$\frac{1}{\Delta t}\left(v^{\frac12,1}-v^{\frac12,0}\right)+\frac{1}{\Delta r}\left(v^{0,\frac12}-v^{1,\frac12}\right)+\frac{1}{2(R-.5\Delta r)}v^{\frac12,-\frac12}=0; \tag{4.0.199}$$

Differencing of (2.1.19):

(4.0.200)
$$\frac{1}{4(\Delta t)^2}A_1+\frac{1}{4(\Delta r)^2}A_2+A_3+\frac{3}{8(R-\Delta r)}(\hat{A}_4+\hat{A}_5)+\frac{1}{(R-\Delta r)^2}A_6=0,$$

(4.0.201)
$$A_1=\left((v^{0,1}-2v^{0,0}+v^{0,-1})+2(v^{1,1}-2v^{1,0}+v^{1,-1})+(v^{2,1}-2v^{2,0}+v^{2,-1})\right),$$

(4.0.202)
$$A_2=\left((v^{0,1}-2v^{1,1}+v^{2,1})+2(v^{0,0}-2v^{1,0}+v^{2,0})+(v^{0,-1}-2v^{1,-1}+v^{2,-1})\right),$$

$$A_3=\frac{1}{4\Delta t\Delta r}\left((v^{0,1}-v^{2,1})-(v^{0,-1}-v^{2,-1})\right), \tag{4.0.203}$$

$$A_4=\frac{1}{\Delta t}((v^{0,1}+2v^{1,1}+v^{2,1})-(v^{0,-1}+2v^{1,-1}+v^{2,-1})), \tag{4.0.204}$$

$$A_5=\frac{1}{\Delta r}((v^{0,1}+2v^{0,0}+v^{0,-1})-(v^{2,1}+2v^{2,0}+v^{2,-1})), \tag{4.0.205}$$

(4.0.206)
$$A_6=\frac{5}{64}\left(v^{0,1}+v^{2,1}+v^{0,-1}+v^{2,-1}+2(v^{1,1}+v^{0,0}+v^{1,-1}+v^{2,-1})-\frac{12}{5}v^{1,0}\right);$$

Differencing of (2.1.25):

$$\frac{1}{\Delta t}\left(v^{\frac12,1}-v^{\frac12,0}\right)+\frac{1}{\Delta r}\left(v^{0,\frac12}-v^{-1,\frac12}\right)+\frac{n}{(R-.5\Delta r)}v^{\frac12,-\frac12}=0; \tag{4.0.207}$$

Differencing of (3.0.42):

$$A_1+A_2+A_3=0, \tag{4.0.208}$$

(4.0.209)
$$A_1=\frac{1}{2\Delta t\Delta r}\left((v^{0,1}-v^{1,1})-(v^{0,-1}-v^{1,-1})\right)+\frac{1}{(\Delta t)^2}(v^{\frac12,1}-2v^{\frac12,0}+v^{\frac12,-1}),$$

$$A_2=\frac{\delta_n(t)}{4\Delta r}\left((v^{0,1}-v^{1,1})+2(v^{0,0}-v^{1,0})+(v^{0,-1}-v^{1,-1})\right), \tag{4.0.210}$$

$$A_3=\frac{(\frac{1}{2(R-.5\Delta r)}+\delta_n(t))}{2\Delta t}(v^{\frac12,1}-v^{\frac12,-1})+\delta_n(t)G_n(t)v^{\frac12,0}. \tag{4.0.211}$$

Our implementation of (2.1.28), which also followed this form, is equivalent (for $c_1=c_2=1$) to (3.0.42) with $G_n=0$ and $\delta_n=\frac{1}{2(R-.5\Delta r)}$.