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Supplement to

RATE OF CONVERGENCE OF A STOCHASTIC PARTICLE METHOD FOR THE KOLMOGOROV EQUATION WITH VARIABLE COEFFICIENTS

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10. Complementary proofs

Proof of Lemma 2.2. Let L be the infinitesimal generator of (ξ_t); then the function

$$q(t,y) = \int_{\mathbb{R}} p_t(x,y)\,dx$$

satisfies the equation (cf. the Problem 10 of Ch.6 in [6])

$$\begin{cases} \dfrac{\partial q}{\partial t}(t,y) = L^* q\,(t,y) = \dfrac{1}{2}\dfrac{\partial^2}{\partial y^2}(\sigma_0^2(y)q(t,y)) - \dfrac{\partial}{\partial y}(b_0(y)q(t,y)), \\ \lim_{t\to 0} q(t,y) = 1. \end{cases}$$

Denote by $(\alpha_t(y))$ the diffusion issued from y with infinitesimal generator $(\sigma_0(x)\sigma_0'(x) - b_0(x))\frac{\partial}{\partial x} + \frac{1}{2}\sigma_0^2(x)\frac{\partial^2}{\partial x^2}$. We have, denoting $a_0(x) = \sigma_0^2(x)$ (see again the Ch.6 in [6], e.g.)

$$q(t,y) = \mathbb{E}\exp\left(\int_0^t \left(\frac{1}{2}a_0''(\alpha_s(y)) - b_0'(\alpha_s(y))\right) ds\right). \ \blacksquare$$

Proof of Lemma 2.6. We will only treat the first case; the **(H5)** case just calls for local and easy modifications.

We introduce the process $(Z_t(x))$, solution to

$$dZ_t = b(Z_t)\,dt + \sigma(Z_t)\,dB_t \quad , \qquad Z_0(x) = x. \tag{10.1}$$

Since $\frac{f \circ u}{u}$ is bounded, the Feynman–Kac formula

$$u(t,x) = \mathbb{E}u_0(Z_t(x))\exp\left(\int_0^t \frac{f \circ u}{u}(s, Z_s(x))\,ds\right)$$

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yields (using (2.3))

$$
\begin{aligned}
\int_0^{+\infty} u(t,x)\,dx &\le C \int_0^{+\infty} \boldsymbol{E} u_0(Z_t(x))\,dx \\
&\le C \int_0^{+\infty} \int_{\boldsymbol{R}} u_0(y) \frac{1}{\sqrt{t}} \exp\left(-\frac{(y-x)^2}{2\lambda t}\right) dy\,dx \\
&\le C \int_{-\infty}^{0} \int_{-\frac{y}{\sqrt{\lambda t}}}^{+\infty} e^{-\frac{\theta^2}{2}}\,d\theta\,dy + C \int_0^{+\infty} \frac{u_0(y)}{\sqrt{t}} \int_0^{+\infty} \exp\left(-\frac{(y-x)^2}{2\lambda t}\right) dx\,dy \\
&\le C \int_{-\infty}^{0} \exp\left(-\frac{y^2}{2\lambda t}\right) dy + C \int_0^{+\infty} u_0(y)\,dy,
\end{aligned}
$$

and we apply the preceding Remark.

For the term $\int_{-\infty}^{0}(1-u(t,x))dx$, denote $v(t,x) = 1-u(t,x)$ and note that

$$
\begin{cases}
\dfrac{\partial v}{\partial t} = Lv - f(u) = Lv + \dfrac{f(1)-f(u)}{1-u} v. \\
v_0 = 1 - u_0.
\end{cases}
$$

By observing that the function $\frac{f(1)-f(u)}{1-u}$ is bounded, we get

$$
\int_{-\infty}^{0} v(t,x)\,dx \le C \int_{-\infty}^{0} (1 - \boldsymbol{E} u_0(Z_t(x))\,dx,
$$

from which we proceed as before. ∎

Proof of Proposition 3.2. Set $u_0^{(\varepsilon)} = u(\varepsilon, x)$, where u is the solution to the nonlinear problem (2.1) with u_0 as initial datum; from the Appendix (see below) we know that $1-u_0^{(\varepsilon)}$ is a regular cumulative function (of a probability distribution) such that $u_0^{(\varepsilon)} \to u_0$ for all continuity points of u_0. For any $t>0$, using the representation (3.1), we have

$$
\begin{aligned}
u(t+\varepsilon, x) &=: u^{(\varepsilon)}(t,x) \\
&= \int_{-\infty}^{+\infty} \underbrace{\boldsymbol{E}\left[H(X_t(z)-x)\exp\left(\int_0^t f'\circ u^{(\varepsilon)}(s, X_s(z))\,ds\right)\right]}_{\psi_\varepsilon(z)} d(1-u_0^{(\varepsilon)}(z)). \quad (10.2)
\end{aligned}
$$

We can write

$$
u(t+\varepsilon, x) = \underbrace{\int_{-\infty}^{+\infty} \left(\psi_\varepsilon(z) - \psi_0(z)\right) d(1-u_0^{(\varepsilon)}(z))}_{A_\varepsilon} + \underbrace{\int_{-\infty}^{+\infty} \psi_0(z)\,d(1-u_0^{(\varepsilon)}(z))}_{B_\varepsilon}.
$$

Let $\zeta_{0,t}$ denote the flow of diffeomorphisms generated by (3.2); we observe that

$$
|\psi_0(z) - \psi_0(z')| \le C \int_0^t \boldsymbol{E}|X_s(z)) - X_s(z')|ds + C\boldsymbol{E}|H(z - \zeta_{0,t}^{-1}(x)) - H(z' - \zeta_{0,t}^{-1}(x))|.
$$

For any $t>0$, the law of $\zeta_{0,t}^{-1}(z)$ has a smooth density (the coefficients of the SDE solved by $(\zeta_{0,t}^{-1}(z))$ (see again [8]) satisfy the sufficient conditions given in [6]), so that we have

$$
\boldsymbol{E}|H(z - \zeta_{0,t}^{-1}(x)) - H(z' - \zeta_{0,t}^{-1}(x))| \le C(t)|z - z'|.
$$

Besides, $\boldsymbol{P}$-almost surely, $z \to \exp\left(\int_0^t f'\circ u(s, X_s(z))\,ds\right)$ is a continous function. Therefore, $\psi_0(z)$ is continuous in z and also, as $u_0^{(\varepsilon)}(z) \to u_0(z)$ at the continuity points of $u_0(z)$, $B_\varepsilon \to \sum_{i=1}^N \omega_0^i\, \psi_0(x_0^i)$.

Moreover $A_\varepsilon \to 0$ for, for any $0<\delta<t$,

$$
\begin{aligned}
\left|\psi_\varepsilon(z) - \psi_0(z)\right| &\le \boldsymbol{E}\left|\exp\left(\int_0^t f'\circ u^{(\varepsilon)}(s, X_s(z))\,ds\right) - \exp\left(\int_0^t f'\circ u(s, X_s(z))\,ds\right)\right| \\
&\le C\,\boldsymbol{E}\int_0^t \left|u^{(\varepsilon)}(s, X_s(z)) - u(s, X_s(z))\right| ds \\
&\le C\left\{\boldsymbol{E}\int_0^\delta \left|u^{(\varepsilon)}(s, X_s(z)) - u(s, X_s(z))\right| ds + \|u^{(\varepsilon)} - u\|_{L^\infty([\delta,t]\times\boldsymbol{R})}\right\} \\
&\le C\left\{\delta + \|u^{(\varepsilon)} - u\|_{L^\infty([\delta,t]\times\boldsymbol{R})}\right\},
\end{aligned}
$$

from which

$$
A_\varepsilon \le \int_{-\infty}^{+\infty} \left|\psi_\varepsilon(z) - \psi_0(z)\right| d(1-u_0^{(\varepsilon)}(z)) \le C\left\{\delta + \|u^{(\varepsilon)} - u\|_{L^\infty([\delta,t]\times\boldsymbol{R})}\right\}.
$$

We apply now the last remark of the Appendix (see below) to bound the right-hand side by $2C\delta$ for any ε small enough. ∎

Proof of Proposition 3.4. Let i_0 be such that $u_0^{-1}(\frac{i_0-1}{N}) > 0$ and $u_0^{-1}(\frac{i_0}{N}) \le 0$. Then (the factor $\frac{1}{2N}$ instead of $\frac{1}{N}$ is due to the definition of x_0^N),

$$
\frac{1}{2N}\sum_{i=1}^{i_0-1} |x_0^i|^2 \le \int_0^{\frac{i_0-1}{N}} |u_0^{-1}(s)|^2\,ds \quad \text{and} \quad \frac{1}{N}\sum_{i=i_0}^{N} |x_0^i|^2 \le \int_{\frac{i_0}{N}}^{T} |u_0^{-1}(s)|^2\,ds.
$$

We deduce, using **(H4)**, that

$$
\frac{1}{N}\sum_{i=1}^{N} |x_0^i|^2 \le 2\int_0^1 |u_0^{-1}(s)|^2\,ds \le -2\int_{\boldsymbol{R}} u_0'(y)\,y^2\,dy \le C.
$$

To get the other part of the statement, we just apply (2.8). ∎

Proof of Lemma 5.3. For $j \neq i$, $k \neq i$, $j \neq k$, the pairs $((X^i),(\bar X^i))$, $((X^j),(\bar X^j))$ and $((X^k),(\bar X^k))$ are mutually independent; hence

$$\begin{aligned}
&\boldsymbol{E}|H(\bar X^j_p - \bar X^i_p) - H(X^j_{ph} - \bar X^i_p)| \cdot |H(\bar X^k_p - \bar X^i_p) - H(X^k_{ph} - \bar X^i_p)| \\
&\quad = \boldsymbol{E}\Big(\boldsymbol{E}(|H(\bar X^j_p - y) - H(X^j_{ph} - y)| \cdot |H(\bar X^k_p - y) - H(X^k_{ph} - y)|)\Big|_{y=\bar X^i_p}\Big) \\
&\quad = \boldsymbol{E}\Big(\{\boldsymbol{E}(|H(X^j_p - y) - H(X^j_{ph} - y)|) \cdot \boldsymbol{E}(|H(\bar X^k_p - y) - H(X^k_{ph} - y)|)\}\Big|_{y=\bar X^i_p}\Big).
\end{aligned}$$

Let A^j denote

$$A^j := \boldsymbol{E}|H(\bar X^j_p - y) - H(X^j_{ph} - y)|,$$

and suppose we have shown

$$\exists C > 0, \quad \forall h < 1, \quad \forall p, \quad \forall j, \quad \forall y \ : A^j \leq C\sqrt{h}(1 + |y|). \qquad (10.3)$$

Then we would have

$$\begin{aligned}
\boldsymbol{E}\,|H(\bar X^j_p - \bar X^i_p) - H(X^j_{ph} - \bar X^i_p)||H(\bar X^k_p - \bar X^i_p) - H(X^k_{ph} - \bar X^i_p)| &\leq \int_{\boldsymbol{R}} A^j A^k \, d\boldsymbol{P}_{\bar X^i_p}(y) \\
&\leq Ch \int_{\boldsymbol{R}} (1 + |y|^2)\, d\boldsymbol{P}_{\bar X^i_p}(y).
\end{aligned}$$

Thus, we could conclude by applying the inequality (2.8).

We now show (10.3).

Let $\beta := -b + \sigma\sigma'$ (cf. (3.6)) and set

$$\begin{aligned}
\psi(x,z) &:= x + \beta(x)h + \sigma(x)z, \\
\phi(x,z) &:= \psi(x,z) + \frac{1}{2}\sigma(x)\sigma'(x)(z^2 - 1).
\end{aligned}$$

We remark that $\bar X^j_p = \phi(\bar X^j_{p-1}, B^j_p - B^j_{p-1})$ and

$$\begin{aligned}
A^j \ &\leq \ \boldsymbol{E}|H(\bar X^j_p - y) - H(\psi(\bar X^j_{p-1}, B^j_p - B^j_{p-1}) - y)| \\
&\quad + \boldsymbol{E}|H(\psi(\bar X^j_{(p-1)}, B^j_p - B^j_{p-1}) - y) - H(\psi(X^j_{(p-1)h}, B^j_p - B^j_{p-1}) - y)| \\
&\quad + \boldsymbol{E}|H(\psi(X^j_{(p-1)h}, B^j_p - B^j_{p-1}) - y) - H(\phi(X^j_{(p-1)h}, B^j_p - B^j_{p-1}) - y)| \\
&\quad + \boldsymbol{E}|H(\phi(X^j_{(p-1)h}, B^j_p - B^j_{p-1}) - y) - H(X^j_{ph} - y)| \\
&=: \ A^j_1 + A^j_2 + A^j_3 + A^j_4.
\end{aligned}$$

We first show $A^j_1 \leq C\sqrt{h}$.

We have (using the fact that the function H is increasing and takes values equal to 0 1)

$$\begin{aligned}
&\leq \ C\boldsymbol{E}\sum_n \int_n^{n+1} |H(\phi(\bar X^j_{p-1}, \sqrt{h}z) - y) - H(\psi(\bar X^j_{p-1}, \sqrt{h}z) - y)| e^{-\frac{z^2}{2}}\, dz \\
&\leq \ C\boldsymbol{E}\sum_n e^{-\frac{n^2}{2}} \int_n^{n+1} \left\{H(\psi(X^j_{p-1}, \sqrt{h}z) + C(n+1)^2 h - y) - H(\psi(\bar X^j_{p-1}, \sqrt{h}z) - y)\right\} \\
&\qquad\qquad \mathbf{1}_{[\psi(\bar X^j_{p-1}, \sqrt{h}z) - y < 0]}\, dz \\
&\quad + C\boldsymbol{E}\sum_n e^{-\frac{n^2}{2}} \int_n^{n+1} \left\{H(\psi(\bar X^j_{p-1}, \sqrt{h}z) - y) - H(\psi(\bar X^j_{p-1}, \sqrt{h}z) - C(n+1)^2 h - y)\right\} \\
&\qquad\qquad \mathbf{1}_{[\psi(\bar X^j_{p-1}, \sqrt{h}z) - y > 0]}\, dz \\
&\leq \ C\boldsymbol{E}\sum_n e^{-\frac{n^2}{2}} \int_{\boldsymbol{R}} |H(\psi(\bar X^j_{p-1}, \sqrt{h}z) + C(n+1)^2 h - y) - H(\psi(X^j_{p-1}, \sqrt{h}z) - y)|\, dz \\
&\quad + C\boldsymbol{E}\sum_n e^{-\frac{n^2}{2}} \int_{\boldsymbol{R}} |H(\psi(\bar X^j_{p-1}, \sqrt{h}z) - C(n+1)^2 h - y) - H(\psi(\bar X^j_{p-1}, \sqrt{h}z) - y)|\, dz.
\end{aligned}$$

Then we divide the arguments of the function H by the positive (see **(H2)**) quantity $\sqrt{h}\sigma(\bar X^j_{p-1})$, and we use (2.14) to conclude

$$A^j_1 \leq \boldsymbol{E}\sum_n e^{-\frac{n^2}{2}} \frac{C(n+1)^2 h}{\sigma(\bar X^j_{p-1})\sqrt{h}} \leq C\sqrt{h}.$$

We now show $A^j_2 \leq C\sqrt{h}(1 + |y|)$.

We have

$$A^j_2 \leq \boldsymbol{E} \int |H(\psi(X^j_{p-1}, \sqrt{h}z) - y) - H(\psi(X^j_{(p-1)h}, \sqrt{h}z) - y)|\, dz.$$

Note that the function $\frac{1}{\sigma}$ is Lipschitz ; dividing the arguments of H by, respectively, $\sqrt{h}\sigma(\bar X^j_{p-1})$ and $\sqrt{h}\sigma(X^j_{p-1})$, and using again the equality (2.14), we get

$$A^j_2 \leq C\sqrt{h} + \frac{C}{\sqrt{h}}(1 + |y|)\boldsymbol{E}|\bar X^j_{p-1} - X^j_{(p-1)h}|.$$

We use Proposition 2.3 to conclude.

To show $A^j_3 \leq C\sqrt{h}$, we repeat the inequalities for A^j_1, substituting $X^j_{(p-1)h}$ to $\bar X^j_{p-1}$.

Finally, it remains to check $A^j_4 \leq C\sqrt{h}$.

For any $h > 0$ and any y,

$$\begin{aligned}
&\{(x_1, x_2) \in \boldsymbol{R}^2 : |H(x_1 - y) - H(x_2 - y)| = 1\} \subset \\
&\qquad \{(x_1, x_2) \in \boldsymbol{R}^2 : |x_1 - x_2| > h\} \cup \{(x_1, x_2) \in \boldsymbol{R}^2 : |x_2 - y| \leq h\}.
\end{aligned}$$

Furthermore, a Taylor expansion shows

$$X^j_{ph} = \phi(X^j_{(p-1)h}, B^j_{ph} - B^j_{(p-1)h}) + r^j_p$$

with $\boldsymbol{E}|r^j_p|^2 \leq Ch^3$. Therefore,

$$A^j_4 \leq \boldsymbol{P}(|X^j_{ph} - y| \leq h) + \boldsymbol{P}(|r^j_p| \geq h).$$

We use the fact that the density of the law of $X^j_{(p-1)h}$ is bounded from above by $\frac{C}{\sqrt{(p-1)h}} < \frac{C}{\sqrt{h}}$ (see (2.3)) for the first term of the right-hand side, and the Bienaymé-Chebyschev inequality for the second, to get the conclusion. ∎

Proof of Lemma 5.5. Consider v^N, the solution of (2.1) with an initial condition equal to $\tilde{u}_0$ (see the Appendix below for the existence), and let $v := u - v^N$; v is a solution to

$$\begin{cases} \dfrac{\partial v}{\partial t} = L\, v + \dfrac{f(u) - f(v^N)}{u - v^N} v, \\ v_0 = u_0 - \tilde{u}_0. \end{cases}$$

By the Feynman-Kac formula, with (Z_t) defined as in (10.1),

$$|v(t,x)| \leq C\, \boldsymbol{E}|(u_0 - \tilde{u}_0)(Z_t(x))| \leq C\, \|u_0 - \tilde{u}_0\|_{L^\infty(\boldsymbol{R})} \leq \frac{C}{N}.$$

Therefore, since by Proposition 3.2, $v^N(t,x) = \sum_{j=1}^N \omega^j_0\, \boldsymbol{E}\left[H(X^j_t - x) \exp\left(\int_0^t f' \circ v^N(s, X^j_s)\, ds\right)\right]$, we get

$$\begin{aligned} \|u(t,\cdot) - u^N(t,\cdot)\|_{L^\infty(\boldsymbol{R})} &\leq \frac{C}{N} + \|v^N(t,\cdot) - u^N(t,\cdot)\|_{L^\infty(\boldsymbol{R})} \\ &\leq \frac{C}{N} + \left\| \sum_{j=1}^N \omega^j_0 \boldsymbol{E}\left[H(X^j_t - \cdot)\left(\exp\left(\int_0^t f' \circ v^N(s, X^j_s) ds\right) - \exp\left(\int_0^t f' \circ u(s, X^j_s) ds\right)\right)\right]\right\|_{L^\infty(\boldsymbol{R})} \\ &\leq \frac{C}{N} + C \sup_{0 \leq s \leq t} \|v(s,\cdot)\|_{L^\infty(\boldsymbol{R})} \leq \frac{C}{N}. \ \blacksquare \end{aligned}$$

Proof of Lemma 5.6. From the definition (5.1), it follows that

$$T^N_p \leq \frac{C}{N^2} \boldsymbol{E}\left[\sum_{j=1}^N \left|\exp \int_0^{ph} f' \circ u(s, X^j_s)\, ds - \prod_{k=0}^{p-1}(1 + h\, f' \circ u(kh, X^j_{ph}))\right|\right]^2.$$

Now, we write (for any h small enough)

$$\prod_{k=0}^{p-1}(1 + h\, f' \circ u(kh, X^j_{ph})) = \exp\left[\sum_{k=0}^{p-1} \log(1 + h\, f' \circ u(kh, X^j_{ph}))\right] \leq C,$$

and, using the inequality $|e^a - e^b| \leq e^{\max\{|a|,|b|\}}|b - a|$, we get

$$T^N_p \leq \frac{C}{N^2} \boldsymbol{E}\left[\sum_{j=1}^N \left|\int_0^{ph} f' \circ u(s, X^j_s)\, ds - h \sum_{k=1}^{p-1} f' \circ u(kh, X^j_{ph})\right|\right]^2 + C\, h^2.$$

We expand the right-hand side, and we observe that, for some uniformly bounded function ψ:

$$\boldsymbol{E}\left|\int_{kh}^{(k+1)h} f' \circ u(s, X^j_s)\, ds - h\, f' \circ u(kh, X^j_{ph})\right|^2 \leq \boldsymbol{E}\left|\int_{kh}^{(k+1)h} \psi(s, X^j_s)\,(W_s - W_{kh})\, ds\right|^2 + Ch^2,$$

so that, using the Cauchy–Schwarz inequality in each term of the expansion, we get

$$T^N_p \leq C\, h^2. \ \blacksquare$$

Proof of Lemma 5.7. We have

$$S^N_p \leq 2\, \boldsymbol{E}\left|\sum_{\substack{j=1 \\ j \neq i}}^N \omega^j_0 \left[H(X^j_{ph} - \bar{X}^i_p)\, e^{\int_0^{ph} f' \circ u(s, X^j_s)\, ds} - \boldsymbol{E} H(X^j_{ph} - \bar{X}^i_p)\, e^{\int_0^{ph} f' \circ u(s, X^j_s)\, ds}\right]\right|^2 + \frac{C}{N^2}$$

and, by the independence property of the (X^j)'s,

$$S^N_p \leq \frac{C}{N^2} \boldsymbol{E} \sum_{\substack{j=1 \\ j \neq i}}^N H(X^j_{ph} - \bar{X}^i_p)\, \exp\left(2 \int_0^{ph} f' \circ u(s, X^j_s)\, ds\right) + \frac{C}{N^2} \leq \frac{C}{N}. \ \blacksquare$$

Finally, we obtain, combining Lemmas 5.5, 5.6, 5.7,

$$U^i_p = \boldsymbol{E}|u^*_p(\bar{X}^i_p) - u(ph, X^i_p)|^2 \leq \frac{C}{N} + C\, h^2. \ \blacksquare \tag{10.4}$$

End of the proof of Proposition 6.5. We observe that

$$T_N(y) \leq \sum_{i<j} \sqrt{\boldsymbol{E}\{\omega^i_0\, H(x^i_0 - Z_s(y)) - \boldsymbol{E}[\omega^i_0 H(x^i_0 - Z_s(y))]\}^2} \sqrt{\boldsymbol{E}\{\omega^j_0\, H(x^j_0 - Z_s(y)) - \boldsymbol{E}[\omega^j_0 H(x^j_0 - Z_s(y))]\}^2}.$$

By using again (3.10) and Lemma 2.1(iii), we have

$$T_N(y) \leq \frac{C}{N^2}\sum_{i<j}\exp\left(-\frac{(y-x_0^i)^2}{4\lambda s}\right)\exp\left(-\frac{(y-x_0^j)^2}{4\lambda s}\right), \tag{10.5}$$

hence

$$\begin{aligned}
&\int_{\boldsymbol{R}} T_N(y)\,P_{h-s}(x,dy)\\
&\leq \frac{C}{N^2}\sum_{i<j}\int_{\boldsymbol{R}}\exp\left(-\frac{(y-x_0^i)^2}{4\lambda s}\right)\exp\left(-\frac{(y-x_0^j)^2}{4\lambda s}\right)P_{h-s}(x,dy)\\
&\leq \frac{C}{N^2}\sum_{i<j}\left[\int_{\boldsymbol{R}}\exp\left(-\frac{(y-x_0^i)^2}{2\lambda s}\right)P_{h-s}(x,dy)\int_{\boldsymbol{R}}\exp\left(-\frac{(y-x_0^j)^2}{2\lambda s}\right)P_{h-s}(x,dy)\right]^{\frac12}\\
&\leq \frac{C}{N^2}\sum_{i<j}\left[\frac{\sqrt{s}}{\sqrt{h}}\exp\left(-\frac{(x-x_0^i)^2}{2\lambda s}\right)\right]^{\frac12}\left[\frac{\sqrt{s}}{\sqrt{h}}\exp\left(-\frac{(x-x_0^j)^2}{2\lambda s}\right)\right]^{\frac12}\\
&\leq \frac{C}{N^2}\sum_{i<j}\exp\left(-\frac{(x-x_0^i)^2}{4\lambda h}\right)\exp\left(-\frac{(x-x_0^j)^2}{4\lambda h}\right),
\end{aligned}$$

from which[6]:

$$\begin{aligned}
\int_{\boldsymbol{R}}\int_{\boldsymbol{R}} T_N(y)\,P_{h-s}(x,dy)\,dx &\leq \frac{C}{N^2}\sum_{i<j}\int_{\boldsymbol{R}}\exp\left(-\frac{(x-x_0^i)^2}{4\lambda h}\right)\exp\left(-\frac{(x-x_0^j)^2}{4\lambda h}\right)dx\\
&\leq \frac{C}{N^2}\sqrt{h}\sum_{i<j}\exp\left(-\frac{(x_0^i-x_0^j)^2}{8\lambda h}\right)\\
&\leq C\sqrt{h}. \quad \blacksquare
\end{aligned}$$

Proof of Lemma 6.7. First we integrate between 0 and $+\infty$, and we use the equality (6.6), which implies $\boldsymbol{E}u_0(Z_h(x)) \leq Cu(h,x)$. We thus have

$$\int_0^{+\infty}\psi_{h,\theta}(x)\,dx \leq C\int_0^{+\infty}u(h,x)\,dx.$$

For the case of integration between $-\infty$ and 0, we note

$$\frac{f(y)}{y(1-y)} \leq C,$$

hence

$$\begin{aligned}
\int_{-\infty}^0 \psi_{h,\theta}(x)\,dx &\leq C\int_{-\infty}^0 \boldsymbol{E}(1-u(\theta,Z_{h-\theta}(x)))\,dx\\
&\leq C\int_{-\infty}^0\int_{\boldsymbol{R}}(1-u(\theta,y))\frac{1}{\sqrt{h-\theta}}\exp\left(-\frac{(y-x)^2}{2\lambda(h-\theta)}\right)dy\,dx\\
&\leq C\int_{\boldsymbol{R}}(1-u(\theta,y))\int_{\frac{y}{\sqrt{\lambda(h-\theta)}}}^{+\infty}\exp\left(-\frac{\alpha^2}{2}\right)d\alpha\,dy\\
&\leq C\int_{-\infty}^0(1-u(\theta,y))\,dy + C\int_0^{+\infty}\exp\left(-\frac{y^2}{2\lambda(h-\theta)}\right)dy,
\end{aligned}$$

from which by Lemma 2.6 we get the conclusion. $\blacksquare$

End of the proof of Proposition 7.4. We must finish our treatment of A. We observe:

* For $k=i$ we have $A_k \leq 1$.
* For $k \neq i$ suppose (without any loss of generality) that the particles have been labelled in the increasing order of their positions (at time ph); we just look at $k<i$, the other case being treated by symmetry; therefore, $H(\bar{X}_p^k - \bar{X}_p^i) = 0$, and we consider

$$\begin{aligned}
\boldsymbol{E}^{\mathcal{G}_{ph}}H(\eta_s^k(\bar{X}_p^k)-\eta_s^i(\bar{X}_p^i)) &\leq \frac{C}{s}\int_{H^2}H(y-z)\exp\left(-\frac{(y-\bar{X}_p^k)^2+(z-\bar{X}_p^i)^2}{2\lambda s}\right)dy\,dz\\
&= C\int_{y-z>\frac{\bar{X}_p^i-\bar{X}_p^k}{\sqrt{\lambda s}}}\exp\left(-\frac{y^2+z^2}{2}\right)dy\,dz\\
&\leq C\exp\left(-\frac{(\bar{X}_p^i-\bar{X}_p^k)^2}{4\lambda s}\right),
\end{aligned}$$

thanks to the following easy inequality:

$$\int_{x_1-x_2>\rho}\exp\left(-\frac{x_1^2+x_2^2}{2}\right)dx_1\,dx_2 \leq C\,e^{-\frac{\rho^2}{4}}.$$

Therefore,

$$\begin{aligned}
A &\leq C\,h\,\boldsymbol{E}\sum_{i=1}^N\omega_p^i|\bar{X}_p^i|\left\{\sum_{k=1}^N\omega_p^k\exp\left(-\frac{(\bar{X}_p^k-\bar{X}_p^i)^2}{4\lambda h}\right)+\frac{C}{N}\right\}\\
&\leq \frac{C\,h}{N^2}\boldsymbol{E}\sum_{\substack{i,k=1\\ i\neq k}}^N|\bar{X}_p^i|\exp\left(-\frac{(\bar{X}_p^k-\bar{X}_p^i)^2}{4\lambda h}\right)+\frac{C\,h}{N^2}\sum_{i=1}^N\boldsymbol{E}|\bar{X}_p^i|.
\end{aligned}$$

[6] See Remark 6.2.

From the elementary inequality

$$\left| e^{-\frac{x^2}{2h}} - e^{-\frac{y^2}{2h}} \right| \leq \frac{|x-y|}{\sqrt{h}}$$

and (2.10), (3.14), we get

$$A \leq \frac{C\,h}{N^2}\, \boldsymbol{E} \sum_{\substack{i,k=1\\ i\neq k}}^{N} |X_p^i| \exp\left(-\frac{(X_p^k - X_p^i)^2}{4\lambda h}\right) + C\,h\sqrt{h} + \frac{C\,h}{N}.$$

Now we use the independence of X^i and X^k for $i \neq k$, and (2.3), to get

$$\begin{aligned} A \;&\leq\; \frac{C\,h}{N^2} \sum_{\substack{i,k=1\\ i\neq k}}^{N} \int_{\boldsymbol{R}^2} \frac{|x|}{\lambda p h} e^{-\frac{(y-x)^2}{4\lambda h}} e^{-\frac{(x-x_0^k)^2}{2\lambda p h}} e^{-\frac{(y-x_0^i)^2}{2\lambda p h}}\, dx\, dy + C\,h\sqrt{h} + \frac{C\,h}{N} \\ &\leq\; \frac{C\,h\sqrt{h}}{N^2} \sum_{\substack{i,k=1\\ i\neq k}}^{N} \int_{\boldsymbol{R}} \frac{|x|}{\lambda h\sqrt{p(p+2)}} \exp\left(-\frac{(x-x_0^k)^2}{2\lambda p h}\right) \exp\left(-\frac{(x-x_0^i)^2}{2\lambda(p+2)h}\right) dx + C\,h\sqrt{h} + \frac{C\,h}{N}. \end{aligned}$$

Using the definition of the x_0^k's and **(H4)**, we observe (see the proof of (3.13) and of Lemma 7.2 for analogous computations) that

$$\frac{1}{N}\sum_{k=1}^{N} \frac{1}{\sqrt{\lambda p h}} \exp\left(-\frac{(x-x_0^k)^2}{2\lambda p h}\right) \leq C,$$

so that, using again the same arguments, we obtain

$$A \leq \frac{C\,h\sqrt{h}}{N} \sum_{i=1}^{N} \int_{\boldsymbol{R}} \frac{|x|}{\sqrt{\lambda(p+2)h}} \exp\left(-\frac{(x-x_0^i)^2}{2\lambda(p+2)h)}\right) dx + C\,h\sqrt{h} + \frac{C\,h}{N} \leq C\,h\sqrt{h} + \frac{C\,h}{N}. \;\blacksquare$$

A. Appendix

In this section, we assume (H1)-(H3). Moreover, we consider the differential operator L, defined in §2, as an abstract unbounded closed operator in a suitable Banach space (of functions) $\mathbf{X}$.

A.1. A result for linear equations.

Let $g(\cdot)$ be a function in $L^\infty(0,T;L^\infty(\boldsymbol{R}))$, and consider the following abstract linear equation in $\mathbf{X} = L^\infty(\boldsymbol{R})$:

$$\begin{cases} u'(t) = L\,u(t) + g(t), \\ u(0) = u_0. \end{cases} \tag{A.1}$$

It is well known that, when $g \equiv 0$, L is the infinitesimal generator of an analytic semigroup e^{Lt} in $L_\pi^2(\boldsymbol{R})$, the space of (classes of equivalence of) functions that are square integrable with respect to a weight $\pi(x)$[7] and also in $C_{b.u}(\boldsymbol{R})$ [8] (cf. e.g., Stewart [15], Cannarsa-Vespri [2]). Hence, e^{Lt} is well defined for any $u_0 \in L^\infty(\boldsymbol{R})$, but is not strongly continuous, that is, $\|e^{Lt}u_0 - u_0\|_{L^\infty(\boldsymbol{R})}$ does not tend to 0 for any $u_0 \in L^\infty(\boldsymbol{R})$. One only has: $e^{Lt}u_0$ is a smooth function of x and $e^{Lt}u_0(x) \to u_0(x)$ almost everywhere (see also Theorem 2.11 in Hida [7]).

In the nonhomogeneous case $g \neq 0$, the function

$$u(t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)} g(s)\, ds \tag{A.2}$$

is called the *mild* solution to (A.1). In fact we need some regularity on g (for example, g satisfies a Hölder condition, i.e., $g \in C^\theta([0,T], L^\infty(\boldsymbol{R}))$) to ensure that (A.2) is the classical solution to (A.1) (see, for example, Da Prato-Sinestrari [11], or Theorem 3.5 in Pazy [10]).

Remark. In our case, we can also use the results of the variational theory, as illustrated in Bensoussan and Lions [1], chapter 6, in the weighted spaces $L_\pi^2(\boldsymbol{R})$).

A.2. Existence and uniqueness of the solution of the nonlinear equation.

Consider now the nonlinear problem with initial datum in $L^\infty(\boldsymbol{R})$:

$$\begin{cases} \dfrac{\partial u}{\partial t} = L\,u + f(u), \\ u(0) = u_0. \end{cases} \tag{A.3}$$

Theorem A.1. *Assume* (H_1)–(H_3). *Then there exists a unique solution* $u \in C^{1,2}(]0,T] \times \boldsymbol{R})$ *to (A.3). Moreover, for* $t \to 0$, $u(t)$ *tends to* u_0 *in the continuity points of* u_0.

We only outline the proof; we refer to Rothe [13]. Actually in [13] the results are stated for problems in bounded domains I of $\boldsymbol{R}$; but, as the proofs are based on semigroup results that are still true in $\boldsymbol{R}$ — as claimed in the preceding section —, they can be extended to cover the case of problems in the whole space.

Indeed, let us consider the following integral equation:

$$u(t) = e^{Lt}u_0 + \int_0^t e^{L(t-s)} f(u(s))\, ds \;=: \mathcal{G}(u)(t) \quad , \qquad t \in [0,T]. \tag{A.4}$$

Then, *under* $(H_1)-(H_3)$, *there exists a unique solution* $u \in L^\infty(0,T;L^\infty(\boldsymbol{R}))$ *to (A.4)*. For this, we refer to Theorem 1 of [13] (p. 111); the proof uses the classical Picard method:

[7] For instance, $\pi(x) = (1+x^2)^{-s}$, with $s > 0$.
[8] I.e. the space of bounded and uniformly continuous functions.

the sequence $u_n = \mathcal{G}(u_{n-1})$ converges to the solution (obviously unique) u of (A.4) as $f(\cdot)$ is Lipschitz.

Moreover, *for any* $\delta > 0$, $u(t)$ *is in* $C^{1+\frac{\alpha}{2},2+\alpha}([\delta, T] \times \boldsymbol{R})$ *and satisfies the first equation of (A.3).* We refer to Theorem 2 of [13] (p. 120).

Remark. Theorem 3 of [13] (p. 123) gives a comparison result, based on the maximum principle, for "smooth" initial data; but, even when u_0 is only assumed to satisfy Hypothesis (**H5**), it is easy to verify that, as $0 \leq u_0 \leq 1$, we also have $0 \leq u(t) \leq 1$ by an approximation argument, as in the proof of Theorem 2 of [13], and taking in account that $u \equiv 0$ and $u \equiv 1$ are solutions to (A.3).

Moreover, when $t \to 0$, then $u(t,x) \to u_0(x)$ in all continuity points of u_0: we use $e^{Lt}u_0 \to u_0$ in all continuity points of u_0 (see Hida [7], Theorem 2.11, e.g.) and one can easily show

$$\|u(t) - e^{Lt}u_0\|_{L^\infty(\boldsymbol{R})} = \mathcal{O}(t).$$

Finally, if the initial data u_{0n}, u_0 are in $C_{b,u}(\boldsymbol{R})$ and $\|u_{0n} - u_0\|_{L^\infty(\boldsymbol{R})} \to 0$, then we also have

$$\sup_{t\in[0,T]} \|u_n(t) - u(t)\|_{L^\infty(\boldsymbol{R})} \to 0,$$

where, by $u_n(t)$ and $u(t)$, we mean the solutions to (A.3) with initial data respectively u_{0n} and u_0.