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A FRAMEWORK FOR BLOCK ILU FACTORIZATIONS USING BLOCK-SIZE REDUCTION

TONY F. CHAN AND PANAYOT S. VASSILEVSKI

ABSTRACT. We propose a block ILU factorization technique for block tridiagonal matrices that need not necessarily be M-matrices. The technique explores reduction by a coarse–vector restriction of the block size of the approximate Schur complements computed throughout the factorization process. Then on the basis of the Sherman-Morrison-Woodbury formula these are easily inverted. We prove the existence of the proposed factorization techniques in the case of (nonsymmetric, in general) M-matrices. For block tridiagonal matrices with positive definite symmetric part we show the existence of a limit version of the factorization (exact inverses of the reduced matrices are needed). The theory is illustrated with numerical tests.

1. INTRODUCTION

Consider a block tridiagonal M-matrix, or a block tridiagonal positive definite matrix, in both cases not necessarily symmetric,

$$A = \begin{pmatrix} A_{11} & A_{12} & & & 0 \\ A_{21} & A_{22} & A_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-1,n-2} & A_{n-1,n-1} & A_{n-1,n} \\ 0 & & & A_{n,n-1} & A_{nn} \end{pmatrix};$$

the blocks $A_{i,j}$ are of dimension $n_i \times n_j$, and also $A_{i,j}$ are assumed to be sparse. This paper deals with the construction of block ILU (incomplete or approximate LU) factorizations of sparse matrices of this form that typically arise in the finite difference or finite element discretization of second-order elliptic problems. There are already several block ILU techniques proposed in the literature, e.g., Kettler [15], Axelsson, Brinkkemper, and Il'in [3], Meurant [18]. General approaches were proposed in Concus, Golub, and Meurant [10],

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Axelsson [1], see also Axelsson [2], Axelsson and Polman [5]. These methods work extremely well for 2-D (two-dimensional) problems; they exhibit very good vectorization properties and reasonably fast convergence, see, e.g., Meurant [18], Concus, Golub, and Meurant [10], Axelsson and Eijkhout [4]. However, for 3-D problems their performance (both for blocks corresponding to grid lines as well as to grid planes) is not as satisfactory, cf., e.g., Axelsson and Eijkhout [4]. There are recent developments in the construction of special approximations to the approximate Schur complements computed throughout the factorization process, cf. Axelsson and Polman [6], Wittum [24]. For example, in Wittum [24] the method of deriving approximate inverses gives very accurate results but the cost is relatively high. Also the theory seems to be mostly applicable for 2-D problems.

The purpose of the present paper is to derive a new class of block ILU factorization methods on the basis of reduction of the block size by a "coarse-grid" restriction to derive better and inexpensive approximations of the successive Schur complements during the factorization process. Then the inverses of these Schur complements can be computed on the basis of the well-known Sherman-Morrison-Woodbury formula (cf., e.g., Golub and van Loan [14]). We consider approximations for the inverses of these approximate Schur complements based mostly on sparse approximate inverses of the blocks A_{ii} of the original matrix A. However, approximations based on the sine transform, or approximations based on the probing technique studied in detail by Chan and Mathew [8] and used also by Keyes and Gropp [16], Axelsson and Polman [6] and Wittum [24], as well as certain polynomial approximations are possible as well. Most of these cases will not be considered here. Our major concern in the present paper is the derivation and the proof of the existence of the proposed approximate block factorization matrices.

Finally, we point out that the new method proposed in the present paper offers a potential for solving coupled systems of differential equations because it does not require the M-matrix property (or more generally, the H-matrix property, see Polman [19] and Kolotilina and Polman [17]), which was a limitation for the earlier (block) ILU factorization methods.

The outline of the paper is as follows. Some preliminary facts are presented in §2. In §3 a general existence result is provided for generally nonsymmetric M-matrices for a specific choice of the restriction matrices needed for the reduction of the block size. The assumptions made in §3 are verified in §4. In §5 some approximations of the inverses of the approximate Schur complements (that are computed throughout the factorization process) based on their representation obtained using the Sherman-Morrison-Woodbury formula are discussed. In §6 the existence of the factorization in the limit case, i.e., when the exact inverses of the reduced Schur complements are used, for block tridiagonal matrices with positive definite symmetric part is proved. Finally in the last section the new method of block ILU matrices is illustrated by numerical examples for M-matrices resulting from model second-order elliptic problems for 2-D domains. The non-M-matrix case is considered in a forthcoming report.

2. Preliminaries

In this section we present the commonly used block ILU factorization technique (cf., e.g., Concus, Golub, and Meurant [10] and Axelsson and Polman

[5]), and then we introduce our new method pointing out the main differences.

First we outline a general block ILU scheme.

Let $\{A_{ii}, A_{i,i-1}, A_{i,i+1}\}$ belong to a class of sparse matrices with a simple structure (e.g., band, or with a fixed number of nonzero diagonals, etc.).

(A General Block ILU Scheme). Set

$$Z_1 = A_{11}, \quad X_1 = \text{Approx}_1\left(Z_1^{-1}\right);$$

for $i = 2, \dots, n$ compute

$$Z_i = A_{ii} - A_{i,i-1}X_{i-1}A_{i-1,i}$$

and let

$$X_i = \text{Approx}_1\left(Z_i^{-1}\right).$$

Finally, let

$$Y_i = \text{Approx}_2(Z_i), \quad i = 1, 2, \dots, n.$$

Then the block ILU factorization matrix is defined to be

$$C = \begin{pmatrix} Y_1 & & & 0 \\ A_{21} & Y_2 & & \\ & \ddots & \ddots & \\ 0 & & A_{nn-1} & Y_n \end{pmatrix} \begin{pmatrix} I & Y_1^{-1}A_{12} & & 0 \\ & I & Y_2^{-1}A_{23} & \\ & & \ddots & \ddots \\ 0 & & & I \end{pmatrix}. \quad \square \tag{2.1}$$

The approximations $X_i = \text{Approx}_1\left(Z_i^{-1}\right)$ and Y_i^{-1} based on $\text{Approx}_2(\cdot)$ may be the same. The role of $\text{Approx}_1(\cdot)$ is to control a prespecified sparsity structure of the approximate Schur complements $\{Z_i\}$, and the $\text{Approx}_2(\cdot)$ is meant to either control a prescribed sparsity pattern of Y_i, and hence make them easily factored or, if the blocks Y_i^{-1} are explicitly formed, make their application to a vector easily computed. For example, in the case of a block tridiagonal matrix arising from the discretization of 2-D (two-dimensional) second-order elliptic equations on a rectangular domain on a uniform mesh, using piecewise linear basis functions, the blocks $\{A_{ii}\}$ are (scalar) tridiagonal and the blocks $A_{i,i-1}$, $A_{i-1,i}$ are diagonal. Then it is natural to keep the blocks $\{Z_i\}$ banded. To achieve this, one can use for $\text{Approx}_1(\cdot)$ various banded approximations of the inverses of the banded (by construction) approximate Schur complements $\{Z_i\}$. For more details, cf., e.g., Concus, Golub, and Meurant [10], Axelsson and Polman [5], or Vassilevski [21], [22]. The motivation for banded approximations of $\{Z_i^{-1}\}$ is that the inverses of band matrices have certain decay rate, cf., e.g., Demko, Moss, and Smith [11], Vassilevski [21]. The decay rate is substantial if these band matrices are sufficiently diagonally dominant. This is the case for the blocks $\{A_{ii}\}$ of A in the particular example of 2-D second-order elliptic problems. The second approximation, $\text{Approx}_2(\cdot)$ may be needed in a more general situation when the blocks $A_{i,i-1}$, $A_{i-1,i}$ are sparse but not diagonal. Then, in general, these blocks create additional fill-in in $\{Z_i\}$. That is why we may allow different approximations for the factorization process than in the computation of the final blocks $\{Y_i\}$ or $\{Y_i^{-1}\}$. One possibility is to choose

$$X_i = \text{Approx}_1\left(Z_i^{-1}\right) \equiv \text{banded approximation to } Z_i^{-1},$$

and

$$Y_i = \text{Approx}_2(Z_i) \equiv \text{banded approximation to } Z_i.$$

For more difficult problems, like those arising in the discretization of 3-D (three-dimensional) elliptic differential equations (including systems of differential equations), the approach of preserving certain sparsity patterns of the incomplete factorization matrix is not as clearly motivated. For more details concerning finite difference elliptic equations in 3-D, cf., e.g., Axelsson and Eijkhout [4]. More importantly, the incomplete factorization methods based on preserving certain sparsity patterns have proven existence basically only for M-matrices (or slightly more generally for H-matrices; see Polman [19] and Kolotilina and Polman [17]), and this is not the case, e.g., for matrices arising in the finite element or finite difference discretization of coupled systems of elliptic partial differential equations or for higher-order finite elements for single elliptic equations.

We now present the new method. It can be written in the general approximate block factorization scheme already presented. Its first main idea is in the construction of the blocks $\{X_i\}$ based on block-size reduction using certain restriction matrices. More precisely, let $\{R_i\}_{i=1}^n$ be a sequence of restriction matrices that transform a vector of the dimension of A_{ii} to a lower-dimensional vector space, say, of a small (fixed) size m.

In the application to discretization matrices A, the reduction matrices $\{R_i\}$ have a natural meaning. They can be viewed as transformation matrices of a fine-grid vector to a coarse-grid vector. (The vectors correspond either to a number of grid lines or to a number of grid-planes in 3-D.) In other words, R_i^T are sparse matrices and typically with nonnegative entries (say, for piecewise constant or piecewise linear interpolation). Then the new approximate factorization scheme takes the form:

Algorithm 1 (Block ILU scheme using block-size reduction). Set

$$Z_1 = A_{11}, \quad \tilde{Z}_1 = R_1 Z_1 R_1^T;$$

for $i = 2, \dots, n$ compute

$$\begin{aligned} V_{i-1} &= \text{Approx}\left(\tilde{Z}_{i-1}^{-1}\right), \\ X_{i-1} &= R_{i-1}^T V_{i-1} R_{i-1} \equiv \text{Approx}_1\left(Z_{i-1}^{-1}\right), \\ Z_i &= A_{ii} - A_{i,i-1} X_{i-1} A_{i-1,i}, \\ \tilde{Z}_i &= R_i Z_i R_i^T. \end{aligned}$$

Finally, let

$$Y_i = \text{Approx}_2(Z_i), \quad i = 1, 2, \dots, n. \quad \square$$

Here " Approx(·) " stands for an approximation to a given matrix.

It is clear that Algorithm 1 is a particular case of the general block ILU scheme.

Note that Algorithm 1 resembles in some sense the multigrid method. The block Z_i is first projected into a coarse space of smaller dimension (restriction) and then after finding an (approximate) inverse the result is taken back into a space of higher dimension (interpolation). However, this is performed only at the block level, similarly to the semicoarsening in multigrid, cf., e.g., Hackbusch [13].

The second main idea is that since $\{Z_i\}$ are low-rank updates of $\{A_{ii}\}$, one can construct approximation matrices $\{Y_i^{-1}\}$ on the basis of the expression of the exact inverses of $\{Z_i\}$ provided by the Sherman-Morrison-Woodbury formula (cf. Golub and van Loan [14, p. 51]).

More precisely, we have (see §5 for a detailed derivation)

$$\begin{aligned} Z_i^{-1} = {} & A_{ii}^{-1} + A_{ii}^{-1}A_{i,i-1}R_{i-1}^T \\ & \times\left[V_{i-1}^{-1} - R_{i-1}A_{i-1,i}A_{ii}^{-1}A_{i,i-1}R_{i-1}^T\right]^{-1} \\ & \times R_{i-1}A_{i-1,i}A_{ii}^{-1}. \end{aligned}$$

We may have to further approximate A_{ii}^{-1} (e.g., in the case of 3-D elliptic difference equations) by some B_{ii}^{-1}, and V_{i-1}^{-1} by $\tilde{Z}_{i-1}$, since $V_{i-1} = \text{Approx}\left(\tilde{Z}_{i-1}^{-1}\right)$, thus obtaining

$$\begin{aligned} Y_i^{-1} = {} & B_{ii}^{-1} + B_{ii}^{-1}A_{i,i-1}R_{i-1}^T \\ & \times\left[\tilde{Z}_{i-1} - R_{i-1}A_{i-1,i}B_{ii}^{-1}A_{i,i-1}R_{i-1}^T\right]^{-1} \\ & \times R_{i-1}A_{i-1,i}B_{ii}^{-1}. \end{aligned}$$

That is, $Y_i = \text{Approx}_2(Z_i)$ is defined by the above representation of Y_i^{-1}.

Note that $\tilde{Z}_{i-1} - R_{i-1}A_{i-1,i}B_{ii}^{-1}A_{i,i-1}R_{i-1}^T$ is an $m \times m$ matrix, i.e., of low dimension and can be easily inverted or approximately inverted, and the latter gives other approximate inverses Y_i^{-1} to Z_i. This will be considered in more detail in §5. One limiting choice, for example, could be $Y_i = Z_i$.

The purpose of constructing the approximate block factorization matrices C is to use them in a preconditioned conjugate gradient method. At every step of the iteration method one has to solve a system

$$C\mathbf{v} = \mathbf{w},$$

for some (residual) vector $\mathbf{w}$. Since C is factored, the above system is solved in the usual forward and backward recurrences.

(i) **Forward.** Solve

$$\begin{pmatrix} Y_1 & & & 0 \\ A_{21} & Y_2 & & \\ & \ddots & \ddots & \\ 0 & & A_{n,n-1} & Y_n \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_n \end{pmatrix} = \begin{pmatrix} \mathbf{w}_1 \\ \mathbf{w}_2 \\ \vdots \\ \mathbf{w}_n \end{pmatrix}$$

in the following steps :

$$\begin{aligned} \mathbf{z}_1 &= Y_1^{-1}\mathbf{w}_1, \\ \mathbf{z}_i &= Y_i^{-1}\left(\mathbf{w}_i - A_{i,i-1}\mathbf{z}_{i-1}\right), \quad i \geq 2. \end{aligned}$$

(ii) **Backward.** Solve

$$\begin{pmatrix} I & Y_1^{-1}A_{12} & & 0 \\ & I & Y_2^{-1}A_{23} & \\ & & \ddots & \ddots \\ 0 & & & I \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_n \end{pmatrix} = \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \\ \vdots \\ \mathbf{z}_n \end{pmatrix}$$

in the following steps :

$$\begin{aligned} \mathbf{v}_n &= \mathbf{z}_n, \\ \mathbf{v}_i &= \mathbf{z}_i - Y_i^{-1}A_{i,i+1}\mathbf{v}_{i+1} \quad \text{for } i = n-1 \text{ down to } 1. \end{aligned}$$

Note that solving systems with C involves only solving systems with the blocks Y_i and matrix-vector products with the sparse matrices $A_{i,i-1}$, $A_{i,i+1}$. Note also that if the blocks B_{ii}^{-1} were explicitly given and the blocks

$$\left(\tilde{Z}_{i-1} - R_{i-1}A_{i-1,i}B_{ii}^{-1}A_{i,i-1}R_{i-1}^T\right)^{-1}$$

were also explicitly computed, then the above solution process is based only on operations that are of the form vector-vector additions and matrix-vector products.

The objective of this paper is to study the existence of the above described block ILU factorization process. We also study the performance characteristics of the new method, depending upon the approximations of the inverses of the diagonal blocks A_{ii} of A. We used sparse approximations B_{ii}^{-1} to A_{ii}^{-1} but many other choices are also possible.

3. Existence theory for M-matrices

In this section we present the existence theory for the block ILU factorization process outlined in the previous section.

We make the following assumptions. We assume that the original, in general, nonsymmetric matrix A is an M-matrix, i.e., the off-diagonal entries of A are nonpositive and A^{-1} has nonnegative entries, cf. Varga [20]. Another equivalent definition of an M-matrix is that in addition to the nonpositiveness of the off-diagonal entries of A, there exists a positive vector $\mathbf{c}$ (i.e., with positive entries) such that $A\mathbf{c}$ is also positive. We also assume that the restriction matrices R_i have nonnegative entries. Finally, we make the following natural (for the applications) assumptions:

Assumption (I). We assume that the "coarse" matrix

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & & 0 \\ \tilde{A}_{12} & \tilde{A}_{22} & \tilde{A}_{23} & \\ & \ddots & \ddots & \\ 0 & & \tilde{A}_{n,n-1} & \tilde{A}_{nn} \end{pmatrix} = RAR^T, \tag{3.1}$$

where

$$R = \begin{pmatrix} R_1 & & & 0 \\ & R_2 & & \\ & & \ddots & \\ 0 & & & R_n \end{pmatrix},$$

is also an M-matrix. □

We will also need the principal submatrices $\tilde{A}^{(i)}$ of $\tilde{A}$ given by

$$\tilde{A}^{(i)} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & \\ & \ddots & \ddots & \\ 0 & & \tilde{A}_{i,i-1} & \tilde{A}_{ii} \end{pmatrix}. \tag{3.2}$$

There is also a technical assumption for the restriction matrices $\{R_i\}$.

Assumption (II). We assume that the intermediate coarse matrices $\hat{A}^{(i)}$ defined below are also M-matrices. □

Denote

$$\hat{R} = \begin{pmatrix} R_1 & & & & & \\ & \ddots & & & & \\ & & R_i & & & \\ & & & I & & \\ & & & & \ddots & \\ & & & & & I \end{pmatrix}.$$

Then,

(3.3)

$$\hat{A}^{(i+1)} \equiv \hat{R}A\hat{R}^T$$
$$= \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & & & & & \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & & & & \\ & \ddots & \ddots & \ddots & & & \\ & & \tilde{A}_{i,i-1} & \tilde{A}_{ii} & R_iA_{i,i+1} & & \\ & & & A_{i+1,i}R_i^T & A_{i+1,i+1} & A_{i+1,i+2} & \\ & & & & \ddots & \ddots & \ddots \\ & & & & A_{n-1,n-2} & A_{n-1,n-1} & A_{n-1,n} \\ & & & & & A_{n,n-1} & A_{nn} \end{pmatrix}.$$

The above assumptions hold, for example, in the case when A is a finite element matrix obtained from discretization of second-order selfadjoint elliptic problems on right triangles and piecewise linear basis functions. For the restriction matrices R_i in this case one can choose a piecewise constant restriction. A similar example can be obtained from cell-centered finite difference approximations of second-order elliptic problems, cf., e.g., Ewing, Lazarov, and Vassilevski [12], and Vassilevski, Petrova, and Lazarov [23]. In §4 we, however, present a general choice of the restriction matrices $\{R_i\}$ such that for any M-matrix A, also $\tilde{A}$ and $\hat{A}^{(i)}$ are M-matrices.

We specify now the algorithm of block ILU factorization. More precisely, in Algorithm 1 we restrict the class of matrices $\{V_{i-1}\}$ to satisfy certain constraints which make sense for M-matrices. Namely, we have

Algorithm 1′ (Block ILU factorization of M-matrices). Set

$$Z_1 = A_{11}, \quad \tilde{Z}_1 = R_1Z_1R_1^T.$$

For $i = 2, \ldots, n$ compute V_{i-1} such that

$$0 \le V_{i-1} \le \tilde{Z}_{i-1}^{-1} \quad \text{(componentwise)},$$

$$\begin{aligned} X_{i-1} &= R_{i-1}^T V_{i-1} R_{i-1} \equiv \mathrm{Approx}_1\left(Z_{i-1}^{-1}\right), \\ Z_i &= A_{ii} - A_{i,i-1}X_{i-1}A_{i-1,i}, \\ \tilde{Z}_i &= R_iZ_iR_i^T. \quad \square \end{aligned} \tag{3.4}$$

The construction of the blocks $Y_i = \mathrm{Approx}_2(Z_i)$ or rather Y_i^{-1} will be discussed in §5.

In practice, one can specify certain sparsity patterns of $\{V_{i-1}\}$ by letting some of the entries of V_{i-1} be zero. When we do not have any restriction of the block size (i.e., when $R_i = I$—the identity matrix), this algorithm reduces to the block ILU methods studied in Concus, Golub, and Meurant [10], Axelsson and Polman [5].

Our first goal is to show that $\tilde{Z}_i$ and Z_i are M-matrices and therefore the above choice of V_{i-1} justified; i.e., the block ILU factorization matrix C, (2.1), of A is well defined as long as the approximations Y_i to Z_i are invertible. In particular, if A is a symmetric and positive definite M-matrix, this will imply that the blocks Z_i are positive definite (and symmetric) and hence the block ILU factorization matrix C is positive definite as long as the approximations Y_i to Z_i are positive definite. The non-M-matrix case will be considered in §6.

We next adopt the following convention. A matrix, or a vector, is called nonnegative if its entries are nonnegative. We say that the matrices W and G satisfy the inequality $W >, \geq, <,$ or $\leq G$ if this is true componentwise. The same relations we may use for vectors.

First, we show that $\tilde{Z}_i = R_i Z_i R_i^T$ are M-matrices.

Lemma 1. *For any choice*

$$0 \leq V_{i-1} \leq \tilde{Z}_{i-1}^{-1},$$

starting with $i = 2$, the next approximate coarse Schur complement $\tilde{Z}_i$ is an M-matrix and the choice of the next V_i $(0 \leq V_i \leq \tilde{Z}_i^{-1})$ justified.

Proof. Note that

$$\tilde{Z}_1 = R_1 Z_1 R_1^T = R_1 A_{11} R_1^T = \tilde{A}_{11}$$

is the first block on the diagonal of the coarse matrix $\tilde{A}$, which is an M-matrix, hence $\tilde{Z}_1 = \tilde{A}_{11}$ is an M-matrix as a principal submatrix of $\tilde{A}$. We also note that

$$\tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \ddots & \\ & \ddots & \ddots & \\ 0 & & \tilde{A}_{n,n-1} & \tilde{A}_{nn} \end{pmatrix},$$

with

$$\begin{aligned} \tilde{A}_{i,i-1} &= R_i A_{i,i-1} R_{i-1}^T, \\ \tilde{A}_{ii} &= R_i A_{ii} R_i^T, \\ \tilde{A}_{i,i+1} &= R_i A_{i,i+1} R_{i+1}^T. \end{aligned} \tag{3.5}$$

Next, we will need the matrices $\{\tilde{D}_i\}$ from the exact block factorization of $\tilde{A}$; namely,

$$\tilde{A} = \begin{pmatrix} \tilde{D}_1 & & & 0 \\ \tilde{A}_{21} & \tilde{D}_2 & & \\ & \ddots & \ddots & \\ 0 & & \tilde{A}_{n,n-1} & \tilde{D}_n \end{pmatrix} \begin{pmatrix} I & \tilde{D}_1^{-1}\tilde{A}_{12} & & 0 \\ & I & \tilde{D}_2^{-1}\tilde{A}_{23} & \\ & & \ddots & \ddots \\ 0 & & & I \end{pmatrix}, \tag{3.6}$$

i.e.,

$$\tilde{D}_i = \tilde{A}_{ii} - \tilde{A}_{i,i-1}\tilde{D}_{i-1}^{-1}\tilde{A}_{i-1,i}, \qquad \tilde{D}_1 = \tilde{A}_{11}. \tag{3.7}$$

Note that the blocks $\{\tilde{D}_i\}$ are M-matrices, since they are Schur complements of corresponding main submatrices of $\tilde{A}$, which is assumed to be an M-matrix; namely (see (3.2))

$$\tilde{D}_i = \tilde{A}_{ii} - \left(0, \ldots, \tilde{A}_{i,i-1}\right) \tilde{A}^{(i-1)^{-1}} \begin{pmatrix} 0 \\ \vdots \\ \tilde{A}_{i-1,i} \end{pmatrix}. \tag{3.8}$$

Our main goal is to show that

$$\tilde{Z}_i \geq \tilde{D}_i. \tag{3.9}$$

We then show that (3.9) implies the desired result, i.e., that $\tilde{Z}_i$ is an M-matrix.

First note that

$$\tilde{Z}_i \leq \tilde{A}_{ii}, \tag{3.10}$$

because

$$\begin{aligned} \tilde{Z}_i &= R_i Z_i R_i^T \\ &= R_i A_{ii} R_i^T - R_i A_{i,i-1} R_{i-1}^T V_{i-1} R_{i-1} A_{i-1,i} R_i^T \\ &= \tilde{A}_{ii} - \{\text{nonnegative term}\}. \end{aligned}$$

The last is true because of the choice of $V_{i-1} \geq 0$, the assumption $R_i \geq 0$, and since $A_{i,i-1} \leq 0$, $A_{i-1,i} \leq 0$ as off-diagonal blocks of the M-matrix A.

Inequality (3.10) implies that $\tilde{Z}_i$ has nonpositive off-diagonal entries, since $\tilde{A}_{ii}$ has nonpositive off-diagonal entries (which are off-diagonal entries of $\tilde{A}$ and the latter was assumed to be an M-matrix). Then inequality (3.9) suffices to guarantee the M-matrix property of $\tilde{Z}_i$ since $\tilde{D}_i$ is an M-matrix (as a Schur complement of $\tilde{A}^{(i)}$, which is an M-matrix as a principal submatrix of $\tilde{A}$; see (3.8)). The latter implies that there exists a positive vector $\tilde{\mathbf{c}}_i$ such that $\tilde{D}_i\tilde{\mathbf{c}}_i > 0$. Then (3.9) implies that $\tilde{Z}_i\tilde{\mathbf{c}}_i > 0$, which together with the nonpositiveness of the off-diagonal entries of $\tilde{Z}_i$ shows the desired M-matrix property of $\tilde{Z}_i$.

The M-matrix property of $\tilde{Z}_i$ and $\tilde{D}_i$ and (3.9) show that $\tilde{D}_i^{-1}\left(\tilde{Z}_i - \tilde{D}_i\right)\tilde{Z}_i^{-1} \geq 0$, i.e.,

$$\tilde{D}_i^{-1} \geq \tilde{Z}_i^{-1}. \tag{3.11}$$

We now prove (3.9) by induction. We have that (3.11) holds for $i := i-1$, i.e., we have $\tilde{D}_{i-1}^{-1} \geq \tilde{Z}_{i-1}^{-1}$, which follows from the induction assumption (3.9) for $i := i-1$ and its corollary that $\tilde{Z}_{i-1}$ is an M-matrix, which we already demonstrated. This also justifies the choice of V_{i-1}. Then using (3.11) (for $i := i-1$), the inequalities $A_{i,i-1} \leq 0$, $A_{i-1,i} \leq 0$, and the constraint on V_{i-1}, we get

$$\begin{aligned} \tilde{D}_i &= \tilde{A}_{ii} - \tilde{A}_{i,i-1}\tilde{D}_{i-1}^{-1}\tilde{A}_{i,i-1} \\ &\leq \tilde{A}_{ii} - \tilde{A}_{i,i-1}\tilde{Z}_{i-1}^{-1}\tilde{A}_{i-1,i} \\ &\leq \tilde{A}_{ii} - \tilde{A}_{i,i-1}V_{i-1}\tilde{A}_{i-1,i} \\ &= \tilde{Z}_i, \end{aligned}$$

which completes the proof of (3.9) and the proof of the lemma as well. □

Remark 1. Note that if $V_i^{-1} = \tilde{Z}_i$, $i = 1, 2, \ldots, n$, then the last series of inequalities implies that $\tilde{D}_i = \tilde{Z}_i$ (for all $i = 2, \ldots, n$). In other words, $V_i =$

$\tilde{Z}_i^{-1}$ are the inverses of the Schur complements of the exact block factorization of the coarse matrix $\tilde{A}$; see (3.6), (3.7).

Lemma 2. *The approximate Schur complements* $\{Z_i\}$ *are M-matrices.*

Proof. Consider now the following block matrix,

$$\begin{pmatrix} \tilde{D}_i & R_i A_{i,i+1} \\ A_{i+1,i} R_i^T & A_{i+1,i+1} \end{pmatrix}. \tag{3.12}$$

Note that it is a principal submatrix of the Schur complement of $\hat{A}^{(i+1)}$ when the block $\tilde{A}^{(i-1)}$ is eliminated (see (3.3), (3.2)). By assumption, then, we get that (3.12) is an M-matrix. This implies that there exists a positive vector $\begin{bmatrix} \tilde{\mathbf{c}}_i \\ \mathbf{c}_{i+1} \end{bmatrix}$ such that

$$\begin{pmatrix} \tilde{D}_i & R_i A_{i,i+1} \\ A_{i+1,i} R_i^T & A_{i+1,i+1} \end{pmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_i \\ \mathbf{c}_{i+1} \end{bmatrix} > 0.$$

Using now inequality (3.9), we obtain that

$$\begin{bmatrix} \tilde{\mathbf{b}}_i \\ \mathbf{b}_{i+1} \end{bmatrix} \equiv \begin{pmatrix} \tilde{Z}_i & R_i A_{i,i+1} \\ A_{i+1,i} R_i^T & A_{i+1,i+1} \end{pmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_i \\ \mathbf{c}_{i+1} \end{bmatrix} > 0.$$

This inequality and the fact that $\tilde{Z}_i$ is an M-matrix imply that

$$\begin{pmatrix} \tilde{Z}_i & R_i A_{i,i+1} \\ A_{i+1,i} R_i^T & A_{i+1,i+1} \end{pmatrix} \tag{3.13}$$

is also an M-matrix. Hence, since $\tilde{Z}_i$ is an M-matrix, we see that

$$0 < \begin{bmatrix} \tilde{Z}_i^{-1} \tilde{\mathbf{b}}_i \\ \mathbf{b}_{i+1} \end{bmatrix} = \begin{pmatrix} I & \tilde{Z}_i^{-1} R_i A_{i,i+1} \\ A_{i+1,i} R_i^T & A_{i+1,i+1} \end{pmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_i \\ \mathbf{c}_{i+1} \end{bmatrix},$$

i.e., we have that

$$\begin{pmatrix} I & \tilde{Z}_i^{-1} R_i A_{i,i+1} \\ A_{i+1,i} R_i^T & A_{i+1,i+1} \end{pmatrix}$$

is an M-matrix as well. Finally, based on the constraint on V_i, namely,

$$0 \le V_i \le \tilde{Z}_i^{-1},$$

we see that

$$\begin{pmatrix} I & V_i R_i A_{i,i+1} \\ A_{i+1,i} R_i^T & A_{i+1,i+1} \end{pmatrix} \tag{3.14}$$

is also an M-matrix. This follows from the inequalities (note that $A_{i,i+1} \le 0$ and $R_i \ge 0$)

$$\begin{aligned} 0 < \begin{bmatrix} \tilde{Z}_i^{-1} \tilde{\mathbf{b}}_i \\ \mathbf{b}_{i+1} \end{bmatrix} &\le \begin{bmatrix} \tilde{Z}_i^{-1} \tilde{\mathbf{b}}_i \\ \mathbf{b}_{i+1} \end{bmatrix} + \begin{bmatrix} (V_i - \tilde{Z}_i^{-1}) R_i A_{i,i+1} \tilde{\mathbf{c}}_{i+1} \\ 0 \end{bmatrix} \\ &= \begin{pmatrix} I & V_i R_i A_{i,i+1} \\ A_{i+1,i} R_i^T & A_{i+1,i+1} \end{pmatrix} \begin{bmatrix} \tilde{\mathbf{c}}_i \\ \mathbf{c}_{i+1} \end{bmatrix}. \end{aligned}$$

Hence, the Schur complement of the last matrix (3.14),

$$I - G = I - V_i R_i A_{i,i+1} A_{i+1,i+1}^{-1} A_{i+1,i} R_i^T, \tag{3.15}$$

is an M-matrix. In particular, we get that

$$(I - G)^{-1} \ge 0. \tag{3.16}$$

Next we show that Z_{i+1}^{-1} is nonnegative. To this end, we use the Sherman-Morrison-Woodbury formula for Z_{i+1}^{-1} in the following form (see (5.1), §5 for a detailed derivation),

$$\begin{aligned} Z_{i+1}^{-1} = & A_{i+1,i+1}^{-1} + A_{i+1,i+1}^{-1} A_{i+1,i} R_i^T \\ & \times \left(I - V_i R_i A_{i,i+1} A_{i+1,i+1}^{-1} A_{i+1,i} R_i^T \right)^{-1} \\ & \times V_i R_i A_{i,i+1} A_{i+1,i+1}^{-1}. \end{aligned}$$

With the help of the inequalities (since A is an M-matrix)

$$A_{i,i+1}, A_{i+1,i} \leq 0, \quad A_{i+1,i+1}^{-1} \geq 0,$$

and (3.16) (see (3.15)), the nonnegativity of Z_{i+1}^{-1} follows.

We also have by construction that

$$Z_i \leq A_{ii},$$

i.e., the off-diagonal entries of Z_i are nonpositive (because the off-diagonal entries of A_{ii} are nonpositive; A and hence A_{ii} are M-matrices). The last two arguments imply that $\{Z_i\}$ are M-matrices. □

Thus, we have proved the following main result:

Theorem 1. *Algorithm* 1′ *(block ILU) is well defined; it gives the block-factored matrix C,* (2.1), *constructed on the basis of the M-matrices Z_i and $\tilde{Z}_i$ for any choice of the matrices V_i that satisfy the inequalities $0 \leq V_i \leq \tilde{Z}_i^{-1}$. In particular, when A is a symmetric and positive definite M-matrix, the blocks Z_i and $\tilde{Z}_i$ are positive definite and hence C is positive definite for any choice of symmetric and positive definite approximations Y_i for Z_i.*

Remark 2. We remark that the above result is valid even if we do not have any reduction of the block size. In that case, $R_i = I$—the identity matrix. And this is a known result already presented in Concus, Golub, and Meurant [10]; see also Axelsson [1], [2] and Axelsson and Polman [5].

4. Verification of Assumptions (I), (II)

In this section we verify Assumptions (I), (II) for a specific choice of the restriction matrices $\{R_i\}$ in the case of a general block tridiagonal M-matrix A.

In this section, A can be any block matrix $\{A_{i,j}\}$ with blocks $A_{i,j}$ of size $n_i \times n_j$ for some integers n_i. We assume that A is an M-matrix and also that there is an explicitly given positive vector

$$\mathbf{v} = \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_n \end{bmatrix}$$

such that

$$A\mathbf{v} > 0. \tag{4.1}$$

Note that for M-matrices there always exists a vector $\mathbf{c} > 0$ (e.g., corresponding to the Perron root of $A^{-1} \geq 0$ or simply $\mathbf{c} = A^{-1} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$) such that $A\mathbf{c} > 0$, and it is in general expensive to be computed. However, in the case of strictly diagonally dominant matrices one can simply choose

$$\mathbf{v}_i = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{n_i}, \tag{4.2}$$

and (4.1) will hold. This is one of the examples that can be used in practice. Moreover, when A has a positive definite symmetric part, any positive vector $\mathbf{v}$ is appropriate, as we shall see in a moment (see Theorem 4).

Consider now any partitioning of $\mathbf{v}_i$,

$$\mathbf{v}_i = \begin{bmatrix} \mathbf{v}_1^{(i)} \\ \mathbf{v}_2^{(i)} \\ \vdots \\ \mathbf{v}_m^{(i)} \end{bmatrix}, \tag{4.3}$$

for some integer $m \le n_i$. We remark that m can vary with i, but in order to simplify the notation we shall not indicate this explicitly.

For the given positive vector $\mathbf{v}_i$ partitioned as above we define the following restriction matrix,

$$R_i = \begin{pmatrix} \mathbf{v}_1^{(i)^T} & 0 & \dots & 0 \\ 0 & \mathbf{v}_2^{(i)^T} & & \\ \vdots & & \ddots & \\ 0 & & & \mathbf{v}_m^{(i)^T} \end{pmatrix}, \tag{4.4}$$

which is an $m \times n_i$ matrix. Note that in practice, for strictly diagonally dominant matrices A, $\{R_i\}$ can be constructed on the basis of the vectors (4.2).

Consider now the matrix $\hat{A}^{(i+1)}$ defined by (3.3), i.e.,

$$\begin{aligned} \hat{A}^{(i+1)} &= \begin{pmatrix} R_1 & & & 0 \\ & \ddots & & \\ & & R_i & \\ 0 & & & I \end{pmatrix} A \begin{pmatrix} R_1^T & & & 0 \\ & \ddots & & \\ & & R_i^T & \\ 0 & & & I \end{pmatrix} \\ &= \hat{R} A \hat{R}^T. \end{aligned} \tag{4.5}$$

Consider also the following vector,

$$\mathbf{w} = \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_i \\ \mathbf{v}_{i+1} \\ \vdots \\ \mathbf{v}_n \end{bmatrix},$$

where

$$\mathbf{e}_i = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^m.$$

Note that

$$R_i^T \mathbf{e}_i = \mathbf{v}_i. \tag{4.6}$$

Our next goal is to show that $\hat{A}^{(i+1)}\mathbf{w}$ is a positive vector. This will imply that $\hat{A}^{(i+1)}$ is an M-matrix, since by construction its off-diagonal entries are nonpositive. This is seen from the fact that these off-diagonal entries are obtained by linear combinations of the off-diagonal entries of A which are nonpositive (A is an M-matrix) with coefficients from $\{R_i\}$ which are nonnegative. Note also that $\{R_i\}$ have block diagonal form.

We consider

$$\left(\hat{A}^{(i+1)}\mathbf{w}\right)_j$$

for three different cases. First let $1 \le j < i$. Then

$$\begin{aligned}\left(\hat{A}^{(i+1)}\mathbf{w}\right)_j &= R_j\left[A_{j,j-1}R_{j-1}^T\mathbf{e}_{j-1} + A_{jj}R_j^T\mathbf{e}_j + A_{j,j+1}R_{j+1}^T\mathbf{e}_{j+1}\right]\\ &= R_j\left[A_{j,j-1}\mathbf{v}_{j-1} + A_{jj}\mathbf{v}_j + A_{j,j+1}\mathbf{v}_{j+1}\right]\\ &= R_j\left(A\mathbf{v}\right)_j\\ &\ge 0.\end{aligned}$$

Next, for $j = i$ we have

$$\begin{aligned}\left(\hat{A}^{(i+1)}\mathbf{w}\right)_i &= R_i\left[A_{i,i-1}R_{i-1}^T\mathbf{e}_{i-1} + A_{ii}R_i^T\mathbf{e}_i + A_{i,i+1}\mathbf{v}_{i+1}\right]\\ &= R_i\left[A_{i,i-1}\mathbf{v}_{i-1} + A_{ii}\mathbf{v}_i + A_{i,i+1}\mathbf{v}_{i+1}\right]\\ &= R_i\left(A\mathbf{v}\right)_i\\ &\ge 0.\end{aligned}$$

And for $j > i$,

$$\left(\hat{A}^{(i+1)}\mathbf{w}\right)_j = (A\mathbf{v})_j \ge 0.$$

In the above we have used equality (4.6).

We note that $\hat{A}^{(i+1)}\mathbf{w} > 0$, i.e., we have strict inequality if $A\mathbf{v} > 0$. Thus, we can formulate the following result.

Theorem 2. *Consider the positive vector* $\mathbf{v}$ *such that* $A\mathbf{v} > 0$. *The block entries* $\mathbf{v}_i$ *are partitioned into the form* (4.3). *For the restriction matrices defined by* (4.4), *the intermediate coarse matrices* $\hat{A}^{(i+1)}$, (3.3), *and the coarse matrix* $\tilde{A} = \hat{A}^{(n)}$ *are M-matrices.*

The above assumptions can be relaxed as follows. In some cases it is possible to directly verify that the intermediate coarse matrices $\hat{A}^{(i+1)}$ are nonsingular. Then we can assume that (4.1) holds in a weaker form; namely, $A\mathbf{v} \ge 0$ for a given positive vector $\mathbf{v}$.

We next need the following auxiliary result for M-matrices. It can be proven following the lines from Theorem (2.3) (the section on semipositivity and diagonal dominance, p. 138) in the book of Berman and Plemmons [7].

Lemma 3. *Let* $A = (a_{i,j})_{i,j=1}^n$ *be nonsingular with nonpositive off-diagonal entries. Let also for some given positive vector* $\mathbf{v} = (v_i)$, $A\mathbf{v} \ge 0$. *Then* A *is an M-matrix.*

We can prove now the following result.

Theorem 3. *Let $\mathbf{v} = (\mathbf{v}_i) > 0$ be such that $A\mathbf{v} \geq 0$. Consider R_i defined by* (4.4), *constructed on the basis of the partitioning* (4.3) *of $\mathbf{v}_i$. Finally assume that the resulting intermediate coarse matrices $\hat{A}^{(i+1)}$ are nonsingular. Then we have that $\hat{A}^{(i+1)}$ are M-matrices.*

Proof. The proof repeats the proof of Theorem 2. In particular, we get that

$$\hat{A}^{(i+1)}\mathbf{w} \geq 0 \quad \text{for } \mathbf{w} = \begin{bmatrix} \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_i \\ \mathbf{v}_{i+1} \\ \vdots \\ \mathbf{v}_n \end{bmatrix}, \qquad \mathbf{e}_j = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \in \mathbb{R}^m.$$

The desired result then follows from Lemma 3 applied to the matrix $\hat{A}^{(i+1)}$ and the vector $\mathbf{w}$, since we have assumed that $\hat{A}^{(i+1)}$ is nonsingular. □

The final result concerns M-matrices with positive definite symmetric part.

Theorem 4. *Assume that A is an M-matrix with positive definite symmetric part* (*i.e., $A + A^T$ is positive definite*). *Then for any choice of positive vectors $\mathbf{v} = (\mathbf{v}_i)$ with block entries $\mathbf{v}_i$ partitioned into the form* (4.3) *and corresponding restriction matrices R_i defined by* (4.4), *the intermediate coarse matrices $\hat{A}^{(i+1)}$ are M-matrices.*

Proof. Since A has a positive definite symmetric part then $\hat{A}^{(i+1)} = \hat{R}A\hat{R}^T$ (see (4.5)) will be nonsingular if $\hat{R}$ is of full rank. This is the case for our restriction matrices for any choice of nonzero vectors $\mathbf{v}_1^{(i)}, \ldots, \mathbf{v}_m^{(i)}$ in (4.3). Note that $R_i R_i^T$ are diagonal with positive entries on the main diagonal, hence $\hat{R}\hat{R}^T$ is invertible. Thus, $\hat{R}^T\mathbf{z} = 0$ implies $\hat{R}\hat{R}^T\mathbf{z} = 0$ and hence $\mathbf{z} = 0$. We also have that $\hat{A}^{(i+1)}$ admits $L_1 L_2 \cdots D^{-1} \cdots U_2 U_1$ (i.e., standard Cholesky factorization) since it has positive definite symmetric part, which is seen from the identity

$$\hat{A}^{(i+1)} + \hat{A}^{(i+1)^T} = \hat{R}(A + A^T)\hat{R}^T. \tag{4.7}$$

The factors L_i (unit lower triangular) and U_i (unit upper triangular) have nonpositive off-diagonal entries since $\hat{A}^{(i+1)}$ has nonpositive off-diagonal entries. This shows that L_i and U_i are M-matrices. We also have that $D \geq 0$ (the diagonal matrix from the factorization of $\hat{A}^{(i+1)}$) since $\hat{A}^{(i+1)}$ is positive definite. Thus, we have that $\hat{A}^{(i+1)^{-1}} = U_1^{-1}U_2^{-1} \cdots D \cdots L_2^{-1}L_1^{-1} \geq 0$, i.e., the M-matrix property of $\hat{A}^{(i+1)}$ is verified. □

5. Approximations of the Schur complements

In this section we derive some approximations Y_i^{-1} to the inverses of the approximate Schur complements Z_i, since the latter can be expensive to compute. Our main focus will be on approximations based on the exact representation of Z_i^{-1} given by the Sherman-Morrison-Woodbury formula. We assume that A as well as the intermediate coarse matrices $\hat{A}^{(i+1)}$ (see (3.3)) are M-matrices and that R_i are nonnegative.

We start with the exact block factorization of $\tilde{A}$,

$$\tilde{A} = \begin{pmatrix} \tilde{D}_1 & & & 0 \\ \tilde{A}_{21} & \tilde{D}_2 & & \\ & \ddots & \ddots & \\ 0 & & \tilde{A}_{n,n-1} & \tilde{D}_n \end{pmatrix} \begin{pmatrix} I & \tilde{D}_1^{-1}\tilde{A}_{12} & & 0 \\ & I & \ddots & \\ & & \ddots & \\ 0 & & & I \end{pmatrix}.$$

We note that with $\tilde{D}_i$ replaced by $\tilde{Z}_i$ or by V_i^{-1} we obtain approximate block LU factorizations of $\tilde{A}$ already studied in Concus, Golub, and Meurant [10] and Axelsson [2], Axelsson and Polman [5]. In practice it makes sense to factor $\tilde{A}$ exactly if m, the dimension of the blocks $\tilde{D}_i$, is reasonably small.

We recall the formula for the blocks Z_i, (3.4),

$$Z_i = A_{ii} - A_{i,i-1}R_{i-1}^T V_{i-1} R_{i-1} A_{i-1,i}.$$

Note that Z_i are not sparse matrices because of the factor R_{i-1}^T. That is why we cannot in general expect that Z_i or Z_i^{-1} can be well approximated by sparse matrices. However, as already mentioned, we can derive exact expressions for the inverse of Z_i exploiting the fact that

$$A_{i,i-1}R_{i-1}^T V_{i-1} R_{i-1} A_{i-1,i}$$

is a low-rank matrix. Then, using the Sherman-Morrison-Woodbury formula (cf., Golub and van Loan [14, p. 51])

$$(M - FG)^{-1} = M^{-1} + M^{-1}F\left(I - GM^{-1}F\right)^{-1} GM^{-1}$$

for

$$\begin{aligned} M &= A_{ii}, \\ F &= A_{i,i-1}R_{i-1}^T, \\ G &= V_{i-1}R_{i-1}A_{i-1,i}, \end{aligned}$$

we get

$$\begin{aligned} Z_i^{-1} &= A_{ii}^{-1} + A_{ii}^{-1}A_{i,i-1}R_{i-1}^T \\ &\quad \times \left(I - V_{i-1}R_{i-1}A_{i-1,i}A_{ii}^{-1}A_{i,i-1}R_{i-1}^T\right)^{-1} \\ &\quad \times V_{i-1}R_{i-1}A_{i-1,i}A_{ii}^{-1} \\ &= A_{ii}^{-1} + A_{ii}^{-1}A_{i,i-1}R_{i-1}^T \\ &\quad \times \left(V_{i-1}^{-1} - R_{i-1}A_{i-1,i}A_{ii}^{-1}A_{i,i-1}R_{i-1}^T\right)^{-1} \\ &\quad \times R_{i-1}A_{i-1,i}A_{ii}^{-1}. \end{aligned} \tag{5.1}$$

The last identity is our starting point for deriving approximations Y_i^{-1} to Z_i^{-1}.

Note first that V_{i-1} was intended as an approximation to $\tilde{Z}_{i-1}^{-1}$, hence it is natural to replace V_{i-1}^{-1} by $\tilde{Z}_{i-1}$. In the applications this makes some difference in storage, since $\tilde{A}$ is sparse and $\tilde{Z}_{i-1}$ is kept sparse (say, banded) by construction.

Next we can use

$$\tilde{A}_{i-1,i}\tilde{A}_{ii}^{-1}\tilde{A}_{i,i-1}$$

as an approximation to

$$R_{i-1}A_{i-1,i}A_{ii}^{-1}A_{i,i-1}R_{i-1}^T.$$

Moreover, we may want to approximate A_{ii}^{-1} by some sparse or readily available (in an algorithmic sense by its actions) matrix B_{ii}^{-1}. In some of the applications, A_{ii} is strictly diagonally dominant, hence A_{ii}^{-1} has a good decay rate behavior (for more details, cf., e.g., Demko, Moss, and Smith [11] or Vassilevski [21]). For M-matrices it is natural then to assume that

$$0 \le B_{ii}^{-1} \le A_{ii}^{-1},$$

with most of the entries of B_{ii}^{-1} being zero.

We summarize the approximations to Z_i^{-1}:

Approximation (1):

$$\begin{aligned} Y_i^{-1} = & B_{ii}^{-1} + B_{ii}^{-1} A_{i,i-1} R_{i-1}^T \\ & \times \left(\tilde{Z}_{i-1} - R_{i-1} A_{i-1,i} B_{ii}^{-1} A_{i,i-1} R_{i-1}^T \right)^{-1} \\ & \times R_{i-1} A_{i-1,i} B_{ii}^{-1}. \end{aligned}$$

Approximation (2):

$$\begin{aligned} Y_i^{-1} = & B_{ii}^{-1} + B_{ii}^{-1} A_{i,i-1} R_{i-1}^T \\ & \times \left(\tilde{Z}_{i-1} - \tilde{A}_{i-1,i} \tilde{A}_{ii}^{-1} \tilde{A}_{i,i-1} \right)^{-1} \\ & \times R_{i-1} A_{i-1,i} B_{ii}^{-1}. \end{aligned}$$

We recall that in order to solve the systems with the block ILU factorization matrix C, (2.1), we have to solve systems with the blocks Y_i, i.e., to solve systems of the form $Y_i \mathbf{v}_i = \mathbf{w}_i$ or to compute $\mathbf{v}_i = Y_i^{-1} \mathbf{w}_i$. The expressions from the above approximations (1) and (2) for Y_i^{-1} show that we need to be able to efficiently solve systems with B_{ii} when B_{ii} are computed, or to have the actions of B_{ii}^{-1} readily available. Finally, note that we also need the blocks

$$T_{i-1} \equiv \tilde{Z}_{i-1} - R_{i-1} A_{i-1,i} B_{ii}^{-1} A_{i,i-1} R_{i-1}^T, \tag{5.2}$$

or

$$T_{i-1} \equiv \tilde{Z}_{i-1} - \tilde{A}_{i-1,i} \tilde{A}_{ii}^{-1} \tilde{A}_{i,i-1}, \tag{5.3}$$

to be easily inverted (or factored). This is the case (for (5.2)), assuming that $\{R_i\}$ are of the form (4.4), since $\tilde{Z}_{i-1}$ is kept sparse and if $G_i \equiv B_{ii}^{-1}$ is a sparse approximation to A_{ii}^{-1}. For such approximations, see Concus, Golub, and Meurant [10], Axelsson, Brinkkemper, and Il'in [3], Axelsson [1], and Vassilevski [21] for band matrices, and Demko, Moss, and Smith [11] and Vassilevski [22] for more general sparse matrices.

However, for both approximations (1) and (2), when the dimension m of the blocks $\tilde{Z}_{i-1}$ is reasonably small, it may make sense to invert (or factor) the blocks (5.2) and (5.3) exactly. Another alternative for the second approximation is to further approximate $\tilde{A}_{ii}^{-1}$ by some easier (for computations) matrix $\tilde{B}_{ii}^{-1}$, similarly to the approximation B_{ii}^{-1} of A_{ii}^{-1}.

We summarize below the algorithm for the computation of the blocks $\{Y_i^{-1}\}$ based on approximation (1).

Algorithm 2 (Approximate block-factorization of A).

(1) Factor the coarse matrix $\tilde{A}$:

$$\tilde{Z}_1 = \tilde{A}_{11}.$$

For $i = 1, 2, \ldots, n-1$, let V_i be such that

$$0 \le V_i \le \tilde{Z}_i^{-1},$$

then form

$$\tilde{Z}_{i+1} = \tilde{A}_{i+1,i+1} - \tilde{A}_{i+1,i} V_i \tilde{A}_{i,i+1};$$

(2) compute an approximate inverse to A_{ii}:

$$0 \le B_{ii}^{-1} \le A_{ii}^{-1};$$

(3) update $\tilde{Z}_i$, i.e., compute (see (5.2))

$$T_i = \tilde{Z}_i - R_i A_{i,i+1} B_{i+1,i+1}^{-1} A_{i+1,i} R_i^T;$$

and let

(4)

$$U_i = \text{Approx}\left(T_i^{-1}\right)$$

be some approximation of T_i^{-1} such that the actions of U_i on a vector can be computed inexpensively. Then

(5)

$$Y_{i+1}^{-1} = B_{i+1,i+1}^{-1} + B_{i+1,i+1}^{-1} A_{i+1,i} R_i^T U_i R_i A_{i,i+1} B_{i+1,i+1}^{-1}. \quad \square$$

A similar algorithm is obtained when we use approximation (2):

Algorithm 3 (Approximate block-factorization of A).

(1) Factor the coarse matrix $\tilde{A}$:

$$\tilde{Z}_1 = \tilde{A}_{11}.$$

For $i = 1, 2, \ldots, n-1$, let V_i be such that

$$0 \le V_i \le \tilde{Z}_i^{-1},$$

then form

$$\tilde{Z}_{i+1} = \tilde{A}_{i+1,i+1} - \tilde{A}_{i+1,i} V_i \tilde{A}_{i,i+1};$$

(2) update $\tilde{Z}_i$, i.e., compute (see (5.3))

$$T_i = \tilde{Z}_i - \tilde{A}_{i,i+1} \tilde{A}_{i+1,i+1}^{-1} \tilde{A}_{i+1,i};$$

and let

(3)

$$U_i = \text{Approx}\left(T_i^{-1}\right);$$

be some approximation of T_i^{-1} such that the actions of U_i on a vector can be computed inexpensively. Then

(4)

$$Y_{i+1}^{-1} = B_{i+1,i+1}^{-1} + B_{i+1,i+1}^{-1} A_{i+1,i} R_i^T U_i R_i A_{i,i+1} B_{i+1,i+1}^{-1}. \quad \square$$

Remark 3. Based on the identity (which results from another application of the Sherman-Morrison-Woodbury formula)

$$\left(\tilde{D}_i - \tilde{A}_{i,i+1} \tilde{A}_{i+1,i+1}^{-1} \tilde{A}_{i+1,i}\right)^{-1} = \tilde{D}_i^{-1} + \tilde{D}_i^{-1} \tilde{A}_{i,i+1} \tilde{D}_{i+1}^{-1} \tilde{A}_{i+1,i} \tilde{D}_i^{-1},$$

one may be motivated to use the following approximation U_i of T_i^{-1} (see (5.3)),

$$U_i = V_i + V_i\tilde{A}_{i,i+1}V_{i+1}\tilde{A}_{i+1,i}V_i,$$

since V_i was intended as an approximation of $\tilde{Z}_i^{-1}$ and $\tilde{Z}_i$ was an approximation of the exact coarse Schur complement $\tilde{D}_i$. This approximation avoids step (3) of Algorithm 2 (or step (2) of Algorithm 3), i.e., the approximate inversion of T_i, since the factors V_i are already computed (see step (1) of Algorithm 2 and Algorithm 3). However, one action of Y_i^{-1} in this case is, in general, more expensive than in the previous cases when we had the blocks U_i explicitly computed. □

Next, we show that the above algorithm is well defined; namely, that the blocks T_i defined by both (5.2) and (5.3) are M-matrices. For (5.2) this is the case under the assumption that

$$0 \le B_{ii}^{-1} \le A_{ii}^{-1}. \tag{5.4}$$

Note that the above inequality makes sense, since A_{ii} is an M-matrix as a principal submatrix of A, which was assumed to be an M-matrix.

The proof follows easily from the fact that (see (3.13) in the proof of Lemma 2)

$$\begin{pmatrix} \tilde{Z}_{i-1} & R_{i-1}A_{i-1,i} \\ A_{i,i-1}R_{i-1}^T & A_{ii} \end{pmatrix}$$

is an M-matrix, and therefore its Schur complement

$$\tilde{Z}_{i-1} - R_{i-1}A_{i-1,i}A_{ii}^{-1}A_{i,i-1}R_{i-1}^T$$

is an M-matrix as well. Finally, using inequality (5.4), we get that

$$T_{i-1} = \tilde{Z}_{i-1} - R_{i-1}A_{i-1,i}B_{ii}^{-1}A_{i,i-1}R_{i-1}^T$$

is an M-matrix. Then it is natural to assume that U_{i-1} satisfies

$$0 \le U_{i-1} \le T_{i-1}^{-1}.$$

This is the case for the approximate inverses of banded matrices used in Concus, Golub, and Meurant [10], Axelsson [1], Axelsson and Polman [5], or Vassilevski [21], [22].

In a similar (even easier) way one can show that

$$\tilde{Z}_{i-1} - \tilde{A}_{i-1,i}\tilde{A}_{ii}^{-1}\tilde{A}_{i,i-1}$$

is an M-matrix. Consider the M-matrix (a Schur complement of $\tilde{A}^{(i)}$, which is an M-matrix as a principal submatrix of the coarse matrix $\tilde{A}$)

$$\begin{pmatrix} \tilde{D}_{i-1} & \tilde{A}_{i-1,i} \\ \tilde{A}_{i,i-1} & \tilde{A}_{ii} \end{pmatrix}$$

and then use inequality (3.9) (for $i := i-1$) to see that

$$\begin{pmatrix} \tilde{Z}_{i-1} & \tilde{A}_{i-1,i} \\ \tilde{A}_{i,i-1} & \tilde{A}_{ii} \end{pmatrix}$$

is an M-matrix. Hence, T_{i-1} defined by (5.3) is an M-matrix as a Schur complement of the last matrix. We summarize the above results in the following theorem.

Theorem 5. *The Algorithms* 2 *and* 3 *are well defined*; *the blocks* T_i *needed in the Sherman-Morrison-Woodbury formula,* (5.2) *or* (5.3), *are M-matrices under the assumptions made in* §3. *For* T_i *defined by* (5.2) *we have the additional requirement that the approximate inverses* B_{ii}^{-1} *of the blocks on the diagonal of* A *satisfy the inequality* $0 \le B_{ii}^{-1} \le A_{ii}^{-1}$. *Also,* T_i^{-1} *can be further approximated with sparse matrices or more generally by matrices* U_i *that satisfy the inequality* $0 \le U_i \le T_i^{-1}$.

Proof. The last statement of the theorem follows from the fact that $T_i^{-1} \ge 0$, and hence the inequality $0 \le U_i \le T_i^{-1}$ makes sense. □

6. Positive definite matrices

We now present the existence result for a limit case of the approximate block factorization method developed in the present paper for generally nonsymmetric block tridiagonal matrices that have positive definite symmetric part. The result is valid for arbitrary choice of full-rank restriction matrices $\{R_i\}$, see Algorithm 1′. We emphasize that here we do not need the M-matrix property. However, at this time we require that the blocks $\{\tilde{Z}_i\}$ are exactly inverted, i.e., $V_i = \tilde{Z}_i^{-1}$. The symmetric positive definite case will be studied in more detail in a forthcoming report (see some preliminary results in [9]).

We are given the positive definite (i.e., with positive definite symmetric part) block tridiagonal matrix

$$A = \begin{pmatrix} A_{11} & A_{12} & & & 0 \\ A_{21} & A_{22} & A_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & A_{n-1,n-2} & A_{n-1,n-1} & A_{n-1,n} \\ 0 & & & A_{n,n-1} & A_{nn} \end{pmatrix};$$

we let

$$\tilde{A} = (R_i A_{i,j} R_j^T) = \left(\tilde{A}_{i,j}\right) \tag{6.1}$$

be the coarse matrix for some restriction matrices $\{R_i\}_{i=1}^n$. The $\{R_i\}$ can be any full-rank matrices and therefore they are more general than those of §4. The full rank of $\{R_i\}$ guarantees that $\tilde{A}$ has positive definite symmetric part, which we will further refer to as positive definiteness.

The approximate block factorization (or block ILU) process, which is a limit case of Algorithm 1′ ($V_{i-1} = \tilde{Z}_{i-1}^{-1}$), is described by the following algorithm:

Algorithm 4. Set

$$Z_1 = A_{11};$$

for $i = 1$ to $n-1$ compute

$$\begin{aligned} \tilde{Z}_i &= R_i Z_i R_i^T, \\ Z_{i+1} &= A_{i+1,i+1} - A_{i+1,i} R_i^T \tilde{Z}_i^{-1} R_i A_{i,i+1}. \quad \square \end{aligned} \tag{6.2}$$

Note that at this time we use the exact inverses of $\tilde{Z}_i$. Hence, this algorithm is of practical interest when m, the size of the coarse blocks $\tilde{Z}_i$, is relatively small.

The block ILU matrix is then defined as follows:

$$C = \begin{pmatrix} Z_1 & & & 0 \\ A_{21} & Z_2 & & \\ & \ddots & \ddots & \\ 0 & & A_{n,n-1} & Z_n \end{pmatrix} \begin{pmatrix} I & Z_1^{-1}A_{12} & & 0 \\ & I & Z_2^{-1}A_{23} & \\ & & \ddots & \ddots \\ 0 & & & I \end{pmatrix},$$

i.e., $Y_i = Z_i$ (see (2.1) and the General Block ILU Scheme).

We now prove the main existence result.

Theorem 6. *Let A be a block tridiagonal matrix with positive definite symmetric part, i.e., let*

$$\frac{1}{2}(A + A^T)$$

be positive definite. Then for any choice of full-rank restriction matrices $\{R_i\}$, Algorithm 4 is well defined; namely, the blocks Z_i and $\tilde{Z}_i = \tilde{D}_i$ are positive definite (i.e., with positive definite symmetric part). (Recall that $\tilde{D}_i$ are the blocks from the exact block factorization of the coarse matrix $\tilde{A}$, see (3.6).)

Proof. We first note that $\tilde{Z}_i = \tilde{D}_i$ (see Remark 1 at the end of Lemma 1). This shows that the $\tilde{Z}_i$ have positive definite symmetric part as Schur complements of principal submatrices of the coarse matrix $\tilde{A}$, which has positive definite symmetric part owing to the full rank of R.

Consider now the intermediate coarse matrix $\hat{A}^{(i+1)}$ defined by (3.3). By (4.7), since $\{R_i\}$ are of full rank, it follows that $\hat{A}^{(i+1)}$ is positive definite. Therefore, any principal submatrix of $\hat{A}^{(i+1)}$ is positive definite. This implies that the matrix

$$\begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} & & & 0 \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & & \\ & \ddots & \ddots & \ddots & \\ & & \tilde{A}_{i,i-1} & \tilde{A}_{ii} & R_iA_{i,i+1} \\ 0 & & & A_{i+1,i}R_i^T & A_{i+1,i+1} \end{pmatrix}$$

is positive definite. Recall (see (3.5)) that $\tilde{A}_{i,j} = R_iA_{i,j}R_j^T$. This implies that its Schur complement (see (3.2) and (3.8) with $\tilde{D}_i = \tilde{Z}_i$),

$$\begin{aligned} A_{i+1,i+1} &- \left(0, \dots, A_{i+1,i}R_i^T\right) \tilde{A}^{(i)^{-1}} \begin{pmatrix} 0 \\ \vdots \\ R_iA_{i,i+1} \end{pmatrix} \\ &= A_{i+1,i+1} - A_{i+1,i}R_i^T\tilde{Z}_i^{-1}R_iA_{i,i+1} \\ &= Z_{i+1}, \end{aligned}$$

has positive definite symmetric part. □

7. Numerical results

In this section we show some numerical illustration of the block ILU method proposed in the present paper. Also, we compare its performance with a variant of the smoothing–correction scheme of Wittum [24] and the more classical block ILU methods (cf., e.g., Concus, Golub, and Meurant [10], Axelsson [1], Axelsson and Polman [5], Vassilevski [21], etc.).

The matrix A is obtained by finite element discretization of the following second-order elliptic equation,

$$-\nabla \cdot (a(x, y)\nabla u) = f(x, y), \quad (x, y) \in \Omega = (0, 1)^2,$$

and

$$\begin{array}{lll} u|_{x=0}, & u|_{y=0} & \text{given}, \\ \frac{\partial u}{\partial x}|_{x=1}, & \frac{\partial u}{\partial y}|_{y=1} & \text{given}. \end{array}$$

The problem is discretized by piecewise linear basis functions on isosceles right triangles with meshsize $h = 1/n$, $n \geq 1$ an integer. The block tridiagonal structure of A in this case refers to an ordering of the meshpoints, say, along the vertical grid lines. In this case, A is a symmetric positive definite M-matrix.

The test problems are as follows.

Problem 1. $u = e^{-x}\left(x^2 + y^2\right)$, $a(x, y) = \frac{1}{1+x^2+y^2}$.

Problem 2 (a discontinuous coefficient).

$$u = (x-1/2)(y-1/2)\Phi(x, y)a^{-1}, \quad a(x, y) = \begin{cases} 1, & x < 1/2 \text{ or } y < 1/2, \\ 10^3, & x > 1/2 \text{ and } y > 1/2, \end{cases}$$

$$\Phi(x, y) = \sin\left(\frac{1}{2}\pi x\right) / \left(1 + x^2 + y^2\right).$$

The first preconditioner tested was

$$C_I = (I - LY)\, Y^{-1}\left(I - YL^T\right),$$

where the block $-L$ is the strictly block lower triangular part of A and Y is a block diagonal matrix. The blocks $\{Y_i\}_{i=1}^n$ on the main diagonal of Y were computed by the following algorithm (cf., e.g., Concus, Golub, and Meurant [10], and Axelsson and Polman [5]):

$$\begin{aligned} Y_1 &= \left[A_{11}^{-1}\right]_p, \\ Y_i &= \left[(A_{ii} - A_{i,i-1}Y_{i-1}A_{i-1,i})^{-1}\right]_p \quad \text{for } i > 1, \end{aligned}$$

where $\left[B^{-1}\right]_p$ stands for a $(2p+1)$-banded approximation of B^{-1} for a given matrix B. We chose the so-called CHOL approximation, cf., e.g., Concus, Golub, and Meurant [10] and Vassilevski [21]. It is defined on the basis of the exact $LD^{-1}U$ factorization of B by

$$\left[B^{-1}\right]_p \equiv \left[U^{-1}\right]^{(p)} D \left[L^{-1}\right]^{(p)},$$

where $[V]^{(p)}$ stands for the exact $2p+1$ inner-most banded part of V. In the case of banded matrices B, the CHOL approximation of B^{-1} can be computed inexpensively (cf., e.g., Vassilevski [21] for arbitrary $p \geq 1$).

The second preconditioner C_{II} was based on the smoothing-correction technique proposed by Wittum [24]. It is defined as follows:

$$\begin{aligned} C_{\mathrm{II}}^{-1} = &\left[\left(Y^{(1)} - L\right) Y^{(1)^{-1}} \left(Y^{(1)} - L^T\right)\right]^{-1} \\ &+ \left[\left(Y^{(2)} - L\right) Y^{(2)^{-1}} \left(Y^{(2)} - L^T\right)\right]^{-1}. \end{aligned}$$

Now the blocks $\left\{Y_i^{(s)}\right\}$ of $Y^{(s)}$, $s = 1, 2$, are tridiagonal and computed by the following algorithm:

$$Y_i^{(s)}: \quad \begin{aligned} Y_1^{(s)} &= A_{11}, \\ Y_i^{(s)}\mathbf{v}^{(r)} &= \left(A_{ii} - A_{i,i-1}Y_{i-1}^{(s)^{-1}}A_{i-1,i}\right)\mathbf{v}^{(r)} \quad \text{for } r = 1, 2 \text{ and all } i > 1, \end{aligned}$$

for the following vectors

$$\begin{aligned} \mathbf{v}^{(1)} &= \left\{\sin\left((i - \tfrac{1}{2})\pi h\right)\right\}_{i=1}^{n}, \\ \mathbf{v}^{(2)} &= \left\{\sin\left((i - \tfrac{1}{2})\pi 2h\right)\right\}_{i=1}^{n}, \end{aligned}$$

for $s = 1$, and

$$\begin{aligned} \mathbf{v}^{(1)} &= \left\{\sin\left((i - \tfrac{1}{2})\pi \nu h\right)\right\}_{i=1}^{n}, \\ \mathbf{v}^{(2)} &= \left\{\sin\left((i - \tfrac{1}{2})\pi(\nu + 1)h\right)\right\}_{i=1}^{n}. \end{aligned}$$

for $s = 2$. In our test we have chosen

$$\nu = \left[\frac{n}{7}\right].$$

The existence of a symmetric tridiagonal matrix T such that its actions on given two vectors is the same as the actions of another symmetric matrix W on the same vectors is shown in Wittum [24]. We remark that in Wittum [24] the above two factorizations were used in a stationary iterative procedure (called smoothing-correction scheme), whereas we use them as an additive preconditioner in the CG method.

Finally, the third preconditioner tested was the one proposed in the present paper, namely, the approximate block factorization matrix $C = C_{\text{III}}$, (2.1), with piecewise constant restriction matrices R_i, i.e.,

$$R_i = \begin{pmatrix} \mathbf{e}_1^T & & & 0 \\ & \mathbf{e}_2^T & & \\ & & \ddots & \\ 0 & & & \mathbf{e}_m^T \end{pmatrix},$$

where

$$\mathbf{e}_i = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^{n_0} \quad \text{and} \quad n_0 = \frac{n}{m}.$$

The dimension of the coarse vectors m was varied throughout the test.

The blocks $\{Y_i\}$ of C in (2.1) are defined by approximation (I) from §5, where

$$B_{ii}^{-1} = \left[A_{ii}^{-1}\right]_p,$$

i.e., the $(2p+1)$-banded CHOL approximation to A_{ii}^{-1}. These approximations are accurate since A_{ii}, the blocks on the diagonal of A, are strictly diagonally dominant and hence A_{ii}^{-1} have good decay-rate behavior (cf., e.g., Demko, Moss, and Smith [11], or Vassilevski [21]). We varied also the semibandwidth

$p \geq 1$ throughout the test. The coarse blocks computed throughout Algorithm 2 in the present test were inverted exactly (i.e., $V_i = \tilde{Z}_i^{-1}$).

We solved the corresponding discrete problems

$$A\mathbf{x} = \mathbf{b}$$

by the preconditioned conjugate gradient method, using the above described preconditioners C. The initial iterate was

$$\mathbf{x}_0 = C^{-1}\mathbf{b}.$$

Let $\mathbf{r}_0 = \mathbf{b} - A\mathbf{x}_0$ be the initial residual, $\mathbf{r}$ the current one, and set

$$\Delta_0 = \mathbf{r}_0^T C^{-1}\mathbf{r}_0, \quad \Delta = \mathbf{r}^T C^{-1}\mathbf{r}.$$

The stopping criterion was

$$\Delta < \epsilon = 10^{-18}.$$

In Tables 1–9 we show the number of iterations, *iter*, required to achieve the desired accuracy ϵ and $\rho = \left(\frac{\sqrt{\Delta}}{\sqrt{\Delta_0}}\right)^{\frac{1}{iter}}$, the average reduction factor.

We see from Tables 1–2 that preconditioner C_{III} performed such that the number of iterations increased about linearly with $\frac{n}{m}$. This is clearly seen, as for a fixed m when n is doubled the number of iterations has also been (nearly) doubled.

The preconditioner C_{III} exhibited a quite robust performance with respect to discontinuous coefficients (see Problem 2). Also, we see that if the blocks A_{ii}^{-1} were approximated accurately enough (in our case $p = 4$ was a good choice), the performance of the approximate preconditioner was close to the exact case. However, if the approximations of A_{ii}^{-1} were rough, then it did not pay off to choose a very fine coarse-vector space, i.e., an m close to n. This was the case for semibandwidth $p = 1$, see Tables 3–4.

The new method showed better convergence results than the classical block ILU preconditioner C_{I}, see Tables 8–9, and for $\frac{n}{m} \leq 4$ was competitive with Wittum's preconditioner C_{II}, see Table 7. This is of course only in terms of number of iterations and average reduction factors. On a particular computer one has to take into account the timings, which will crucially depend on the implementation and on the computer itself. To be more precise, we have to note that the preconditioner C_{III} has the most expensive inverse action (in terms of number of operations). The test should be seen only as a demonstration of the potential of the new preconditioner. We also note that it is directly applicable for matrices that arise in the discretization of 3-D elliptic partial differential equations. For Wittum's preconditioner it is not clear how to build the blocks $\{Y_i^{(s)}\}$ of $Y^{(s)}$ (whether they should be tridiagonal or not). It is also seen that the preconditioners C_{I} and C_{III} are vectorizable.

Finally we remark that at this point we have not explored various possible modification techniques in the factorization process similar to the MINV preconditioners studied by Concus, Golub, and Meurant [10].

TABLE 1. Preconditioner C_{III} : Number of iterations and average reduction factors; exact blocks A_{ii}^{-1} , Problem 1

n	m	2	4	8	16	32	64
128	*iter*	102	71	42	27	17	11
	ρ	0.822	0.754	0.617	0.468	0.310	0.152
64	*iter*	52	38	24	16	10	
	ρ	0.680	0.590	0.440	0.287	0.140	
32	*iter*	28	21	14	9		
	ρ	0.480	0.375	0.240	0.119		
15	*iter*	15	12	8			
	ρ	0.256	0.175	0.089			

TABLE 2. Preconditioner C_{III} : Number of iterations and average reduction factors; exact blocks A_{ii}^{-1} , Problem 2

n	m	2	4	8	16	32	64
128	*iter*	107	73	44	26	16	10
	ρ	0.840	0.776	0.655	0.483	0.312	0.160
64	*iter*	55	38	25	16	10	
	ρ	0.715	0.614	0.484	0.309	0.147	
32	*iter*	29	21	14	9		
	ρ	0.527	0.407	0.266	0.123		
15	*iter*	16	12	8			
	ρ	0.301	0.20	0.10			

TABLE 3. Preconditioner C_{III} : Number of iterations and average reduction factors; $p = 1$, Problem 1

n	m	4	8	16	32	64
128	*iter*	112	90	78	71	69
	ρ	0.830	0.791	0.765	0.743	0.737
64	*iter*	60	48	43	41	
	ρ	0.706	0.647	0.614	0.601	
32	*iter*	32	27	25		
	ρ	0.528	0.466	0.430		
15	*iter*	17	16			
	ρ	0.302	0.278			

TABLE 4. Preconditioner C_{III}: Number of iterations and average reduction factors; $p = 1$, Problem 2

n	m	4	8	16	32	64
128	*iter*	138	104	86	80	80
	ρ	0.870	0.831	0.80	0.789	0.786
64	*iter*	70	55	47	46	
	ρ	0.762	0.704	0.667	0.655	
32	*iter*	37	30	28		
	ρ	0.602	0.545	0.506		
16	*iter*	20	18			
	ρ	0.395	0.369			

TABLE 5. Preconditioner C_{III}: Number of iterations and average reduction factors; $p = 4$, Problem 1

n	m	4	8	16	32	64
128	*iter*	73	44	28	19	12
	ρ	0.760	0.632	0.486	0.332	0.185
64	*iter*	39	25	17	11	
	ρ	0.594	0.448	0.300	0.160	
32	*iter*	21	15	10		
	ρ	0.383	0.248	0.125		
16	*iter*	12	9			
	ρ	0.181	0.091			

TABLE 6. Preconditioner C_{III}: Number of iterations and average reduction factors; $p = 4$, Problem 2

n	m	4	8	16	32	64
128	*iter*	79	53	34	23	15
	ρ	0.787	0.698	0.581	0.434	0.275
64	*iter*	43	29	19	13	
	ρ	0.651	0.526	0.388	0.240	
32	*iter*	23	16	11		
	ρ	0.453	0.326	0.199		
16	*iter*	13	8			
	ρ	0.233	0.149			

TABLE 7. Wittum's Preconditioner: Number of iterations and average reduction factors

	Problem 1		Problem 2	
n	ρ	*iter*	ρ	*iter*
128	0.285	18	0.460	25
64	0.250	16	0.373	20
32	0.242	16	0.356	19
16	0.193	13	0.312	17

TABLE 8. Classical Block ILU Preconditioner: Number of iterations and average reduction factors; $p = 1$

	Problem 1		Problem 2	
n	ρ	*iter*	ρ	*iter*
128	0.886	173	0.902	188
64	0.789	93	0.821	96
32	0.630	48	0.674	47
16	0.425	26	0.492	27

TABLE 9. Classical Block ILU Preconditioner: Number of iterations and average reduction factors; $p = 4$

	Problem 1		Problem 2	
n	ρ	*iter*	ρ	*iter*
128	0.699	57	0.735	61
64	0.500	31	0.560	32
32	0.283	17	0.337	18
16	0.145	11	0.251	12

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Department of Mathematics, University of California at Los Angeles, 405 Hilgard Avenue, Los Angeles, California 90024-1555

E-mail address: `chan@math.ucla.edu`

Department of Mathematics, University of California at Los Angeles, 405 Hilgard Avenue, Los Angeles, California 90024-1555

Current address: Center of Informatics and Computer Technology, Bulgarian Academy of Sciences, "Acad. G. Bontchev" street, block 25a, 1113 Sofia, Bulgaria

E-mail address: `panayot@bgearn.bitnet`