MATHEMATICS OF COMPUTATION
VOLUME 64, NUMBER 209
JANUARY 1995, PAGES 89-115

NEGATIVE NORM ESTIMATES FOR FULLY DISCRETE FINITE ELEMENT APPROXIMATIONS TO THE WAVE EQUATION WITH NONHOMOGENEOUS L_2 DIRICHLET BOUNDARY DATA

L. BALES AND I. LASIECKA

ABSTRACT. This paper treats time and space finite element approximations of the solution to the nonhomogeneous wave equation with L_2 boundary terms and smooth right-hand side. For the case of L_2 boundary data, the rates of convergence in negative norms are derived. In the case of smooth forcing term and zero boundary data, optimal rates of convergence in "positive" norms are provided.

1. INTRODUCTION

1.1. **Statement of problem and results.** Let Ω be a bounded domain in R^N with a boundary Γ. We assume that Ω is either a smooth domain, or else a convex polyhedron. Let $T > 0$, and let $y(x, t)$ satisfy the equation

$$\begin{cases} y_{tt} = \Delta y & \text{in } Q \equiv \Omega \times (0, T), \\ y(x, 0) = 0, \quad y_t(x, 0) = 0 & \text{in } \Omega, \\ y|_\Gamma = g \in L_2(\Sigma) & \text{in } \Sigma \equiv \Gamma \times (0, T). \end{cases} \tag{1.1}$$

The main goal of this paper is to construct a fully discrete approximation of problem (1.1) and to derive the rates of convergence in negative norms. This is done in Theorem 1. To this end, we shall need, as a preliminary step, the important new result of Theorem 2, which is also of interest in its own right. Theorem 2 contains "positive norm" error estimates for the time and space finite element method (FEM) applied to a *nonhomogeneous* wave equation with zero boundary data; see problem (1.25) below. To motivate our interest in studying the above problem in negative norms, we recall the recent "sharp" regularity results for problem (1.1) from [16] (which are noted to hold true also for convex domains as well as arbitrary polyhedrons of dimension $N \leq 3$ in [19] and [12]). This result says that the solution of problem (1.1) with $g \in L_2(\Sigma)$ satisfies the optimal regularity properties

$$y \in C[0, T; L_2(\Omega)], \qquad y_t \in C[0, T; H^{-1}(\Omega)]. \tag{1.2}$$

Since we are interested in "nonsmooth" $L_2(\Sigma)$ boundary data, we see that the optimal regularity result (1.2) precludes the possibility of obtaining rates of

Received by the editor November 12, 1991 and, in revised form, September 27, 1993.
1991 *Mathematics Subject Classification*. Primary 65N30.
Research partially supported by the National Science Foundation Grant NSF DMS-9204338.

0025-5718/95 $1.00 + $.25 per page

convergence in positive norms. On the other hand, negative norm estimates will allow us to use postprocessing techniques (see [4, 5, 8, 15, 18]), which, a posteriori, will provide good interior convergence results for the solution y.

In order to introduce the fully discrete scheme, we introduce the following spaces and operators: $T^{-1}: L_2(\Omega) \to L_2(\Omega)$ given by

$$T^{-1}y \equiv -\Delta y, \qquad y \in \mathscr{D}(T^{-1}) \equiv H_0^1(\Omega) \cap H^2(\Omega); \tag{1.3}$$

the space $H \equiv L_2(\Omega) \times H^{-1}(\Omega)$ with the inner product $(u, v)_H \equiv ((u, v)) \equiv (u_1, v_1)_{L_2(\Omega)} + (Tu_2, v_2)_{L_2(\Omega)}$; and the skew adjoint operator $A: H \to H$ with $\mathscr{D}(A) \equiv H_0^1(\Omega) \times L_2(\Omega)$ given by

$$A = \begin{pmatrix} 0 & -I \\ T^{-1} & 0 \end{pmatrix}. \tag{1.4}$$

We shall also introduce the so-called "Dirichlet" map, which is defined by the formula

$$Dg = v \text{ iff } \Delta v = 0 \text{ in } \Omega \text{ and } v|_\Gamma = g.$$

It is well known (see [11, 21]) that

$$D \in \mathscr{L}(L_2(\Gamma) \to L_2(\Omega)). \tag{1.5}$$

From (1.1) we obtain that

$$y_{tt} = \Delta y = \Delta(y - Dg) \quad \text{in } \mathscr{D}'(\Omega);$$

that is,

$$(y_{tt}, \phi)_\Omega = (\Delta(y - Dg), \phi)_\Omega \equiv (y - Dg, \Delta\phi)_\Omega \quad \text{for all } \phi \in \mathscr{D}(\Omega), \tag{1.6}$$

where $\mathscr{D}(\Omega)$ denotes C^∞ functions with compact support in Ω and $(x, y)_\Omega \equiv \int_\Omega xy\, d\Omega$.

We shall show that (1.6) can be extended to all $\phi \in \mathscr{D}(T^{-1})$. This is to say that, for all $\phi \in \mathscr{D}(T^{-1})$,

$$(y_{tt}, \phi)_\Omega = (y - Dg, \Delta\phi)_\Omega = -(y - Dg, T^{-1}\phi)_\Omega. \tag{1.7}$$

We note that $\mathscr{D}(\Omega)$ is *not* dense in $\mathscr{D}(T^{-1})$ in the topology of $\mathscr{D}(T^{-1})$ (i.e., $H^2(\Omega)$). However, to prove (1.7), it suffices to show that for all $\phi \in \mathscr{D}(T^{-1})$ there exists a sequence $\phi_n \in \mathscr{D}(\Omega)$ such that

$$(y - Dg, T^{-1}\phi_n)_\Omega \to (y - Dg, T^{-1}\phi) \quad \text{as } n \to \infty. \tag{1.8}$$

Since the elements of $\mathscr{D}(T^{-1})$ are in $H_0^1(\Omega)$, for all $\phi \in \mathscr{D}(T^{-1})$, we certainly can find $\phi_n \in \mathscr{D}(\Omega)$ such that

$$\phi_n \to \phi \quad \text{in } H^1(\Omega). \tag{1.9}$$

On the other hand, from (1.2), (1.5), and since $y - Dg = 0$ on Γ, we obtain by Green's formula

$$\frac{\partial}{\partial x_i}(y - Dg) \in (H^1(\Omega))', \tag{1.10}$$

where $(H^1(\Omega))'$ is pivotal (with respect to the $L_2(\Omega)$ inner product) to $H^1(\Omega)$. By using the convergence in (1.9) together with the regularity in (1.10), we can pass to the limit in the expression

(1.11)

$$\begin{aligned}(y - Dg, \Delta\phi_n)_\Omega = &- (\nabla(y - Dg), \nabla\phi_n)_{V,V'} \Rightarrow -(\nabla(y - Dg), \nabla\phi)_{V,V'} \\ &= (y - Dg, \Delta\phi)_\Omega,\end{aligned}$$

where $V \equiv H^1(\Omega)$, hence proving (1.8) and (1.7). Setting $u_1 = y$, $u_2 = y_t$ and recalling (1.2), (1.7), we obtain

$$(1.12)\qquad \begin{cases} (u_{1t}, \phi_1)_\Omega = (u_2, \phi_1)_\Omega \quad \text{for all } \phi_1 \in H_0^1(\Omega), \\ (u_{2t}, \phi_2)_\Omega + (u_1 - Dg, T^{-1}\phi_2)_\Omega = 0 \quad \text{for all } \phi_2 \in \mathscr{D}(T^{-1}). \end{cases}$$

Application of Green's formula yields

$$(1.13)\qquad \begin{aligned} (Dg, \psi)_\Omega &= (Dg, T^{-1}T\psi)_\Omega \\ &= -(Dg, \Delta T\psi)_\Omega = -\left(g, \frac{\partial}{\partial\nu}T\psi\right)_\Gamma, \qquad \psi \in L_2(\Omega). \end{aligned}$$

Here, $\frac{\partial}{\partial\nu}$ denotes the outward normal and $(x, y)_\Gamma \equiv \int_\Gamma xy\,d\sigma$. Thus, the solution $u \equiv (y, y_t)$ to (1.1) can be written in a variational form as

$$(1.14)\qquad \begin{cases} (u_{1t}, \phi_1)_\Omega = (u_2, \phi_1)_\Omega, \qquad \phi_1 \in H_0^1(\Omega); \\ (u_{2t}, T\phi_2)_\Omega + (u_1, \phi_2)_\Omega = -\left(g, \frac{\partial}{\partial\nu}T\phi_2\right)_\Gamma, \qquad \phi_2 \in L_2(\Omega). \end{cases}$$

The equations in (1.14) will serve as a basis for our approximation.

Let $0 < h < 1$ be a parameter of the space discretization. Let $S_h^q(\Omega) \subset H_0^1(\Omega)$ be a standard finite element space of piecewise polynomials (or, more generally, piecewise curvilinear elements) of degree q defined on Ω with a quasi-uniform mesh parameter h and with the approximation property

$$(1.15)\qquad \begin{aligned} &\inf_{\lambda\in S_h^q(\Omega)} \left(|u-\lambda|_{L_2(\Omega)} + h|u-\lambda|_{H^1(\Omega)} + h^{3/2}\left|\frac{\partial}{\partial\nu}(u-\lambda)\right|_{L_2(\Gamma)} \right) \\ &\quad \le Ch^s|u|_{H^s(\Omega)}, \qquad 2 \le s \le q+1, \quad u \in H^s(\Omega). \end{aligned}$$

Let $0 = t_0 < \cdots < t_M = T$ be a partition of $[0, T]$, and let $S_k^p[0, T]$ be the finite element space on this partition consisting of continuous piecewise pth-degree polynomials in time with time step $k = \max|t_i - t_{i-1}|$, $1 \le i \le M$ [i.e., $S_k^p[0, T] = \{\chi \in C^0[0, T],\ \chi|_{[t_{i-1}-t_i]} \in P^p([t_{i-1}, t_i])\}$, where $P^p([t_{i-1}, t_i])$ is the set of polynomials of degree $\le p$ on $[t_{i-1}, t_i]]$. Then define

$$W_{hk} = S_h^q(\Omega) \otimes S_k^p[0, T].$$

Let $T_h\colon H^{-1}(\Omega) \to S_h^q(\Omega)$ be an "approximation" of T given by

$$(\nabla T_h f, \nabla\phi^h)_\Omega = (f, \phi^h)_\Omega \quad \text{for all } \phi^h \in S_h^q(\Omega),$$

where $(x, y)_\Omega \equiv \int_\Omega xy$. We shall also use the "discrete" norm on H,

$$|u|_h^2 \equiv |u_1|_{L_2(\Omega)}^2 + |T_h^{1/2}u_2|_{L_2(\Omega)}^2,$$

and the discrete inner product,

$$((u, v))_h \equiv (u_1, v_1)_\Omega + (T_h u_2, v_2)_\Omega.$$

We always have $|u|_h \le |u|_H$. Let $A_h\colon [S_h^q(\Omega)]^2 \to H$ be defined by

$$(1.16)\qquad A_h \equiv \begin{pmatrix} 0 & -I_h \\ T_h^{-1} & 0 \end{pmatrix}.$$

Since $(T_h^{-1}y^h, x^h)_\Omega = (T^{-1}y^h, x^h)_\Omega$, y^h, $x^h \in S_h^q(\Omega)$, it is immediate to verify that for ϕ^h, $\psi^h \in [S_h^q(\Omega)]^2$

$$(1.17)\qquad ((A_h\phi^h, \psi^h))_h = ((A\phi^h, \psi^h))_h = -((\phi^h, A_h\psi^h))_h.$$

Our finite element method for approximating (1.1) is to find $u^{hk} \in W^2_{hk}$ such that

$$(1.18) \quad \begin{cases} \int_0^T (u^{hk}_{1t}, \phi^{hk}_{1t})_\Omega \, dt = \int_0^1 (u^{hk}_{2k}, \phi^{hk}_{1t}) \, dt, \\ \int_0^T [(u^{hk}_{2t}, T_h\phi^{hk}_{2t})_\Omega + (u^{hk}_1, \phi^{hk}_{2t})_\Omega] \, dt = -\int_0^T (g, \frac{\partial}{\partial\nu} T_h \phi^{hk}_{2t})_\Gamma \, dt \\ \qquad\qquad \text{for all } \phi^{hk} \equiv (\phi^{hk}_1, \phi^{hk}_2) \in W^2_{hk}, \\ u^{hk}(t=0) = 0, \end{cases}$$

or equivalently,

$$(1.19) \quad \begin{cases} \int_0^T [((u^{hk}_t, \phi^{hk}_t))_h + ((A_h u^{hk}, \phi^{hk}_t))_h] \, dt = -\int_0^T (g, \frac{\partial}{\partial\nu} T_h (\phi^{hk}_2)_t)_\Gamma \, dt \\ \qquad\qquad \text{for all } \phi^{hk} \in W^2_{hk}, \\ u^{hk}(t=0) = 0. \end{cases}$$

On the other hand, the finite element solution $u^{hk}(t)$ can be computed by marching through successive time levels. Indeed, u^{hk} can be computed on $[t_n, t_{n+1}]$ as the unique solution (as we shall see later) of

$$(1.20) \quad \begin{aligned} &\int_{t_n}^{t_{n+1}} [((u^{hk}_t, w^{hk}))_h + ((A_h u^{hk}, w^{hk}))_h] \, dt \\ &\quad = -\int_{t_n}^{t_{n+1}} \left(g, \frac{\partial}{\partial\nu} T_h w^{hk}_2\right)_\Gamma dt \end{aligned}$$

for all $w^{hk} \in [[S^q_h(\Omega)] \otimes P^{p-1}[t_n, t_{n+1}]]^2$ with $u^{hk}(t_n)$ given.

Notice also that another (equivalent) version of (1.18) or (1.19) is the following:

$$(1.21) \quad \begin{cases} \int_0^T (u^{hk}_{1t}, \phi^{hk}_{1t})_\Omega \, dt = \int_0^T (u^{hk}_{2t}, \phi^{hk}_{1t})_\Omega \, dt, \\ \int_0^T [(u^{hk}_{2t}, \phi^{hk}_{2t})_\Omega + (\nabla u^{hk}_1, \nabla \phi^{hk}_{2t})_\Omega] \, dt = -\int_0^T (g, \frac{\partial}{\partial\nu}\phi^{hk}_{2t})_\Gamma \, dt, \ \phi^{hk} \in W^2_{hk}, \\ u^{hk}(t=0) = 0. \end{cases}$$

Remark 1.1. Note that for the elliptic boundary value problem $-\Delta w = 0$ in Ω, $w = g$ on Γ, Nitsche's method is to find $w^h \in S^q_h(\Omega) \subset H^1(\Omega)$ such that

$$(\nabla w^h, \nabla \phi^h)_\Omega - \left(w^h - g, \frac{\partial \phi^h}{\partial \nu}\right)_\Gamma - \left(\frac{\partial w^h}{\partial \nu}, \phi^h\right)_\Gamma + \beta h^{-1}(w^h - g, \phi^h)_\Gamma = 0,$$

for all $\phi^h \in S^q_h(\Omega) \subset H^1(\Omega)$, where β is a positive constant. Nitsche's method in the space variables combined with time discretization, when applied to (1.1), yields the following algorithm (see Appendix):

$$(1.22) \quad \begin{cases} \int_0^T (u^{hk}_{1t}, \phi^{hk}_{1t})_\Omega \, dt = \int_0^T (u^{hk}_{2t}, \phi^{hk}_{1t})_\Omega \, dt, \\ \int_0^T [(u^{hk}_{2t}, \phi^{hk}_{2t})_\Omega + (\nabla u^{hk}_1, \nabla \phi^{hk}_{2t})_\Omega - (\frac{\partial}{\partial\nu} u^{hk}_1, \phi^{hk}_{2t})_\Gamma - (u^{hk}_1, \frac{\partial}{\partial\nu}\phi^{hk}_{2t})_\Gamma \\ \qquad\qquad + \beta h^{-1}(u^{hk}_1, \phi^{hk}_{2t})_\Gamma] \, dt \\ \quad = \int_0^T [-(g, \frac{\partial}{\partial\nu}\phi^{hk}_{2t})_\Gamma + \beta h^{-1}(g, \phi^{ht}_{2t})_\Gamma] \, dt \quad \text{for all } \phi^{hk} \in W^2_{hk}, \\ u^{hk}(t=0) = 0. \end{cases}$$

If $S_h^q(\Omega) \subset H_0^1(\Omega)$, then (1.22) is equivalent to (1.21). For *smooth* data, the rate of convergence of this restricted version of Nitsche's method is not optimal when compared to Nitsche's method that is not restricted to functions which vanish on the boundary Γ. For simplicity, and since we are considering the case of *nonsmooth* data, we do not incorporate the extra terms in (1.22), which are required for the unrestricted method. In [9], the extra terms were included in the case of parabolic problems with nonhomogeneous Dirichlet boundary data, and optimal rates of convergence for smooth data (and nonsmooth data) were proved. A rather straightforward compilation of the techniques of the present paper with ones in [9] allows one to prove optimal rates of convergence for the algorithm (1.22) when applied to both smooth and nonsmooth boundary data.

In order to formulate our convergence results, we recall some notation. For any $s \in R$, T^{-s} (resp. A^s) denotes fractional powers of the operators T^{-1} (resp. A). Since these are closed and positive normal operators, the fractional powers are well defined (see [22]). Note that the following relation holds:

$$\mathscr{D}(A) = \mathscr{D}(T^{-1/2}) \times L_2(\Omega)$$

and, more generally, for $s \geq 0$

$$\mathscr{D}(A^s) = \mathscr{D}(T^{-s/2}) \times \mathscr{D}(T^{-(s-1)/2}). \tag{1.23}$$

Here, for $s \geq 0$, $\mathscr{D}(T^s) \equiv (\mathscr{D}(T^{-s}))'$, and duality is understood with respect to the $L_2(\Omega)$ inner product. Thus, $\mathscr{D}(T^s)$ can be endowed with the following topology:

$$|u|_{\mathscr{D}(T^s)} \equiv |T^s u|_{L_2(\Omega)} \quad \text{for } s \in R. \tag{1.24}$$

We then introduce the spaces

$$\mathscr{H}^{p,q} \equiv H^1[0, T; \mathscr{D}(A^q)] \cap H^1[0, T; \mathscr{D}(A^p)] \cap H^p[0, T; H] \quad \text{for } p, q \geq 0$$

and $\mathscr{H}^{-p,-q} = (\mathscr{H}^{p,q})'$, where duality is taken with respect to the $\int_0^T (\cdot, \cdot)_H \, dt$ inner product.

Our main results are as follows.

Theorem 1. *Let* $u \equiv (y, y_t)$ *be the solution to* (1.1). *There exists a unique solution* u^{hk} *of* (1.18). *Moreover, the following error estimates hold*:

(i) $|u - u^{hk}|_{\mathscr{H}^{-p,-q}} \leq C|g|_{L_2(\Sigma)} h^{-1/2}[k^p + h^q]$.

(ii) *The same result holds with* $\mathscr{H}^{p,q}$ *replaced by*

$$\mathscr{H}^{p,q} \equiv L_0[0, T; \mathscr{D}(A^{q+1})] \cap L_2[0, T; \mathscr{D}(A^{p+1})] \cap H^{p-1}[0, T; \mathscr{D}(A)].$$

Corollary 1. *Let* $y(t)$ *satisfy* (1.1) *and* y^{hk} *denote the first coordinate of* u^{hk}. *Assume* $p = q$. *Then*

$$|y - y^{hk}|_{H^{-q}} \leq C|g|_{L_2(\Sigma)} h^{-1/2}[h^q + k^q],$$

where H^{-q} *is the dual with respect to the* $L_2[0, T \times \Omega]$ *inner product to*

$$H^q \equiv L_2[0, T; H_0^{q+1}(\Omega)] \cap H^{q-1}[0, T; H_0^1(\Omega)]$$
$$\supset L_2[0, T; H_0^{q+1}(\Omega)] \cap H^q[0, T; L_2(\Omega)] \equiv H^{q+1,q}(Q),$$

where we denote $H_0^p(\Omega) \equiv D(T^{-p/2})$.

As mentioned above, in the process of proving Theorem 1, we need "positive norm" error estimates for the time and space FEM applied to a *nonhomogeneous* wave equation with zero boundary data. Since this result seems to be new, original, and of interest in its own right (see comments in §1.2), we shall describe it below.

Let $y(x, t)$ satisfy the nonhomogeneous wave equation

$$(1.25)\qquad \begin{cases} y_{tt} = \Delta y + F & \text{in } Q, \\ y(t=0) = y_0, \quad y_t(t=0) = y_1 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma, \end{cases}$$

where the forcing term $F \in L_2(Q)$ and the initial data $(y_0, y_1) \in L_2(\Omega) \times H^{-1}(\Omega)$. Our finite element method for approximating (1.25) is to find $u^{hk} \in W_{hk}^2$ such that

$$(1.26)\qquad \begin{cases} \int_0^T [((u_t^{hk}, \phi_t^{hk}))_h + ((A_h u^{hk}, \phi_t^{hk}))_h]\, dt = \int_0^T ((f, \phi_t^{hk}))_h\, dt \\ \qquad\qquad\qquad\qquad \text{for all } \phi^{hk} \in W_{hk}^2, \\ u^{kh}(t=0) \text{ is given}, \end{cases}$$

$$(1.27)\qquad f \equiv (0, F).$$

Notice that (1.26) is equivalent to solving the following system of equations: $u^{hk} \equiv (u_1^{hk}, u_2^{hk})$,

$$(1.28)\qquad \begin{cases} \int_0^T [(u_{1t}^{hk}, \phi_t^{hk})_\Omega - (u_2^{hk}, \phi_t^{hk})_\Omega]\, dt = 0, \\ \int_0^T [(u_{2t}^{hk}, \psi_t^{hk})_\Omega + (\nabla u_1^{hk}, \nabla \psi_t^{hk})_\Omega]\, dt = \int_0^T (F, \psi_t^{hk})_\Omega\, dt, \\ \qquad\qquad\qquad\qquad \forall \phi^{hk}, \psi^{hk} \in W_{hk}. \end{cases}$$

On the other hand, the finite element solution $u^{hk}(t)$ can be computed by marching through successive time levels. Indeed, u^{hk} can be computed on $[t_n, t_{n+1}]$ as the unique solutions (as we shall see later) of

$$(1.29)\qquad \int_{t_n}^{t_{n+1}} [((u_t^{hk}, w^{hk}))_h + ((A_h u^{hk}, w^{hk}))_h]\, dt = \int_{t_n}^{t_{n+1}} ((f, w^{hk}))_h\, dt$$

for all $w^{hk} \in [[S_h^q(\Omega)] \otimes P^{p-1}[t_n, t_{n+1}]]^2$ with $u^{hk}(t_n)$ given.

Following [1], we introduce the "time projection" operator $R^t: H^1[0, T] \to S_k^p$ defined by

$$(1.30)\qquad \begin{cases} \int_0^T (R^t \psi)_t \phi_t^k\, dt = \int_0^T \psi_t \phi_t^k\, dt, \quad \forall \phi^k \in S_k^p, \\ (R^t \psi)(0) = \psi(0). \end{cases}$$

The above definition allows us to define extended projections

$$Q^t: H^1[0, T; H] \to H \otimes (S_k^p)^2$$

given by

$$(1.31)\qquad Q^t \psi = (R^t \psi_1, R^t \psi_2), \qquad \text{where } \psi = (\psi_1, \psi_2).$$

We also introduce "spatial projections"

$$P^x: L_2[0, T; \mathscr{D}(A)] \to [S_h^q(\Omega) \otimes L_2[0, T]]^2$$

defined by

$$(1.32)\qquad P^x \psi \equiv (P^1 \psi_1, P \psi_2),$$

where P is the usual $L_2(\Omega)$ orthogonal projection onto $S_h^q(\Omega)$, and P^1 is a corresponding "elliptic" projection.

The following error estimates can be proved in a standard way:

$$\int_0^T |(Q^t - I)\psi|_H^2\, dt \le Ck^{2(p+1)}|\psi|^2_{H^{p+1}[0,T;H]}, \tag{1.33}$$

$$\int_0^T |(P^x - I)\psi|_h^2\, dt \le Ch^{2(q+1)}|\psi_1|^2_{L_2[0,T;H^{q+1}(\Omega)]}. \tag{1.34}$$

We are now ready to state our main result, valid for (1.25).

Theorem 2. *Let $u \equiv (y, y_t)$ be the solution to* (1.25). *There exists a unique solution $u^{hk} \equiv (y^{hk}, y_t^{hk})$ to* (1.26) (*or equivalently* (1.28)). *Assume also that $u^{hk}(0) = (P^1 y_0, Py_1)$. Then*

$$\max_{t\in[0,T]}(|y^{hk}(t) - (P^1R^t y)(t)|_{L_2(\Omega)} + |y_t^{hk}(t) - (PR^t y_t)(t)|_{H^{-1}(\Omega)})$$
$$\le C[h^{q+1} + k^{p+1}][|y|_{H^{p+1}[0,T;H^1(\Omega)]} + |y_t|_{H^{p+1}[0,T;L_2(\Omega)]} + |y_t|_{L_2[0,T;H^{q+1}(\Omega)]}].$$

If instead $u^{hk}(0) = (Py_0, P^1y_1)$, then

$$\max_{t\in[0,T]}(|y^{hk}(t) - (PR^t y)(t)|_{H^1(\Omega)} + |y_t^{hk} - (P^1R^t y_t)(t)|_{L_2(\Omega)})$$
$$\le C[h^{q+1} + k^{p+1}][|y|_{H^{p+1}[0,T;H^2(\Omega)]} + |y_t|_{H^{p+1}[0,T;H^1(\Omega)]} + |y_{tt}|_{L_2[0,T;H^{q+1}(\Omega)]}].$$

Remark 1.2. By combining the results of Theorem 2 with the convergence properties in (1.33), (1.34), one obtains error estimates for $u = u^{hk}$. Details are straightforward, hence are omitted.

Remark 1.3. Error estimates for the algorithm in (1.28) expressed in terms of $H_0^1(\Omega) \times L_2(\Omega)$ norms are derived in [6].

Remark 1.4. The arguments of this paper apply to any abstract operator A with the properties $\mathrm{Re}(Ax, x)_H = 0$ and $|A^*x|_H \le C|Ax|_H$.

Remark 1.5. Notice that the rates of convergence of approximations are always subject to the regularity of the corresponding solutions. This, in turn, depends on the regularity or type of the boundary Γ. Thus, in practice, whenever the boundary is not C^∞, the rates of convergence may be limited by the properties of the domains. Since questions of regularity are not the main focus or contribution of this paper, we shall not pursue this aspect of the analysis any further. However, the estimates in both theorems should be understood as being subject to the above-mentioned regularity. More precisely, the value of q should be less than or equal to $\alpha_0 - 1$, where α_0 is such that the inclusion $\mathscr{D}(T^{-\alpha/2}) \subset H^\alpha(\Omega)$ for $\alpha \le \alpha_0$ holds (see (4.8)).

1.2. **Literature.** Semidiscrete approximations, combined with a regularization procedure for the wave equation (1.1) with L_2 boundary data, were considered in [16]. Fully discrete single step approximations for second-order hyperbolic equations with *smooth* initial data and *zero* boundary conditions were treated in [8] (and in references therein). "Negative" norm estimates for second-order hyperbolic equations with *zero* boundary data and "*rough*" *initial* conditions have been derived in [4].

The convergence properties of finite element methods—in *time* and *space*—were studied in [1] for parabolic problems defined on polygonal domains, and in [24] for first-order, one-dimensional hyperbolic systems. The error estimates obtained in [1] (also [24]) are optimal in the sense that they reconstruct the best approximation properties of the underlying approximating spaces, modulo, however, regularity of the corresponding solutions, which, in turn, depends on the nature of the corners of the polygon (see Remark 1.4).

As to *space* and *time* FEM applied to the wave equation, in dimension higher than one, the only work we became aware of, after the first draft of the present paper was completed, is [10]. This work considers the nonhomogeneous wave equation (see (1.11)) subject to *zero* boundary conditions defined on a polyhedral domain, with *smooth* initial data and forcing term F. In [10], error estimates are derived under the restriction that $k \leq c_0 h$, where c_0 is a suitably *small* constant.

The result of Theorem 1 (resp. Corollary 1) is the first analysis of fully discrete approximations and, in particular, *space* and *time* FEM, applied to the wave equation with "*rough*" boundary data. Here, the error estimates are optimal, in the sense that they reconstruct the best approximation properties of the finite-dimensional subspaces (modulo, of course, regularity considerations—see Remark 1.5). Moreover, these results hold without assuming any compatibility between the time and space step. Also, in the case of the *nonhomogeneous* wave equation with *smooth data*, our results (Theorem 2) are optimal and do not require compatibility between k and h. Moreover, the rates of convergence obtained in Theorem 2 require one derivative *less* than the corresponding results in [10]. This difference is due to a different choice of space projection operators. Indeed, the projections used in [10] are the same as the ones introduced in [1] for parabolic equations. Instead, in our case, the projection operators which we choose are more tuned to the hyperbolic nature of the problem.

1.3. **Orientation of the paper.** The main strategy in proving Theorem 1 is an estimate of the difference between the adjoint of the solution operator for the continuous equation and the adjoint of the solution operator for the discrete equation, as in [4]. This technique is often referred to in the literature as duality, or solving the dual problem. The amount of technical detail in this work is much greater than for the *homogeneous* equation in [4], owing to a combination of the following factors:

(i) the product space $L_2(\Omega) \times H^{-1}(\Omega)$ and a discrete analog of this space (as in [2] and [4]) are used in the analysis;
(ii) a *nonhomogeneous hyperbolic boundary value* problem with *nonsmooth* boundary data is analyzed;
(iii) time discretization is based on time stepping methods (see [1] or [24]) for *nonhomogeneous* equations.

As will be seen in §3, the formulation of the dual problem leads to an equation whose structure is different from the ones usually encountered in similar situations (for instance *semidiscrete* approximations or time-space approximations of *initial* value problems). The task of proving error estimates for this "dual" problem is another point where new ideas are needed in §4. To appreciate this point, let us mention that for a simpler problem (a *special* case of the dual problem considered in §4), which is just space *and* time FEM approximation of a forward wave equation with *nonhomogeneous* right-hand side (see §2), the error estimates and, in particular, stability estimates do not follow from the usual arguments based on energy methods (as is the case for parabolic equations; see [1]). Indeed, even the one-dimensional hyperbolic problem treated by [24] requires rather delicate arguments, which critically rely on the one-dimensionality of the problem. In the case of higher dimensions, the situation is plainly more difficult: here the key ideas are contributed in §2, where the fundamental stability estimates are obtained for a forward wave equation (see Lemma 2.1). By using these stability estimates, we derive "positive" norm estimates for the dual problem in §4. These estimates, together with the structure of the adjoint ("dual") problem formulated in §3, are used (in §5) to prove the main result of Theorem 1.

2. A PRIORI ESTIMATES

In this section we consider the problem of finding $z^{hk} \in (W_{hk})^2$ such that

$$\begin{aligned} &\int_0^T [((z_t^{hk}, \phi_t^{hk}))_h + ((A_h z^{hk}, \phi_t^{hk}))_h]\, dt \\ &\quad = \int_0^T ((f, \phi_t^{hk}))_h\, dt \quad \text{for all } \phi^{hk} \in (W_{hk})^2, \end{aligned} \tag{2.1}$$

where $z^{hk}(0)$ is given in $(W_{hk})^2$ and $f \in L_2(\Omega) \times H^{-1}(\Omega)$. Notice that in the special case when $f = (0, F)$, (2.1) coincides with (1.26).

The goals of this section are to prove that (2.1) has a unique solution z^{hk} and to prove the following stability estimate.

Lemma 2.1. *Let z^{hk} be a solution to* (2.1). *Then there is a constant C which does not depend on h and k such that*

$$\max_{t\in[0,T]} |z^{hk}(t)|_h \le C \left[|z^{hk}(0)|_h + \left(\int_0^T |f|_h^2 \right)^{1/2} dt \right]. \tag{2.2}$$

Before we prove this estimate, we note that since (2.1) is a system of linear equations with as many equations as unknowns, and uniqueness follows from the stability estimate (2.2), the following corollary is a direct consequence of Lemma 2.1.

Corollary 2. *The system* (2.1) *has a unique solution $z^{hk} \in (W_{hk})^2$.*

Proof of Lemma 2.1. The proof is a consequence of the following propositions.

Proposition 2.1. *For $m = 0, 1, \dots, M-1$,*

$$|z^{hk}(t_{m+1})|_h^2 - |z^{hk}(t_m)|_h^2 \le C \left[\int_{t_m}^{t_{m+1}} |f|_h^2\, dt + \int_{t_m}^{t_{m+1}} |z^{hk}|_h^2\, dt \right]. \tag{2.3}$$

Proof. To prove (2.3), consider (2.1), and take

$$\phi^{hk} \equiv \int_0^t \left\{ \begin{matrix} A_h^{-1} z_t^{hk} & \text{for } t \in [t_m, t_{m+1}] \\ 0 & \text{otherwise} \end{matrix} \right\} dt. \tag{2.4}$$

Thus, $\phi^{hk}(t) \equiv 0$ for $t < t_m$, $\phi^{hk}(t) = \int_{t_m}^t A_h^{-1} z_t^{hk}\, dt$ for $t \in [t_m, t_{m+1}]$, and

$$\phi^{hk}(t) = \int_{t_m}^{t_{m+1}} A_h^{-1} z_t^{hk}\, dt \quad \text{for } t \ge t_{m+1}.$$

Then, using the identities

$$(2.5)\qquad ((A_h\phi^h, A_h^{-1}\psi^h))_h = -((\phi^h, \psi^h))_h \quad \text{for } \phi^h, \psi^h \in (S_h^q)^2,$$

$$(2.6)\qquad ((A_h\phi^h, \phi^h))_h = ((A_h^{-1}\phi^h, \phi^h))_h = 0 \quad \text{for } \phi^h \in (S_h^q)^2,$$

we obtain

$$\int_{t_m}^{t_{m+1}} ((z_t^{hk}, A_h^{-1}z_t^{hk}))_h\,dt - \int_{t_m}^{t_{m+1}} ((z^{hk}, z_t^{hk}))_h\,dt = \int_{t_m}^{t_{m+1}} ((f, A_h^{-1}z_t^{hk}))_h\,dt,$$

and by (2.6),

$$(2.7)\qquad |z^{hk}(t_{m+1})|_h^2 - |z^{hk}(t_m)|_h^2 = -2\int_{t_m}^{t_{m+1}} ((f, A_h^{-1}z_t^{hk}))_h\,dt.$$

We shall also need the test function

$$(2.8)\qquad \phi^{hk}(t) \equiv \int_0^t \begin{bmatrix} A_h^{-2}z_t^{hk}(t) & \text{for } t \in [t_m, t_{m+1}] \\ 0 & \text{otherwise} \end{bmatrix} dt.$$

Consider (2.1) with (2.8):

(2.9)

$$\int_{t_m}^{t_{m+1}} [((z_t^{hk}, A_h^{-2}z_t^{hk}))_h + ((A_hz^{hk}, A_h^{-2}z_t^{hk}))_h]\,dt = \int_{t_m}^{t_{m+1}} ((f, A_h^{-2}z_t^{hk}))_h\,dt.$$

By (2.5) and (2.9),

$$\int_{t_m}^{t_{m+1}} |A_h^{-1}z_t^{hk}|_h^2\,dt \le \int_{t_m}^{t_{m+1}} |z^{hk}|_h|A_h^{-1}z_t^{hk}|_h\,dt + \int_{t_m}^{t_{m+1}} |A_h^{-1}f|_h|A_h^{-1}z_t^{hk}|_h\,dt.$$

Hence,

$$(2.10)\qquad \int_{t_m}^{t_{m+1}} |A_h^{-1}z_t^{hk}|_h^2\,dt \le C\left[\int_{t_m}^{t_{m+1}} [|z^{hk}|_h^2 + |A_h^{-1}f|_h^2]\,dt\right].$$

Combining (2.7), (2.10), and the estimates $|A_h^{-1}f|_h^2 \le C|f|_h^2$ yields the result of Proposition 2.1. □

In order to complete the proof of Lemma 2.1, we need to estimate the term $\int_{t_m}^{t_{m+1}} |z^{hk}|_h^2\,dt$ in (2.3).

Proposition 2.2. *There holds*

$$\int_{t_m}^{t_{m+1}} |z^{hk}|_h^2\,dt \le Ck\left[|z^{hk}(t_m)|_h^2 + k\int_{t_m}^{t_{m+1}} |f|_h^2\,dt\right].$$

Proof. We notice first that (2.1) is equivalent to solving

$$(2.11)\qquad \int_{t_m}^{t_{m+1}} [((z_t^{hk}, w^{hk}))_h + ((A_hz^{hk}, w^{hk}))_h]\,dt = \int_{t_m}^{t_{m+1}} ((f, w^{hk}))_h\,dt$$

for all $w^{hk} \in (S_h^q(\Omega) \otimes P^{p-1}[t_m, t_{m+1}])^2$ with $z^{hk}(t_m)$ given. As in [24], we rescale the problem and define $\hat{z}$ and $\hat{f}$ in $(S_h^q(\Omega) \otimes P^{p-1}[0, 1])^2$ by

$$(2.12)\qquad \hat{z}(t) \equiv z^{hk}(t_m + kt), \qquad \hat{f}(t) = f(t_m + kt) \quad \text{where } t \in [0, 1].$$

By (2.11) and (2.12), $\hat{z}$ satisfies the equation

$$\int_0^1 [((\hat{z}_t, w^{hk}))_h + k((A_h\hat{z}, w^{hk}))_h]\, dt = k\int_0^1 ((\hat{f}, w^{hk}))_h\, dt \tag{2.13}$$

for all $w^{hk} \in (S_h^q(\Omega) \otimes P^{p-1}[0, 1])^2$. To prove the result in Proposition 2.2, it suffices to establish the following estimate:

$$\int_0^1 |\hat{z}(t)|_h^2\, dt \le C\left[|\hat{z}(t=0)|_h^2 + k^2\int_0^1 |\hat{f}|_h^2\, dt\right]. \tag{2.14}$$

Indeed, since

$$\int_0^1 |\hat{z}(t)|_h^2\, dt = \frac{1}{k}\int_{t_m}^{t_{m+1}} |z^{hk}|_h^2\, dt \quad \text{and} \quad \int_0^1 |\hat{f}(t)|_h^2\, dt = \frac{1}{k}\int_{t_m}^{t_{m+1}} |f(t)|_h^2\, dt,$$

the inequality (2.14) is equivalent to

$$\int_{t_m}^{t_{m+1}} |z^{hk}(t)|_h^2\, dt \le Ck\left[|z^{hk}(t_m)|_h^2 + k\int_{t_m}^{t_{m+1}} |f|_h^2\, dt\right]$$

as desired. Thus, it is enough to prove (2.14).

Let $\hat{z}(t, x) = \sum_{j=0}^p \phi_j z_j^h$, where $\{\phi_j\}_{j=0}^p$ is a basis for $P^p[0, 1]$, and $z_j^h \in [S_h^q(\Omega)]^2$, $j = 0, 1, \dots, p$. Consider (2.13), and set $w^{hk} = \phi_i w^h$, where $w^h \in [S_h^q(\Omega)]^2$,

$$\begin{aligned}\int_0^1 \left[\left(\left(\sum_{j=0}^p \phi_j' z_j^h, \phi_i w^h\right)\right)_h + k\left(\left(A_h\sum_{j=0}^p \phi_j z_j^h, \phi_i w^h\right)\right)_h\right] dt \\ = k\int_0^1 ((\hat{f}, \phi_i w^h))_h\, dt\end{aligned} \tag{2.15}$$

for all $w^h \in [S_h^q(\Omega)]^2$, $i = 0, 1, \dots, p-1$. Rearranging yields the identity

(2.16)

$$\begin{aligned}\left(\left(\sum_{j=0}^p\left(\int_0^1 \phi_j'\phi_i\, dt\right) z_j^h, w^h\right)\right)_h + k\left(\left(\sum_{j=0}^p\left(\int_0^1 \phi_j\phi_i\, dt\right) A_h z_j^h, w^h\right)\right)_h \\ = k\left(\left(\int_0^1 \phi_i\hat{f}\, dt, w^h\right)\right)_h\end{aligned}$$

for all $w^h \in [S_h^q(\Omega)]^2$, $i = 0, 1, \dots, p-1$. Therefore, $\hat{z} = \sum_{j=0}^p \phi_j z_j^h$ satisfies

$$\sum_{j=0}^p \left[\left(\int_0^1 \phi_j'\phi_i\, dt\right) z_j^h + \left(\int_0^1 \phi_j\phi_i\, dt\right) kA_h z_j^h\right] = k\int_0^1 \phi_i\hat{f}\, dt, \tag{2.17}$$

$$i = 0, 1, \dots, p-1.$$

Let $\{\phi_i\}_{i=0}^p$ be the Legendre polynomials on the interval $[0, 1]$, i.e., $\phi_i(t) = P_i(2(t - 1/2))$, where $P_i(x)$ is the Legendre polynomial of degree i on the interval $[-1, 1]$. Then we have that

(2.18)
$$\int_0^1 \phi_j'\phi_i\,dt = \begin{cases} 0, & j \le i, \\ \phi_j\phi_i|_0^1 - \int_0^1 \phi_j\phi_i'\,dt = 1 - \phi_j(0)\phi_i(0) = 0 \text{ or } 2, & j > i, \end{cases}$$

since $\phi_i(1) = 1$ and $\phi_i(0) = \pm 1$. Also,

$$\begin{aligned} \int_0^1 \phi_j\phi_i\,dt &= \int_0^1 P_j(2(t-\tfrac{1}{2}))P_i(2(t-\tfrac{1}{2}))\,dt \\ &= \frac{1}{2}\int_{-1}^1 P_j(x)P_i(x)\,dx = \frac{\delta_{ij}}{2i+1}. \end{aligned} \tag{2.19}$$

Since $\hat{z}(t=0) = \sum_{j=0}^p \phi_j(t=0)z_j^h = \sum_{j=0}^p (-1)^j z_j^h$, it follows from (2.17), (2.18), and (2.19) that

(2.20)
$$\begin{pmatrix} I & -I & I & & -I & & \pm I \\ kA_h & 2I & 0 & \cdots & 2I & \cdot & \cdot \\ 0 & \frac{k}{3}A_h & 2I & \cdots & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & & \\ \cdot & & \cdot & \cdots & \cdot & \cdot & 2I \\ \cdot & & & \cdots & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & \cdots & 0 & \frac{k}{2p-1}A_h & 2I \end{pmatrix} \begin{pmatrix} z_0^h \\ z_1^h \\ \vdots \\ z_p^h \end{pmatrix} = \begin{pmatrix} \hat{z}(t=0) \\ k\int_0^1 \phi_0 \hat{f}\,dt \\ \vdots \\ k\int_0^1 \phi_{p-1}\hat{f}\,dt \end{pmatrix}.$$

Using determinants to find the inverse of the matrix in (2.20) (see Hoffman and Kunze [13, Chapter 5, p. 140] for a discussion of determinants of matrices whose entries are elements from a commutative ring with identity), we have that

$$\begin{pmatrix} z_0^h \\ z_1^h \\ \vdots \\ z_p^h \end{pmatrix} = \begin{pmatrix} R_{00} & R_{01} & \cdots & R_{0p} \\ R_{10} & R_{11} & \cdots & R_{1p} \\ \vdots & & & \vdots \\ R_{p0} & R_{p1} & \cdots & R_{pp} \end{pmatrix} \begin{pmatrix} \hat{z}(t=0) \\ k\int_0^1 \phi_0\hat{f}\,dt \\ \vdots \\ k\int_0^1 \phi_{p-1}\hat{f}\,dt \end{pmatrix}. \tag{2.21}$$

From the definition of the determinant, it follows that R_{lj} is a rational function of kA_h with degrees of numerator and denominator less than or equal to p. In fact, since for homogeneous problems $(\hat{f} = 0)$, $\hat{z}(t=1) = r(-kA_h)\hat{z}(t=0)$, where $r(x)$ is the (p, p) diagonal Padé approximation to e^x (see [13]), the denominator of R_{lj} is a constant multiple of the denominator of $r(-kA_h)$. The rational function $r(x)$ has p distinct poles in the right-hand half of the complex plane. It follows that $|R_{lj}(z)| \le C$ for $\operatorname{Re} z \le 0$.

From Lemma 4.2, p. 398, of [3] it follows that if $\operatorname{Re}\alpha \ne 0$ and $R(z) = \frac{a+bz}{1+\alpha z}$, then $|R(-kA_h)|_h \le C$, where C is independent of h and k. Since the numerator (as well as the denominator) of R_{lj} has degree less than or equal to p, it follows that

$$|R_{lj}|_h \le C, \tag{2.22}$$

where C is independent of h and k, by writing the numerator and denominator of R_{lj} as products of linear factors.

Since $\hat{z}(t, x) = \sum_{l=0}^p \phi_l z_l^h$, we have

$$\begin{aligned}\int_0^1 |\hat{z}(t)|_h^2\,dt &\le C\sum_{l=0}^{p}\int_0^1 |\phi_l(t)z_l^h|_h^2\,dt = C\sum_{l=0}^{p}\int_0^1 |z_l^h|_h^2|\phi_l(t)|^2\,dt \\ &\le C\sum_{l=0}^{p}|z_l^h|_h^2.\end{aligned} \tag{2.23}$$

Note that from (2.21)

$$z_l^h = R_{l0}\hat{z}(t=0) + \sum_{j=0}^{p} R_{lj}k\int_0^1 \phi_{j-1}\hat{f}\,dt. \tag{2.24}$$

The relations (2.22), (2.23), and (2.24) imply that

$$\int_0^1 |\hat{z}(t)|_h^2\,dt \le C\left(|\hat{z}(t=0)|_h^2 + \sum_{j=1}^{p}\left|\int_0^1 k\phi_{j-1}\hat{f}\,dt\right|_h^2\right). \tag{2.25}$$

Since

$$\begin{aligned}\left|\int_0^1 k\phi_{j-1}\hat{f}\,dt\right|_h^2 &\le \left(\int_0^1 |k\phi_{j-1}\hat{f}|_h\,dt\right)^2 \\ &\le \int_0^1 |\phi_{j-1}|^2\,dt\int_0^1 |k\hat{f}|_h^2\,dt \le Ck^2\int_0^1 |\hat{f}|_h^2\,dt,\end{aligned} \tag{2.26}$$

it follows that

$$\int_0^1 |\hat{z}(t)|_h^2\,dt \le C\left(|\hat{z}(t=0)|_h^2 + k^2\int_0^1 |\hat{f}|_h^2\,dt\right), \tag{2.27}$$

which completes the proof of (2.14), and hence of Proposition 2.2. □

Proof of Lemma 2.1 (continued). From (2.22), (2.24), and (2.26), it follows that

$$\max_{t\in[0,1]}|\hat{z}(t)|_h \le C\sum_{l=0}^{p}|z_l^h|_h \le C\left[|\hat{z}(t=0)|_h + k\left(\int_0^1 |\hat{f}|_h^2\,dt\right)^{1/2}\right]. \tag{2.28}$$

Hence,

$$\max_{t\in[t_m, t_{m+1}]}|z^{hk}(t)|_h \le C\left[|z^{hk}(t_m)|_h + \sqrt{k}\left(\int_{t_m}^{t_{m+1}} |f|_h^2\,dt\right)^{1/2}\right]. \tag{2.29}$$

By combining the results of Propositions 2.1 and 2.2, we obtain

$$|z^{hk}(t_{m+1})|_h^2 - |z^{hk}(t_m)|_h^2 \le C\left(k|z^{hk}(t_m)|_h^2 + \int_{t_m}^{t_{m+1}} |f|_h^2\,dt\right). \tag{2.30}$$

By the discrete Gronwall's identity we obtain

$$|z^{hk}(t_m)|_h^2 \le C\left[\int_0^{t_m} |f|_h^2\,dt + |z^{hk}(0)|_h^2\right]. \tag{2.31}$$

The conclusion of Lemma 2.1 follows now from (2.31) and (2.29). □

3. The adjoint problem

Following [20] (see also [21, p. 190]), we introduce a dual (pivotal) space $\mathscr{D}(T^{-1})'$, where $\mathscr{D}(T^{-1}) \subset L_2(\Omega) \subset \mathscr{D}(T^{-1})'$ and $L_2(\Omega)$ is identified with its dual. Then the adjoint $(T^{-1})^*$ of the operator T^{-1}, considered as a *bounded* operator $\mathscr{D}(T^{-1}) \to L_2(\Omega)$, is bounded: $L_2(\Omega) \to \mathscr{D}(T^{-1})'$. Since $(T^{-1})^* x = T^{-1}x$ for $x \in \mathscr{D}(T^{-1})$, $(T^{-1})^*$ is a proper extension of T^{-1} and $(T^{-1})^* \in \mathscr{L}(L_2(\Omega) \to \mathscr{D}(T^{-1})')$. For simplicity of notation we shall use the same symbol T^{-1} to denote its extension, such that

$$T^{-1} \in \mathscr{L}(L_2(\Omega) \to \mathscr{D}(T^{-1})'). \tag{3.1}$$

Going back to (1.12) and using the above interpretation of T^{-1}, we obtain that for all $\phi_2 \in \mathscr{D}(T^{-1})$

$$(u_{2t}, \phi_2)_\Omega + (T^{-1}(u_1 - Dg), \phi_2)_\Omega = 0.$$

Here, the inner product $(x, y)_\Omega$ should be understood as a duality pairing between $\mathscr{D}(T^{-1})$ and $\mathscr{D}(T^{-1})'$. Again, for simplicity, we shall use the same notation $(x, y)_\Omega$ for duality pairing. From (3.1), (1.5), and (1.2),

$$T^{-1}(u_1 - Dg) = T^{-1}u_1 - T^{-1}Dg \quad \text{on } \mathscr{D}(T^{-1})'.$$

The above formula yields the following abstract differential equation satisfied by the variable $u = (u_1, u_2)$:

$$\begin{cases} u_t + Au = Bg & \text{on } \mathscr{D}(A)', \\ u(t=0) = 0, \end{cases} \tag{3.2}$$

where, we recall, $\mathscr{D}(A)' = H^{-1}(\Omega) \times \mathscr{D}(T^{-1})'$ and the operator $B \in \mathscr{L}(L_2(\Gamma) \to \mathscr{D}(A)')$ is given by (see (3.1))

$$Bg = \begin{pmatrix} 0 \\ T^{-1}Dg \end{pmatrix}.$$

Using the fact that the operator A generates an unitary group on $\mathscr{D}(A)'$, we can use the variation of parameter formula (on $\mathscr{D}(A)'$) to write an explicit form of the solution to (3.2):

$$u(t) = (Lg)(t) = \int_0^1 e^{-A(t-\tau)} Bg(\tau)\, d\tau. \tag{3.3}$$

For a more detailed exposition of differential equations defined on extended spaces and semigroup formulas defined within the extended framework, we refer the reader to [7].

Thus, with $g \in L_2(\Sigma)$, we know a priori that $u \in C[0, T; \mathscr{D}(A)']$. However, sharp regularity results of [17] (see also [19] and [11] for the case of nonsmooth boundaries) provide us with improved regularity of the operator L, namely that

$$L \in \mathscr{L}(L_2(\Sigma) \to C[0, T; H]). \tag{3.4}$$

For the continuous and discrete dual problems, we will need the adjoint of L evaluated with respect to the $(u, v)_H = ((u, v))$ inner product and with respect to the $((u, v))_h$ inner product. To obtain this, note that for $u \in \mathscr{D}(T^{-1}) = H_0^1(\Omega) \cap H^2(\Omega)$ and $w \in L_2(\Gamma)$

$$\begin{aligned} (D^*(-\Delta)^* u, w)_{L_2(\Gamma)} &= (-\Delta u, Dw)_{L_2(\Omega)} \\ &= -\int_\Gamma \frac{\partial u}{\partial \nu} Dw\, d\sigma + \int_\Gamma u \frac{\partial (Dw)}{d\nu}\, d\sigma - \int_\Omega u\Delta(Dw)\, dx \\ &= -\left(\frac{\partial u}{\partial \nu}, w\right)_{L_2(\Gamma)}, \end{aligned}$$

since $\Delta(Dw) = 0$ in $L_2(\Omega)$ and $Dw = w$ in $L_2(\Gamma)$ by definition of D and $u = 0$ in $L_2(\Gamma)$. It follows that for $\phi \in \mathscr{D}(A) = H_0^1(\Omega) \times L_2(\Omega)$

$$\begin{aligned}(Bg, \phi)_H &= (T^{-1}Dg, \phi_2)_{H^{-1}(\Omega)} = (T^{-1}Dg, T\phi_2)_\Omega = (Dg, T^{-1}T\phi_2)_\Omega \\ &= (Dg, -\Delta T\phi_2)_\Omega = -\left(g, \frac{\partial}{\partial\nu}T\phi_2\right)_\Gamma .\end{aligned} \tag{3.5}$$

Similarly,

$$((Bg, \phi))_h = -\left\langle g, \frac{\partial}{\partial\nu}T_h\phi_2\right\rangle_\Gamma, \qquad \phi \in (S_h^q(\Omega))^2. \tag{3.5i}$$

To compute the adjoint of L, we notice that the operator L can also be represented by $(Lg)(t) = (KBg)(t)$. Here, the operator $K: \mathscr{D}(A)' \to \mathscr{D}(A)'$ is given by $(Kf)(t) \equiv \int_0^t e^{A(-t+\tau)} f(\tau)\, d\tau \equiv z(t)$, and $z(t)$ satisfies

$$\begin{cases} z_t + Az = f, \\ z(0) = 0. \end{cases} \tag{3.6}$$

We compute the adjoint L^* of the operator L, where the adjoint is evaluated with respect to the $\int_0^T ((\cdot, \cdot))\, dt$ inner product, i.e.,

$$\int_0^T (L^*f, g)_\Gamma\, dt = \int_0^T (f, Lg)_H\, dt.$$

By (3.5) and direct computations with the wave operator (notice that the H-adjoint of A is $-A$) we obtain

$$(L^*f)(t) = -\frac{\partial}{\partial\nu}T[K^*f]_2, \tag{3.7}$$

where

$$(K^*f)(t) \equiv \int_t^T e^{-A(-\tau+t)} f(\tau)\, d\tau. \tag{3.8}$$

Note that $w(t) \equiv (K^*f)(t)$ satisfies

$$\begin{cases} w_t + Aw = -f, \\ w(T) = 0. \end{cases} \tag{3.9}$$

It is well known that

$$K^* \in \mathscr{L}(L_2[0, T; \mathscr{D}(A^\alpha)] \to C[0, T; \mathscr{D}(A^\alpha)])^1 \quad \text{for all } \alpha. \tag{3.10}$$

Let the operator $L_{hk}: L_2(\Sigma) \to W_{hk}^2$ be defined by

$$(L_{hk}g)(t) \equiv u^{hk}(t), \tag{3.11}$$

where $u^{hk}(t)$ satisfies (1.18). Let the operator $K_{hk}: L_2(0, T; H) \to W_{hk}^2$ be defined by

$$(K_{hk}f)(t) \equiv z^{hk}(t), \tag{3.12}$$

where $z^{hk}(t)$ satisfies (2.1) with $z^{hk}(0) = 0$. From (2.1) and (3.5i) we deduce that

$$(L_{hk}g)(t) = (K_{hk}Bg)(t). \tag{3.13}$$

[1]Here we adopt for $\alpha \leq 0$ the notation $\mathscr{D}(A^\alpha) \equiv (\mathscr{D}(A^{-\alpha}))'$, where duality is with respect to the $L_2(\Omega) \times H^{-1}(\Omega)$ inner product.

We shall compute the adjoint L_{hk}^* of L_{hk} with respect to the $\int_0^T((\ ,\ ))_h\,dt$ inner product, i.e.,

$$\int_0^T ((L_{hk}g, u))_h\,dt = \int_0^T ((g, L_{hk}^* u))_h\,dt. \tag{3.14}$$

To accomplish this, it suffices to compute K_{hk}^* with respect to the $\int_0^T((\ ,\ ))_h\,dt$ inner product. Indeed, by (3.5i) and (3.13),

$$(L_{hk}^* f)(t) = -\frac{\partial}{\partial\nu} T_h[K_{hk}^* f]_2, \tag{3.15}$$

where $[\psi]_2$ denotes the second coordinate of ψ.

Proposition 3.1. *The operator* $K_{hk}^*: L_2(0, T; H) \to (\dot{W}_{hk})^2$ *is given by* $(K_{hk}^* f)(t) = w^{hk}(t)$, *where* $w^{hk}(t) \in (\dot{W}_{hk})^2$ *satisfies*

(3.16)

$$\int_0^T [((w^{hk}, \phi_t^{hk}))_h - ((A_h w^{hk}, \phi^{hk}))_h]\,dt + \left(\left(\int_0^T [f + A_h w^{hk}]\,dt, \phi^{hk}(t=0)\right)\right)_h$$
$$= \int_0^T ((f, \phi^{hk}))_h\,dt \quad \textit{for all } \phi^{hk} \in [W_{hk}]^2.$$

Notice that at each time level, (3.16) consists of $\dim(S_h^q(\Omega)) \times p$ equations with $\dim(S_h^q(\Omega)) \times p$ unknowns. ($\dim(S_h^q(\Omega))$ equations corresponding to $\phi_t = 0$ are redundant.)

Proof of Proposition 3.1. We need to prove that K_{hk}^* defined by (3.16) is the adjoint (with respect to the $\int_0^T((\ ,\ ))_h\,dt$ inner product) of the operator K_{hk} defined by (3.12); i.e., we need to show that

$$\int_0^T ((K_{hk} f, v))_h\,dt = \int_0^T ((f, K_{hk}^* v))_h\,dt$$

for all $f \in L_2(0, T; H)$.

Consider (2.1) with the test function $\phi_t^{hk} = w^{hk} \in (\dot{W}_{hk})^2$,

$$\int_0^T ((z_t^{hk}, w^{hk}))_h\,dt + \int_0^T ((A_h z^{hk}, w^{hk}))_h\,dt = \int_0^T ((f, w^{hk}))_h\,dt. \tag{3.17}$$

Consider next (3.16) (with $f = v$) and with $\phi^{hk} = z^{hk}$,

$$\begin{aligned}&\int_0^T [((w^{hk}, z_t^{hk}))_h - ((A_h w^{hk}, z^{hk}))_h]\,dt \\ &+ \left(\left(\int_0^T (-v + A_h w^{hk})\,dt, z^{hk}(t=0)\right)\right)_h = \int_0^T ((v, z^{hk}))_h\,dt.\end{aligned} \tag{3.18}$$

Since $z^{hk}(t=0) = 0$, by (1.17), we obtain

$$\int_0^T ((f, I_{hk}^*(v)))_h\,dt = \int_0^T ((f, w^{hk}))_h\,dt = \int_0^T ((v, z^{hk}))_h\,dt$$
$$= \int_0^T ((v, K_{hk} f))_h\,dt$$

as desired. □

Since the adjoints L^* (resp. K^*) are computed with respect to the $\int_0^T((\ ,\ ))\,dt$ inner product, while L^*_{hk} (K^*_{hk}) with respect to the $\int_0^T((\ ,\ ))_h\,dt$ inner product, we need to estimate the difference between these two (different) adjoints. Proposition 3.2 below is proved along the same lines as in [4].

Proposition 3.2. *There holds*

$$\begin{aligned}\int_0^T(((L-L_{hk})g\,,f))\,dt-\int_0^T(g\,,(L^*-L^*_{hk})f)_\Gamma\,dt\\ =\int_0^T\left(L^*_{hk}\begin{pmatrix}0\\(I-T_h^{-1}T)f_2\end{pmatrix},g\right)_\Gamma dt.\end{aligned}\tag{3.19}$$

Proof. Following [4], we have

$$\begin{aligned}&\int_0^T(((L-L_{hk})g\,,f))\,dt\\
&=\int_0^T(((L-L_{hk})g\,,f))\,dt+\int_0^T((L_{hk}g\,,f))_h\,dt-\int_0^T((L_{hk}g\,,f))_h\,dt\\
&=\int_0^T((L_{hk}g\,,f))_h\,dt-\int_0^T((L_{hk}g\,,f))\,dt\\
&\quad-\int_0^T((L_{hk}g\,,f))_h\,dt+\int_0^T((Lg\,,f))\,dt\\
&=\int_0^T((L_{hk}g\,,f))_h\,dt-\int_0^T((L_{hk}g\,,f))\,dt\\
&\quad+\int_0^T(g\,,L^*f)_\Gamma\,dt-\int_0^T(g\,,L^*_{hk}f)_\Gamma\,dt,\\
&\text{since } ((L_{hk}g\,,f))=\left(\left(L_{hk}g\,,\begin{pmatrix}f_1\\T_h^{-1}Tf_2\end{pmatrix}\right)\right)_h=\left(g\,,L^*_{hk}\begin{pmatrix}f_1\\T_h^{-1}Tf_2\end{pmatrix}\right)_\Gamma\\
&=\int_0^T\left[\left(g\,,L^*_{hk}\begin{pmatrix}0\\f_2-T_h^{-1}Tf_2\end{pmatrix}\right)_\Gamma+\int_0^T(g\,,(L^*-L^*_{hk})f)_\Gamma\right]dt,\end{aligned}$$

as desired. □

Now we are in a position to express the error of approximation for u^{hk} (in negative norms) in terms of the error for the adjoint problem.

Lemma 3.1. *Let* u *be the solution of* $u_t+Au=Bg$ *on* $\mathscr{D}(A)'$, $u(t=0)=0$, *and* u^{hk} *be the solution of* (1.18). *Then*

$$\|u-u^{hk}\|_{\mathscr{H}^{-p,-q}}\le|g|_{L_2(\Sigma)}\sup_\psi\frac{|(L^*-L^*_{hk})\psi|_{L_2(\Sigma)}+\left|\frac{\partial}{\partial\nu}T_h\left[K^*_{hk}\begin{pmatrix}0\\\psi_2-T_h^{-1}T\psi_2\end{pmatrix}\right]_2\right|_{L_2(\Sigma)}}{|\psi|_{\mathscr{H}^{p,q}}}$$

Proof. By the result of Proposition 3.2,

$$\|u - u^{hk}\|_{\mathscr{H}^{-p,-q}} = \|(L - L_{hk})g\|_{\mathscr{H}^{-p,-q}} = \sup_{\psi} \frac{\int_0^T(((L - L_{hk})g, \psi))\,dt}{|\psi|_{\mathscr{H}^{p,q}}}$$

$$= \sup_{\psi}\left[\frac{\int_0^T(g, (L^* - L^*_{hk})\psi)_\Gamma\,dt}{|\psi|_{\mathscr{H}^{p,q}}} + \frac{\int_0^T\left(L^*_{hk}\begin{pmatrix}0\\ \psi_2 - T_h^{-1}T\psi_2\end{pmatrix}, g\right)_\Gamma dt}{|\psi|_{\mathscr{H}^{p,q}}}\right],$$

by (3.15)

$$\leq |g|_{L_2(\Sigma)} \sup_{\psi} \frac{\left[|(L^* - L^*_{hk})\psi|_{L_2(\Sigma)} + \left|\frac{\partial}{\partial\nu}T_h\left[K^*_{hk}\begin{pmatrix}0\\ \psi^2 - T_h^{-1}T\psi_2\end{pmatrix}\right]_2\right|_{L_2(\Sigma)}\right]}{|\psi|_{\mathscr{H}^{p,q}}},$$

as desired. □

Our next step is to derive the error estimates for the adjoint problem. This will be done in the next section.

4. Error estimates for the adjoint problem

From (3.7) and (3.15),

$$L^*\psi - L^*_{hk}\psi = -\frac{\partial}{\partial\nu}[T[K^*\psi]_2 - T_h[K^*_{hk}\psi]_2]. \tag{4.1}$$

Let P_h denote the orthogonal projection of $L_2(\Omega)$ onto $S_h^q(\Omega)$. Then

(4.2)

$$|L^*\psi - L^*_{hk}\psi|_{L_2(\Sigma)} \leq \left|\frac{\partial}{\partial\nu}(T - T_h)[K^*\psi]_2\right|_{L_2(\Sigma)} + \left|\frac{\partial}{\partial\nu}T_h[(K^* - K^*_h)\psi]_2\right|_{L_2(\Sigma)}$$

$$\leq \left|\frac{\partial}{\partial\nu}(P_hT - T_h)[K^*\psi]_2\right|_{L_2(\Sigma)} + \left|\frac{\partial}{\partial\nu}(I - P_h)T[K^*\psi]_2\right|_{L_2(\Sigma)}$$

$$+ \left|\frac{\partial}{\partial\nu}T_h[[K^* - K^*_{hk}]\psi]_2\right|_{L_2(\Sigma)}.$$

Since [23, p. 18]

$$\left|\frac{\partial}{\partial\nu}\phi_h\right|_{L_2(\Gamma)} \leq Ch^{-1/2}|\phi_h|_{H^1(\Omega)}, \tag{4.3}$$

we obtain from the estimates for T_h

$$\left|\frac{\partial}{\partial\nu}T_hf\right|_{L_2(\Gamma)} \leq Ch^{-1/2}|T_hf|_{H^1(\Omega)} \leq Ch^{-1/2}|T_h^{1/2}f|_{L_2(\Omega)}. \tag{4.4}$$

By (4.4),

(4.5)

$$\left|\frac{\partial}{\partial\nu}T_h[(K^* - K^*_{hk})\psi]_2\right|_{L_2(\Sigma)} \leq Ch^{-1/2}|T_h^{1/2}[(K^* - K^*_{hk})\psi]_2|_{L_2(Q)}$$

$$\leq Ch^{-1/2}\left[\int_0^T |(K^* - K^*_{hk})\psi|_h^2\,dt\right]^{1/2}.$$

Since, by (1.5),

$$\left|\frac{\partial}{\partial\nu}(I-P_h)\phi\right|_{L_2(\Gamma)} \le Ch^{q+1-3/2}|\phi|_{H^{q+1}(\Omega)}, \tag{4.6}$$

and recalling the regularity of T, we get

$$\begin{aligned}\left|\frac{\partial}{\partial\nu}(I-P_h)T[K^*\psi]_2\right|_{L_2(\Sigma)} &\le Ch^{q-1/2}|T(K^*\psi)_2|_{L_2(0,T;H^{q+1}(\Omega))}\\ &\le Ch^{q-1/2}|[K^*\psi]_2|_{L_2[0,T;H^{q-1}(\Omega)]}.\end{aligned} \tag{4.7}$$

For $\alpha \ge 0$ and smooth domains we always have [21]

$$\mathscr{D}(T^{-\alpha/2}) \subset H^\alpha(\Omega). \tag{4.8}$$

In the case of polygonal domains, the values of α for which (4.8) holds may be restricted by some $\alpha_0 > 0$. For convex domains we always have $\alpha_0 \ge 2$ [11].

By virtue of (3.10), (1.23), and (4.8), we have for $q \le \alpha_0 - 1$

$$|[K^*\psi]_2|_{L_2[0,T;H^{q-1}(\Omega)]} \le C|K^*\psi|_{L_2[0,T;\mathscr{D}(A^q)]} \le C|\psi|_{L_2[0,T;\mathscr{D}(A^q)]}. \tag{4.9}$$

Combining (4.7) and (4.9) yields

$$\left|\frac{\partial}{\partial\nu}(I-P_h)T[K^*\psi]_2\right|_{L_2(\Sigma)} \le Ch^{q-1/2}|\psi|_{L_2[0,T;\mathscr{D}(A^q)]}. \tag{4.10}$$

From (4.3) we have that

$$\left|\frac{\partial}{\partial\nu}(P_hT-T_h)[K^*\psi]_2\right|_{L_2(\Sigma)} \le Ch^{-3/2}|[P_hT-T_h][K^*\psi]_2|_{L_2(Q)}$$

and by the elliptic approximation estimates (see [23])

$$\le Ch^{-3/2}h^{q+1}|[K^*\psi]_2|_{L^2[0,T;H^{q-1}(\Omega)]}.$$

By the same estimates as those leading to (4.10), this is

$$\le Ch^{q-1/2}|\psi|_{L_2[0,T;\mathscr{D}(A^q)]}. \tag{4.11}$$

Combining the results of (4.2), (4.5), (4.10), and (4.11) yields

Lemma 4.1. *For $q \le \alpha_0 - 1$ we have*

$$\begin{aligned}|L^*\psi - L^*_{hk}\psi|_{L_2(\Sigma)} &\le Ch^{q-1/2}|\psi|_{L_2[0,T;\mathscr{D}(A^q)]}\\ &\quad + Ch^{-1/2}\left(\int_0^T |(K^*-K^*_{hk})\psi|_h^2\,dt\right)^{1/2}\end{aligned} \tag{4.12}$$

In order to provide an explicit bound for $L^* - L^*_{hk}$, we need to estimate $|K^* - K^*_{hk}|_h$. Notice first that, by the result of Lemma 2.1 and (3.12) we can infer that

$$\int_0^T |K_{nk}f|_h^2\,dt = \int_0^T |z^{hk}|_h^2\,dt \le C\int_0^T |f|_h^2\,dt. \tag{4.13}$$

Hence, we have

Proposition 4.1. *There holds*

$$\int_0^T |K^*_{hk}\psi|_h^2\,dt \le C\int_0^T |\psi|_h^2\,dt. \tag{4.14}$$

We shall prove

Lemma 4.2. *We have*

$$(4.15)\quad \begin{aligned}&\int_0^T |K_{hk}^*\psi - K^*\psi|_h^2\,dt\\ &\leq C\left[\int_0^T |(P^x - I)(K^*\psi)_t|_h^2\,dt + \int_0^T |(P^t - I)AK^*\psi|_h^2\,dt\right.\\ &\qquad\left.+\int_0^T |(I - P^tP^x)K^*\psi|_h^2\,dt\right],\end{aligned}$$

where P^t, P^x *are defined by* (4.21), (4.22) *below.*

Proof. By (3.8) and Proposition 3.1,

$$K_{hk}^*\psi - K^*\psi = w^{hk} - w,$$

where w satisfies (3.9) and w_{hk} satisfies (3.16). Thus, we need to estimate

$$\int_0^T |w - w^{hk}|_h^2\,dt.$$

We multiply (3.9) by $\phi^{hk} \in W_{hk}^2$, take the inner product $((\ ,\ ))_h$, and integrate by parts. This gives

$$(4.16)\quad \begin{aligned}&\int_0^T ((w, \phi_t^{hk}))_h\,dt + ((w(t=0), \phi^{hk}(0)))_h \int_0^T ((Aw, \phi^{hk}))_h\,dt\\ &= \int_0^T ((f, \phi^{hk}))_h\,dt.\end{aligned}$$

On the other hand, since $w_t = -Aw - f$,

$$(4.17)\quad w(t=0) = \int_0^T (Aw + f)\,dt.$$

From (4.16) and (4.17),

$$(4.18)\quad \begin{aligned}&\int_0^T ((w, \phi_t^{hk}))_h\,dt + \left(\left(\int_0^T (Aw + f)\,dt, \phi^{hk}(0)\right)\right)_h - \int_0^T ((Aw, \phi^{hk}))_h\,dt\\ &= \int_0^T ((f, \phi^{hk}))_h\,dt.\end{aligned}$$

Subtracting equation (4.18) from (3.16) yields

$$(4.19)\quad \begin{aligned}&\int_0^T ((w^{hk} - w, \phi_t^{hk}))_h\,dt + \int_0^T ((Aw - A_hw^{hk}, \phi^{hk}))_h\,dt\\ &\qquad + \left(\left(\int_0^T [A_hw^{hk} - Aw]dt, \phi^{hk}(0)\right)\right)_h = 0.\end{aligned}$$

With $\hat{w}^{hk}$ any element in $[\dot{W}^{hk}]^2$, denote $e \equiv w^{hk} - \hat{w}^{hk}$. From (4.19),

$$\int_0^T ((e, \phi_t^{hk}))_h\,dt - \int_0^T ((A_h e, \phi^{hk}))_h\,dt + \left(\left(\int_0^T A_h e\,dt, \phi^{hk}(0)\right)\right)_h$$
$$(4.20)\qquad = \int_0^T ((w - \hat{w}^{hk}, \phi_t^{hk}))_h\,dt - \int_0^T ((Aw - A\hat{w}^{hk}, \phi^{hk}))_h\,dt + \left(\left(\int_0^T (Aw - A\hat{w}^{hk})\,dt, \phi^{hk}(0)\right)\right)_h .$$

We shall introduce the following approximating operators:

$$P^t : L_2[0, T; H] \to H \otimes (\dot{S}_k^p)^2$$

defined by

$$(4.21)\qquad \int_0^T ((P^t \psi, \phi_t^k))_h = \int_0^T ((\psi, \phi_t^k))_h, \qquad \phi^k \in H \otimes (S_k^p)^2,$$

and

$$P^x : L_2[0, T; \mathscr{D}(A)] \to [S_h^q(\Omega) \otimes L_2[0, T]]^2$$

defined by

$$(4.22)\qquad \int_0^T ((AP^x \psi, \phi^h))_h\,dt \equiv \int_0^T ((A\psi, \phi^h))_h\,dt, \qquad \phi^h \in (S_h^q)^2 .$$

It will be seen later that the definition in (4.22) coincides with the one in (1.32). We select $\hat{w}^{hk} = P^t P^x w \in (\dot{W}_{hk})^2$. Then, by (4.21),

$$(4.23)\qquad \begin{aligned} \int_0^T ((w - \hat{w}^{hk}, \phi_t^{hk}))_h\,dt &= \int_0^T ((w - P^t P^x w, \phi_t^{hk}))_h\,dt \\ &= \int_0^T ((w - P^x w, \phi_t^{hk}))_h\,dt \\ &= -((w(t=0) - P^x w(t=0), \phi^{hk}(t=0)))_h \\ &\quad + \int_0^T (((P^x - I)w_t, \phi^{hk}))_h\,dt . \end{aligned}$$

Similarly, using (4.22) and commutativity of P^t and P^x, we get

$$(4.24)\qquad \begin{aligned} \int_0^T ((Aw - AP^t P^x w, \phi^{hk}))_h\,dt &= \int_0^T ((A(I - P^t)w, \phi^{hk}))_h\,dt \\ &= \int_0^T (((I - P^t)Aw, \phi^{hk}))_h\,dt , \end{aligned}$$

$$(4.25)\qquad \begin{aligned} ((Aw - A\hat{w}^{hk}, \phi^{hk}))_h &= ((Aw - AP^x P^t w, \phi^{hk}))_h \\ &= (((I - P^t)Aw, \phi^{hk}))_h . \end{aligned}$$

Collecting the results of (4.23)–(4.25) and recalling (4.20), we obtain the following error equation:

(4.26)
$$\int_0^T ((e, \phi_t^{hk}))_h\, dt - \int_0^T ((A_h e, \phi^{hk}))_h\, dt + \left(\left(\int_0^T A_h e\, dt, \phi^{hk}(0)\right)\right)_h$$
$$= \int_0^T (((P^x - I)w_t, \phi^{hk}))_h\, dt - \int_0^T (((I - P^t)Aw, \phi^{hk}))_h\, dt$$
$$+ \left(\left(\int_0^T [(I - P^t)Aw]\, dt, \phi^{hk}(t=0)\right)\right)_h$$
$$- (((I - P^x)w(t=0), \phi^{hk}(t=0)))_h .$$

Denote

$$F \equiv (P^x - I)w_t(I - P^t)Aw . \tag{4.27}$$

Then, since $w(T) = 0$,

(4.28)
$$\left(\left(\int_0^T F(t)\, dt, \phi^{hk}(t=0)\right)\right)_h = ((I - P^x)w(t=0), \psi^{hk}(t=0))_h$$
$$- \left(\left(\int_0^T (I - P^t)Aw\, dt, \phi^{hk}(t=0)\right)\right)_h .$$

By (4.26)–(4.28),

(4.29)
$$\int_0^T ((e, \phi_t^{hk}))_h\, dt - \int_0^T ((A_h e, \phi^{hk}))_h\, dt + \left(\left(\int_0^T A_h e\, dt, \phi^{hk}(t=0)\right)\right)_h$$
$$= \int_0^T ((F(t), \phi^{hk}))_h\, dt - \left(\left(\int_0^T F(t)\, dt, \phi^{hk}(t=0)\right)\right)_h .$$

After recalling (3.16), we obtain

Proposition 4.2. *With* $e = w^{hk} - P^t P^x w$, *we have* $e = K_{hk}^*[F]$, *where* F *is defined by* (4.27).

Now the result of Lemma 4.2 follows from Proposition 4.1, Proposition 4.2, and the triangle inequality. □

We establish next the error estimates for the projections P^t and P^x (here the arguments are similar to those of [1]).

Proposition 4.3. *For any* $\overline{p} \le p$, $\overline{q} \le q + 1 \le \alpha_0$, *we have*

$$\int_0^T |(P^t - I)\psi|_h^2\, dt \le Ck^{2\overline{p}} |\psi|^2_{H^{\overline{p}}[0, T; H]}, \tag{4.30}$$

$$\int_0^T |(P^x - I)\psi|_h^2\, dt \le Ch^{2\overline{q}} |\psi|^2_{L_2(0, T; \mathscr{D}(A^{\overline{q}}))} . \tag{4.31}$$

From Proposition 4.3 and Lemma 4.2 we obtain at once

Lemma 4.3. *From any* $\bar{p} \le p$, $\bar{q} \le q + 1 \le \alpha_0$, *we have*
(4.32)
$$\int_0^T |K_{hk}^* \psi - K^* \psi|_h^2 \, dt \le C[k^{2\bar{p}} + h^{2\bar{q}}][|[K^*\psi]_t|^2_{L_2(0,T;\mathscr{D}(A^{\bar{q}}))} + |AK^*\psi|^2_{H^{\bar{p}}(0,T;H)}].$$

Proof of Proposition 4.3. Let $P^t(\psi) \equiv (\psi_1^k, \psi_1^k)$. Let the operator R_k: $(L_2(0, T \times \Omega) \to \dot{S}_k^p \otimes L_2(\Omega)$ be defined as

$$\int_0^T (R_k f, \dot{\phi}_k)_\Omega \, dt = \int_0^T (f, \dot{\phi}_k)_\Omega \, dt, \quad \phi_k \in L_2(\Omega) \otimes (S_k^p[0, T]).$$

Then

$$\psi_1^k = R_k \psi_1 \quad \text{and} \quad T_h^{1/2} \psi_2^k = T_h^{1/2} R_k \psi_2 = R_k T_h^{1/2} \psi_2.$$

Hence,

$$\begin{aligned}\int_0^T |P^t \psi - \psi|_h^2 \, dt &= \int_0^T |\psi_1^k - \psi_1|^2_{L_2(\Omega)} \, dt + \int_0^T |T_h^{1/2}(\psi_2^k - \psi_2)|^2_{L_2(\Omega)} \, dt \\ &= \int_0^T |(I - R_k)\psi_1|^2_{L_2(\Omega)} \, dt + \int_0^T |(I - R_k) T_h^{1/2} \psi_2|^2_{L_2(\Omega)} \, dt \\ &\le C k^{2\bar{p}} [|\psi_2|^2_{H^{\bar{p}}[0,T;L_2(\Omega)]} + |T_h^{1/2} \psi_2|^2_{H^{\bar{p}}[0,T;L_2(\Omega)]}] \\ &\le C k^{2\bar{p}} |\psi|^2_{H^{\bar{p}}[0,T;H]},\end{aligned}$$

which proves (4.30).

As for (4.31), we denote $P^x \psi \equiv (\psi_1^h, \psi_2^h)$. It is straightforward to verify that

$$\psi_1^h = P_h^1 \psi_1 \quad \text{and} \quad \psi_2^h = P_h \psi_2,$$

where P_h is an $L_2(\Omega)$ projection onto $S_h^q(\Omega)$ and P_h^1 is an elliptic H^1 projection onto $S_h^q(\Omega)$. Hence,

$$\begin{aligned}|(P^x - I)\psi|_h &\le |\psi_1^h - \psi_1|_{L_2(\Omega)} + |T_h^{1/2}(\psi_2^h - \psi_2)|_{L_2(\Omega)} \\ &\le |(P_h^1 - I)\psi_1|_{L_2(\Omega)} \le C[h^{q+1} |\psi_1|_{H^{q+1}(\Omega)}],\end{aligned} \tag{4.33}$$

where the last inequality follows from standard elliptic estimates. Combining (4.33) with (4.8), we arrive at (4.31). □

Standard semigroup methods applied to the equation (3.8) yield

$$\begin{aligned}\left| \frac{d}{dt} K^* \psi \right|_{L_2[0,T;\mathscr{D}(A^q)]} &\le C[|\psi|_{L_2[0,T;\mathscr{D}(A)]} + |\psi|_{H^1[0,T;\mathscr{D}(A^q)]}] \\ &\le C|\psi|_{H^1[0,T;\mathscr{D}(A^q)]},\end{aligned} \tag{4.34}$$

$$|AK^*\psi|_{H^p[0,T;H]} \le C[|\psi|_{H^p[0,T;H]} + |\psi|_{H^1[0,T;\mathscr{D}(A^p)]}] \tag{4.35}$$

and by an intermediate derivative theorem

$$\le C[|\psi|_{H^p[0,T;H]} + |\psi|_{L_2[0,T;\mathscr{D}(A^{p+1})]}].$$

Combining the result of Lemma 4.3 with (4.34) and (4.35) yields

Corollary 4.1. *For any* $\overline{p} \leq p$, $\overline{q} \leq q+1 \leq \alpha_0$, *we have*

$$\int_0^T |K_{hk}^*\psi - K^*\psi|_h^2\,dt \leq C[k^{2\overline{p}} + h^{2\overline{q}}][|\psi|_{H^1[0,T;\mathscr{D}(A^{\overline{q}})]} + |\psi|^2_{H^1[0,T;\mathscr{D}(A^{\overline{p}})]} + |\psi|^2_{H^{\overline{p}}[0,T;H]}].$$

Lemma 4.1 combined with Corollary 4.1 yields

Corollary 4.2. *For* $q+1 \leq \alpha_0$, *we have*

$$\begin{aligned}(4.36)\quad & |(L^* - L_{hk}^*)\psi|_{L_2(\Sigma)} \\ &\leq Ch^{q-1/2}|\psi|_{L_2[0,T;\mathscr{D}(A^q)]} \\ &\quad + Ch^{-1/2}[k^p + h^q][|\psi|_{H^1[0,T;\mathscr{D}(A^q)]} + |\psi|_{H^1[0,T;\mathscr{D}(A^p)]} + |\psi|_{H^p[0,T;H]}].\end{aligned}$$

5. Proof of Theorem 1, Corollary 1, and Theorem 2

From Proposition 4.1 and (4.4) we obtain

$$\begin{aligned}&\left|\frac{\partial}{\partial\nu}T_h\left[K_{hk}^*\begin{pmatrix}0\\ \psi_2 - T_h^{-1}T\psi_2\end{pmatrix}\right]_2\right|_{L_2(\Gamma)} \leq Ch^{-1/2}\left|K_{hk}^*\begin{pmatrix}0\\ \psi_2 - T_h^{-1}T\psi^2\end{pmatrix}\right|_h \\ &\quad\leq Ch^{-1/2}|T_h^{1/2}(I - T_h^{-1}T\psi_2)|_{L_2(\Omega)} \leq Ch^{-1/2}|(T_h - T)|_{H_0^1(\Omega)}.\end{aligned}$$

Now the result of Theorem 1 (part i) follows directly from Corollary 4.2, Lemma 3.1, and the fact that for $\overline{q} \leq q$ standard elliptic estimates give, with $q+1 \leq \alpha_0$,

$$\begin{aligned}(5.1)\quad |(T_h - T)\psi_2|_{L_2[0,T;H_0^1(\Omega)]} &\leq Ch^{\overline{q}}|T\psi_2|_{L_2[0,T;H^{\overline{q}+1}(\Omega)]} \\ &\leq Ch^{\overline{q}}|\psi_2|_{L_2[0,T;H^{\overline{q}-1}(\Omega)]} \leq Ch^{\overline{q}}|\psi|_{L_2[0,T;\mathscr{D}(A^{\overline{q}})]}.\end{aligned}$$

As for part (ii), it is enough to notice that, instead of using (4.34)–(4.35), one can use

$$(5.2)\quad \left|\frac{d}{dt}K^*\psi\right|_{L_2[0,T;\mathscr{D}(A^q)]} \leq C|\psi|_{L_2[0,T;\mathscr{D}(A^{q+1})]},$$

$$(5.3)\quad |AK^*\psi|_{H^p[0,T;H]} \leq C[|\psi|_{L_2[0,T;\mathscr{D}(A^{p+1})]} + |\psi|_{H^{p-1}[0,T;\mathscr{D}(A)]}].$$

By the above regularity results and Lemma 4.3, Corollary 4.2 reads now

Corollary 5.1. *For* $q+1 \leq \alpha_0$ *we have*

$$\begin{aligned}(5.4)\quad & |(L^* - L_{hk}^*)\psi|_{L_2(\Sigma)} \\ &\leq Ch^{q-1/2}|\psi|_{L_2[0,T;\mathscr{D}(A^q)]} \\ &\quad + Ch^{-1/2}[k^p + h^q][|\psi|_{L_2[0,T;\mathscr{D}(A^{q+1})]} + |\psi|_{L_2[0,T;\mathscr{D}(A^{p+1})]} \\ &\quad + |\psi|_{H^{p-1}[0,T;\mathscr{D}(A)]}].\end{aligned}$$

Thus, the result of part (ii) of Theorem 1 is a direct consequence of Corollary 5.1 and Lemma 3.1, supported with (5.1).

Proof of Corollary 1. Since $y - y^{hk} = u_1 - u_1^{hk}$, by the same arguments as those in Lemma 3.1 we obtain

$$(5.5)\qquad |y - y^{hk}|_{H^{-p}} \leq \sup_{\psi_1} \frac{|(L^* - L^*_{hk})\binom{\psi_1}{0}|_{L_2(\Sigma)}}{|\psi_1|_{H^p}} |g|_{L_2(\Sigma)}.$$

By the result of Corollary 5.1, specialized to $\psi_2 = 0$, we obtain

$$(5.6)\qquad \begin{aligned} &\left|(L^* - L^*_{hk})\begin{pmatrix}\psi_1\\0\end{pmatrix}\right|_{L_2(\Sigma)} \\ &\leq Ch^{q-1/2}|\psi_1|_{L_2[0,T;\mathscr{D}(T^{-q/2})]} \\ &\quad + Ch^{-1/2}[k^q + h^q][|\psi_1|_{L_2[0,T;\mathscr{D}(T^{-(q+1)/2})]} + |\psi_1|_{H^{p-1}[0,T;H_0^1(\Omega)]}] \\ &\leq Ch^{-1/2}(h^q + k^q)|\psi_1|_{H^{p+1,p}(Q)}. \end{aligned}$$

Corollary 1 now follows from (5.5) and (5.6). □

The proof of Theorem 2 follows from Lemma 2.1, the error estimates in (1.33), (1.34) combined with the standard (see [23] and [1]) "positive norm" estimates for the error function $e^{hk} \equiv u^{hk} - P^x Q^t u$.

APPENDIX

Following Nitsche, we define the bilinear symmetric form

$$a_{\beta_0}(x, y) \equiv (\nabla x, \nabla y)_\Omega - \left(\frac{\partial}{\partial \nu}x, y\right)_\Gamma - \left(x, \frac{\partial}{\partial \nu}y\right)_\Gamma + \beta_0(x, y)_\Gamma,$$

with the constant $\beta_0 > 0$. The solution y to (1.1) (corresponding to smooth boundary data) satisfies $y_{tt} = \Delta y = \Delta(y - Dg)$. Taking the inner product in $L_2(\Omega)$ with an arbitrary test function $\phi \in V$, where $V \equiv \{\phi \in H^1(\Omega);\ \phi|_\Gamma \in H^1(\Gamma)\}$, we obtain

$$(\text{A.1})\qquad (-\Delta(y - Dg), \phi)_\Omega = a_{\beta_0}(y - Dg, \phi).$$

Here we have used Green's formula and the fact that $y - Dg = 0$ on Γ. On the other hand, one more application of Green's formula yields

(A.2)

$$\begin{aligned} a_{\beta_0}(Dg, \phi) &= (\nabla Dg, \nabla\phi)_\Omega - \left(\frac{\partial}{\partial \nu}Dg, \phi\right)_\Gamma - \left(Dg, \frac{\partial}{\partial \nu}\phi\right)_\Gamma + \beta_0(g, \phi)_\Gamma \\ &= (\Delta Dg, \phi)_\Omega - \left(g, \frac{\partial}{\partial \nu}\phi\right)_\Gamma + \beta_0(g, \phi)_\Gamma \\ &= -\left(g, \frac{\partial}{\partial \nu}\phi\right)_\Gamma + \beta_0(g, \phi)_\Gamma. \end{aligned}$$

From (1.1), (A.1), and (A.2) it follows that the variables $u_1 \equiv y$, $u_2 = y_t$ satisfy

$$(\text{A.3})\qquad \begin{cases} (u_{1t}, \phi_2)_\Omega = (u_2, \phi_1)_\Omega \quad \text{for } \phi_1 \in V, \\ (u_{2t}, \phi_2)_\Omega + a_{\beta_0}(u_1, \phi_2) = -\left(g, \frac{\partial}{\partial\nu}\phi_2\right)_\Gamma + \beta_0(g, \phi_2)_\Gamma \quad \text{for } \phi_2 \in V. \end{cases}$$

The variational formulation in (A.3) combined with time discretization is the basis of our approximation. Indeed, let $\beta_0 \equiv \beta h^{-1}$, $\beta > 0$. We seek $u^{hk} = (u_1^{hk}, u_2^{hk}) \in W_{hk}^2$, where $W_{hk} = S_h^q(\Omega) \otimes S_k^p(0, T)$ and $S_h^q(\Omega) \subset V$, such that

$$\int_0^T (u_{1t}^{hk}, \phi_{1t}^{hk})_\Omega\, dt = \int_0^T (u_2^{hk}, \phi_{1t}^{hk})_\Omega\, dt \qquad \text{for all } \phi_1^{hk} \in W_{hk},$$

$$\int_0^T (u_{2t}^{hk}, \phi_{2t}^{hk})_\Omega\, dt + a_{\beta_0}(u_1^{hk}, \phi_{2t}^{hk})\, dt$$
$$= -\int_0^T \left(g, \frac{\partial}{\partial\nu}\phi_{2t}^{hk}\right)_\Gamma dt + \beta h^{-1}\int_0^T (g, \phi_{2t}^{hk})_\Gamma\, dt, \qquad \phi_2^{hk} \in W_{hk}.$$

Bibliography

1. A. K. Aziz and P. Monk, *Continuous finite elements in space and time for the heat equation*, Math. Comp. **52** (1989), 255–274.
2. G. A. Baker and J. H. Bramble, *Semidiscrete and single step fully discrete approximations for second order hyperbolic equations*, RAIRO Numer. Anal. **13** (1979), 75–100.
3. L. Bales, *Semidiscrete and single step fully discrete approximations for second order hyperbolic equations with time-dependent coefficients*, Math. Comp. **43** (1984), 383–414.
4. ——, *Semidiscrete and single step fully discrete finite element approximations for second order hyperbolic equations with nonsmooth solutions*, Math. Models Methods Appl. Sci. **27** (1993), 55–63.
5. ——, *Some remarks on post-processing and negative norm estimates for approximations to nonsmooth solutions of hyperbolic equations*, Comm. Appl. Numer. Methods **9** (1993), 701–710.
6. L. Bales and J. Lasiecka, *Continuous finite elements in space and time for the nonhomogeneous wave equation*, Comput. Math. Appl. **27** (1994), 91–102.
7. A. Bensoussan, G. DaPrato, M. Delfour, and S. K. Mitter, *Representation and control of infinite dimensional systems*, Vol. I, Birkhäuser, Boston, MA, 1992.
8. J. H. Bramble and A. H. Schatz, *Higher order local accuracy by averaging in the finite element method*, Math. Comp. **31** (1977), 94–111.
9. G. Choudury and I. Lasiecka, *Optimal convergence rates for semidiscrete approximations of parabolic problems with nonsmooth boundary data*, Numer. Funct. Anal. Optim. **12** (1991), 469–485.
10. D. French and T. Peterson, *A space-time finite element method for the second order wave equation*, Centre for Mathematics and its Applications, Australian National University, Canberra CMA-MR13-(1991).
11. P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, London, 1985.
12. ——, *Exact controllability of the wave equation in presence of corners and cracks*, Lecture Notes in Pure and Appl. Math., vol. 135, Marcel Dekker, New York, 1991, pp. 213–225.
13. K. Hoffman and R. Kunze, *Linear algebra*, 2nd ed., Prentice-Hall, Englewood Cliffs, NJ, 1971.
14. B. Hulme, *One step polynomial Galerkin methods for initial value problems*, Math. Comp. **26** (1972), 415–426.
15. C. Johnson and U. Nävert, *An analysis of some finite element methods for advection-diffusion problems*, Analytical and Numerical Approaches to Asymptotic Problems in Analysis (L. S. Frank and A van der Sluis, eds.), North-Holland, Amsterdam, 1981, pp. 99–116.
16. I. Lasiecka and J. Sokolowski, *Semidiscrete approximations of hyperbolic boundary value problems with nonhomogeneous Dirichlet boundary conditions*, SIAM J. Math. Anal. **20** (1989), 1366–1387.
17. I. Lasiecka, J. Lions, and R. Triggiani, *Non homogeneous boundary value problems for second order hyperbolic operators*, J. Math. Pure Appl. **65** (1986), 149–192.

18. P. D. Lax and M. S. Mock, *The computation of discontinuous solutions of linear hyperbolic equations*, Comm. Pure Appl. Math. **31** (1978), 423–430.

19. J. L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués*, Vol. I, Masson, Paris, 1988.

20. ——, *Espaces d'interpolation et domaines de puissances fractionnaires d'opérateurs*, J. Math. Soc. Japan **14** (1962), 233–241.

21. J. L. Lions and E. Magenes, *Nonhomogeneous boundary value problems and applications*, Springer-Verlag, New York, 1972.

22. A. Pazy, *Semi-groups of linear operators and applications to partial differential equations*, Springer-Verlag, New York, 1983.

23. V. Thomée, *Galerkin finite element methods for parabolic problems*, Lecture Notes in Math., vol. 1054, Springer-Verlag, Berlin, 1984.

24. R. Winther, *A stable finite element method for initial-boundary value problems for first order hyperbolic systems*, Math. Comp. **36** (1981), 65–86.

Department of Mathematics, Cornell University, Ithaca, New York 14853

Department of Applied Mathematics, University of Virginia, Charlottesville, Virginia 22903

E-mail address: `il2v@amsun24.apma.virginia.edu`