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CHEBYSHEV-TYPE QUADRATURE AND PARTIAL SUMS OF THE EXPONENTIAL SERIES

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ABSTRACT. Chebyshev-type quadrature for the weight functions

$$w_a(t) = \frac{1-at}{\pi\sqrt{1-t^2}}, \qquad -1 < t < 1, \quad -1 < a < 1,$$

is related to a problem concerning partial sums of the exponential series, namely the problem to extend the nth partial sum to a polynomial of degree $2N$ having all zeros on the circle $|z| = |a|N$. Using this connection, we show that the minimal number N of nodes needed for Chebyshev-type quadrature of degree n for $w_a(t)$ satisfies an inequality $C_1 n \le N \le C_2 n$ with positive constants C_1, C_2. As an application we prove that the minimal number N of nodes for Chebyshev-type quadrature of degree n on a torus embedded in $\mathbf{R}^3$ satisfies an inequality $C_1 n^2 \le N \le C_2 n^2$.

1. INTRODUCTION AND STATEMENT OF RESULTS

A Chebyshev-type quadrature formula is a numerical integration formula in which all weights are equal. For an integrable nonnegative weight function $w(t)$ on $[-1, 1]$ with $\int_{-1}^{1} w(t)\,dt = 1$, this is a formula of the type

$$\int_{-1}^{1} f(t)w(t)\,dt \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i) \tag{1.1}$$

with (not necessarily distinct) nodes $x_i \in [-1, 1]$, $i = 1, \dots, N$. We call N the size of (1.1). The degree of (1.1) is the maximal number n such that equality holds for every polynomial $f(t)$ of degree $\le n$. We say that $w(t)$ admits Chebyshev-type quadrature of size N and degree n if there exist N points $x_i \in [-1, 1]$ such that (1.1) has degree n. See [2, 3], for surveys on Chebyshev-type quadrature.

If $N \le n$, then (1.1) is called a Chebyshev quadrature formula. We say that $w(t)$ admits Chebyshev quadrature if a Chebyshev quadrature formula exists for every n. The classical example of a weight function which admits

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Chebyshev quadrature is the function $(1-t^2)^{-1/2}/\pi$, but more examples are known; see [2] and the references therein.

In this paper we consider the weight functions

$$w_a(t) = \frac{1-at}{\pi\sqrt{1-t^2}}, \qquad -1 < t < 1, \quad -1 \le a \le 1.$$

These functions arise in connection with quadrature problems on the surface of a torus; see §5. It has been proved by Xu [16] that $w_a(t)$ admits Chebyshev quadrature if $|a| < \gamma = 0.27846\ldots$, where γ is the unique positive root of $xe^{1+x} = 1$.

We show that for the weight functions $w_a(t)$, the existence of Chebyshev-type quadrature is related to properties of the partial sums $s_n(z)$ of the exponential series,

$$s_n(z) = \sum_{k=0}^{n} \frac{z^k}{k!}.$$

Using a general condition for the existence of Chebyshev-type quadrature (Theorem 1), we find that $w_a(t)$ admits Chebyshev-type quadrature of size N and degree $\ge n$ if and only if $s_n(z)$ can be extended to a real polynomial of degree $2N$ having all its zeros on the circle $|z| = |a|N$. Here we say that $p(z)$ is an extension of $s_n(z)$ if $p(z) = s_n(z) + \mathscr{O}(z^{n+1})$ $(z \to 0)$. Furthermore, if $s_n(z)$ has an extension to a polynomial of degree $2N-n-1$ which has all its zeros in $|z| > |a|N$, then $w_a(t)$ admits Chebyshev-type quadrature of size N and degree $\ge n$.

Thus, we are led to consider extensions of $s_n(z)$ which have their zeros as far from the origin as possible. Our main results are as follows:

- For $0 < R < \frac{1}{2}$ there is a constant c such that every $s_n(z)$ has an extension to a polynomial of degree cn which has no zeros in $|z| < Rcn$ (Theorem 6).
- For $a \in (-1, 1)$ there exist positive constants C_1, C_2 such that $w_a(t)$ admits Chebyshev-type quadrature of degree n and size N where

  $$C_1 n \le N \le C_2 n$$

  (Corollary 7). An application to Chebyshev-type quadrature on the surface of the torus is given in Theorem 8.
- For $|a| = 1$ the corresponding bounds are

  $$C_1 n^3 \le N \le C_2 n^3,$$

  which imply that $s_n(z)$ can be extended to a polynomial of degree $N \approx Cn^3$ which has all its zeros on the circle $|z| = N/2$ (Corollary 5).
- The bound $|a| < \gamma$ for the existence of Chebyshev quadrature is sharp: For $|a| > \gamma$, the weight function $w_a(t)$ does not admit Chebyshev quadrature (Proposition 4).

2. Condition for Chebyshev-type quadrature

Let $w(t)$ be a weight function on $[-1, 1]$ with $\int_{-1}^{1} w(t)\,dt = 1$. Set

$$c_k = 2\int_{-1}^{1} T_k(t)w(t)\,dt, \qquad k \geq 1, \tag{2.1}$$

where $T_k(t)$ is the Chebyshev polynomial of the first kind of degree k, and construct the power series

$$G(z) = \sum_{k=1}^{\infty} \frac{c_k}{k} z^k, \tag{2.2}$$

which is analytic in $|z| < 1$. In view of the formula

$$-\log(1 - 2tz + z^2) = \sum_{k=1}^{\infty} \frac{2}{k} T_k(t) z^k,$$

cf. [14, equation (4.7.25)], we see that

$$G(z) = -\int_{-1}^{1} \log(1 - 2tz + z^2) w(t)\,dt.$$

In terms of the function $G(z)$ we have the following conditions for the existence of Chebyshev-type quadrature. Theorem 1 is a slight modification of results due to Geronimus [4, Theorem 1] and Peherstorfer [8, Theorem 1], [9, Theorem 3]. For convenience of the reader we have included the proof.

Theorem 1. *Let $w(t)$ be a nonnegative integrable function on $[-1, 1]$ such that $\int_{-1}^{1} w(t)\,dt = 1$. Let $G(z)$ be defined by* (2.1) *and* (2.2). *Let $n, N \in \mathbf{N}$. Then the following hold:*

1. *The weight function $w(t)$ admits Chebyshev-type quadrature of size N and degree $\geq n$ with all nodes in the open interval $(-1, 1)$ if and only if there is a real polynomial $P(z)$ of degree $2N$ such that*
   (a) $P(z) = \exp(-NG(z)) + \mathcal{O}(z^{n+1}) \quad (z \to 0)$,
   (b) *all zeros of $P(z)$ are nonreal and have modulus* 1.
2. *If there is a real polynomial $p(z)$ of degree $2N - n - 1$ such that*
   (a) $p(z) = \exp(-NG(z)) + \mathcal{O}(z^{n+1}) \quad (z \to 0)$,
   (b) *all zeros of $p(z)$ have modulus* > 1,
   then $w(t)$ admits Chebyshev-type quadrature of size N and degree $\geq n$.

Proof. Suppose $P(z)$ satisfies 1.(a)(b), so that $P(0) = 1$ and all zeros of $P(z)$ are complex and come in conjugate pairs. Then there are $\phi_j \in (0, \pi)$, $j = 1, \ldots, N$, such that

$$P(z) = \prod_{j=1}^{N} (e^{i\phi_j} - z)(e^{-i\phi_j} - z). \tag{2.3}$$

We will compute the logarithmic derivative of $P(z)$ in two ways. From (a) and (2.2) we have

$$\frac{P'(z)}{P(z)} = -NG'(z) + \mathcal{O}(z^n) = -N\sum_{k=1}^{n} c_k z^{k-1} + \mathcal{O}(z^n) \qquad (z \to 0). \tag{2.4}$$

From (2.3) it follows that

$$\begin{aligned}\frac{P'(z)}{P(z)} &= -\sum_{j=1}^{N}\left[\frac{1}{e^{i\phi_j} - z} + \frac{1}{e^{-i\phi_j} - z}\right] = -2\sum_{j=1}^{N}\sum_{k=1}^{\infty}\cos(k\phi_j)z^{k-1} \\ &= -2\sum_{k=1}^{\infty}\sum_{j=1}^{N} T_k(x_j)z^{k-1},\end{aligned} \tag{2.5}$$

where we have written $\cos\phi_j = x_j$.

Comparing coefficients in (2.4) and (2.5) and using (2.1), we find

$$\frac{1}{N}\sum_{j=1}^{N} T_k(x_j) = \frac{c_k}{2} = \int_{-1}^{1} T_k(t)w(t)dt, \qquad k = 1, \dots, n,$$

that is, the points $x_j \in (-1, 1)$, $j = 1, \dots, N$, are the nodes of a Chebyshev-type quadrature formula for $w(t)$ of degree $\geq n$.

Conversely, if $x_j \in (-1, 1)$, $j = 1, \dots, N$, are the nodes of a Chebyshev-type quadrature formula of degree $\geq n$, then writing $x_j = \cos\phi_j$ and defining $P(z)$ as in equation (2.3), we can easily check that $P(z)$ satisfies 1.(a)(b).

Next, assume that the real polynomial $p(z)$ of degree $2N - n - 1$ satisfies 2.(a)(b). Let $p^*(z) = z^{2N-n-1}p(z^{-1})$ denote the reciprocal polynomial of $p(z)$. Then

$$P(z) := p(z) + z^{n+1}p^*(z)$$

is a real polynomial of degree $2N$ (exactly) which has all its zeros on the unit circle, cf. [10, pp. 88 and 256] for a related result of Schur. Note that $P(\pm 1) = 2p(\pm 1) \neq 0$, so that $P(z)$ satisfies condition 1.(b). Since $p(z)$ satisfies 2.(a), it is clear from the definition of $P(z)$ that $P(z)$ satisfies 1.(a) and the theorem follows. □

3. The weight functions $w_a(t)$

For the weight function

$$w_a(t) = \frac{1 - at}{\pi\sqrt{1 - t^2}}, \qquad t \in (-1, 1), \quad -1 \leq a \leq 1,$$

the function $G(z)$ of formula (2.2) is simply $G(z) = -az$, and the condition 2.(a) of Theorem 1 is

$$p(z) = \exp(aNz) + \mathcal{O}(z^{n+1}) \qquad (z \to 0).$$

Denoting by $s_n(z) = \sum_{j=0}^{n} z^j/j!$ the nth partial sum of the exponential series, we obtain for $a \neq 0$,

$$p(z/aN) = s_n(z) + \mathcal{O}(z^{n+1}) \qquad (z \to 0).$$

A result of Seymour and Zaslavsky [11, Corollary 2] shows that for every n, Chebyshev-type quadrature formulas of degree $\geq n$ exist *in case the size* N *is sufficiently large.* So part 1 of Theorem 1 implies

Corollary 2. *Let* $n \in \mathbf{N}$, $0 < a \leq 1$. *For* N *sufficiently large,* $s_n(z)$ *has an extension to a real polynomial of degree* $2N$ *having all its zeros on the circle* $|z| = aN$.

Note that the bound $a \leq 1$ is sharp. For $a > 1$, it is not possible that every $s_n(z)$ has an extension to a real polynomial of degree $2N$ having all its zeros on $|z| = aN$, since that would imply that Chebyshev-type quadrature of every degree exists for the weight function $(1 - at)/(\pi\sqrt{1 - t^2})$ which assumes negative values in $(-1, 1)$. This is impossible, since a slim high-peaked impulse function, centered at a point where the weight function is negative could be approximated arbitrarily closely by a polynomial of sufficiently high degree whose square could then be taken in the role of f in (1.1). This would produce a negative number on the left, and a nonnegative number on the right.

Part 2 of Theorem 1 gives the following condition for the existence of a Chebyshev-type quadrature for $w_a(t)$.

Corollary 3. *Let* $-1 \leq a \leq 1$. *If* $s_n(z)$ *has an extension to a polynomial of degree* $2N - n - 1$ *which has all its zeros in* $|z| > |a|N$, *then there exists a Chebyshev-type quadrature formula for* $w_a(t)$ *of size* N *and degree* $\geq n$.

The question of Chebyshev quadrature for $w_a(t)$ has been discussed by Xu [16]. He proved that $w_a(t)$ admits Chebyshev quadrature if $|a| < \gamma = 0.2784645\ldots$, where γ is the unique positive solution of $xe^{1+x} = 1$.

Corollary 3 with $N = n + 1$ shows that Chebyshev-type quadrature of size $n + 1$ and degree $\geq n$ is possible if the zeros of $s_{n+1}(z)$ have absolute value $> |a|(n + 1)$. [Take $s_{n+1}(z)$ as the extension of $s_n(z)$.] The behavior of the zeros of $s_n(z)$ has been well studied. It is a classical result of Szegö [13] that accumulation points of the zeros of the normalized partial sums $s_n(nz)$ lie on the curve given by

$$|e^{1-z}z| = 1, \qquad |z| \leq 1.$$

Later, Buckholtz [1] showed that all zeros lie outside this curve. The point on the curve with smallest absolute value is on the negative real axis and is $-\gamma$, which is in accordance with Xu's result. For more details on the zeros of $s_n(z)$, see [15, Chapter 4].

Using Theorem 1, we can prove that Xu's bound $|a| < \gamma$ for the existence of Chebyshev quadrature is sharp.

Proposition 4. *For* $|a| > \gamma$, *the weight function* $w_a(t)$ *does not admit Chebyshev quadrature.*

Proof. Without loss of generality we take $a \in (\gamma, 1)$.

Let $n \in \mathbf{N}$ and suppose that $w_a(t)$ admits Chebyshev quadrature of degree n. By part 1 of Theorem 1 there is a real polynomial $P(z)$ of degree $2n$ having all its zeros on the unit circle and satisfying

$$P(z) = \exp(anz) + \mathcal{O}(z^{n+1}) \qquad (z \to 0).$$

Then $P(z) = P^*(z)$ and it easily follows that

$$P(z) = q_n(z) + z^n q_n^*(z)$$

with

$$q_n(z) = \sum_{k=0}^{n-1} \frac{(anz)^k}{k!} + \frac{1}{2}\frac{(anz)^n}{n!} = \frac{1}{2}(s_n(anz) + s_{n-1}(anz)).$$

In particular, $P(1) = 2q_n(1) > 0$ and $P(-1) = 2q_n(-1)$. If we could show that $q_n(-1) < 0$, then it would follow that $P(z)$ has a zero in the interval $(-1, 1)$, which would be a contradiction. Therefore we will show that $q_n(-1) < 0$ for n sufficiently large (in fact only for n even).

Since for $z \in \mathbf{C}$,

$$e^{-z}s_n(z) = 1 - \frac{1}{n!}\int_0^z e^{-t}t^n\,dt$$

(which can be verified by differentiation), it follows that

$$\begin{aligned}\frac{1}{2}e^{-z}(s_n(z) + s_{n-1}(z)) &= 1 - \frac{1}{2n!}\int_0^z e^{-t}(t^n + nt^{n-1})\,dt \\ &= 1 - \frac{(-1)^n n^{n+1}}{2n!}\int_0^{-z/n} e^{nx}(x^{n-1} - x^n)\,dx,\end{aligned}$$

where we have made the substitution $t = -nx$. Hence

$$e^{-anz}q_n(z) = 1 - \frac{(-1)^n n^{n+1}}{2n!}e^{-n}\int_0^{-az} e^{(1+x)n}x^n\left(\frac{1}{x} - 1\right)dx$$

and we see that $q_n(-1) < 0$ if and only if n is even and

$$\int_0^a e^{(1+x)n}x^n\left(\frac{1}{x} - 1\right)dx > \frac{2n!e^n}{n^{n+1}}. \tag{3.1}$$

For the left-hand side we have

$$\int_0^a e^{(1+x)n}x^n\left(\frac{1}{x} - 1\right)dx > \left(\frac{1}{a} - 1\right)\int_0^a (e^{1+x}x)^n\,dx.$$

Since $a > \gamma$ we have $e^{1+a}a > 1$ and we see that that the left-hand side of (3.1) increases exponentially as $n \to \infty$. Further, the right-hand side tends to 0 for $n \to \infty$, so that for n large enough the inequality (3.1) holds and the proposition follows. □

For the special cases $a = \pm 1$, the weight function $w_a(t)$ is a Jacobi weight function. Chebyshev-type quadrature for $w_a(t)$ is related to Chebyshev-type quadrature for the ultraspherical weight function $2(1 - s^2)^{1/2}/\pi$ because of the relation

$$\frac{1}{\pi}\int_{-1}^1 f(t)(1 - t)^{1/2}(1 + t)^{-1/2}dt = \frac{2}{\pi}\int_{-1}^1 f(2s^2 - 1)(1 - s^2)^{1/2}\,ds.$$

[We have taken $a = +1$.] Using this relation and the symmetry of the weight function $2(1 - s^2)^{1/2}/\pi$ we get the following:

If $x_1, \ldots, x_N$ are the nodes of a Chebyshev-type quadrature formula for $w_1(t)$ of degree n then the $2N$ points

$$\pm\left(\frac{x_1+1}{2}\right)^{1/2}, \pm\left(\frac{x_2+1}{2}\right)^{1/2}, \ldots, \pm\left(\frac{x_N+1}{2}\right)^{1/2}$$

are the nodes of a Chebyshev-type quadrature formula for $2(1-s^2)^{1/2}/\pi$ of degree $2n+1$.

Conversely, if $\pm y_1, \ldots, \pm y_N$ are the nodes of a symmetric Chebyshev-type quadrature formula for $2(1-s^2)^{1/2}/\pi$ of degree $2n+1$, then

$$2y_1^2-1, 2y_2^2-1, \ldots, 2y_N^2-1$$

are the nodes of a Chebyshev-type quadrature formula of degree n for $w_1(t)$.

For the weight function $2(1-s^2)^{1/2}/\pi$, the author [7] has shown that the minimal number N of nodes needed for Chebyshev-type quadrature of degree n satisfies an inequality

$$C_1 n^3 \le N \le C_2 n^3,$$

where C_1, C_2 are positive constants which do not depend on n. Hence also for $w_1(t)$, Chebyshev-type quadrature of degree n is possible with $\approx Cn^3$ nodes, and this is the correct order. Now part 1 of Theorem 1 immediately gives:

Corollary 5. *There exist constants $C_1, C_2 > 0$ such that, for every $n \in \mathbf{N}$, $s_n(z)$ has an extension to a polynomial of degree N with $C_1 n^3 \le N \le C_2 n^3$ whose zeros are nonreal and all lie on the circle $|z| = N/2$. The order n^3 cannot be improved.*

For $\gamma < |a| < 1$ no results on Chebyshev-type quadrature seem to be known. We will show that the minimal number of nodes N needed for Chebyshev-type quadrature of degree n for $w_a(t)$ satisfies an inequality $C_1 n \le N \le C_2 n$. The positive constants C_1 and C_2 depend on a but not on n. To obtain this result, we will construct extensions of $s_n(z)$.

4. Extension of partial sums of the exponential series

We will prove the following theorem.

Theorem 6. *Let $0 < R < \frac{1}{2}$. Then there is a constant $c_0 = c_0(R) \in \mathbf{N}$ such that, for every n and every $c \ge c_0$, $s_n(z)$ has an extension to a polynomial of degree cn which is zero-free in the disc $|z| < Rcn$.*

Remark. a) From the results of Szegö [13] and Buckholtz [1] (see §3) it follows that one can take $c_0 = 1$ in case $R < \gamma = 0.2784645\ldots$.

b) The theorem does not hold for $R \ge \frac{1}{2}$; see Corollary 5.

Proof. Motivated by the relation

$$e^{-z}s_n(z) = 1 - \frac{1}{n!}\int_0^z e^{-t}t^n\,dt,$$

we will study polynomials $S_N(z)$ satisfying

$$e^{-z}S_N(z) = 1 - \frac{1}{A_n}\int_0^z e^{-t}t^n p_m(t)^n\,dt. \tag{4.1}$$

Here $p_m(t)$ will be a polynomial of degree m and

$$A_n = \int_0^\infty e^{-t} t^n p_m(t)^n \, dt. \tag{4.2}$$

It is easy to see that with this choice of A_n, (4.1) defines a polynomial $S_N(z)$ of degree $N := (1+m)n$ and

$$S_N(z) = s_n(z) + \mathcal{O}(z^{n+1}) \qquad (z \to 0).$$

Thus, $S_N(z)$ is an extension of $s_n(z)$ to a polynomial of degree $(1+m)n$.

For $p_m(t)$ we take $q_m(t/N)$, where $q_m(w)$ is a monic polynomial with real coefficients. In the integrals of (4.1) and (4.2) we make the substitution $t = wN$ to obtain

$$e^{-z} S_N(z) = 1 - \frac{1}{B_n} \int_0^{z/N} \left[e^{-(1+m)w} w q_m(w) \right]^n dw. \tag{4.3}$$

with

$$B_n = \int_0^\infty \left[e^{-(1+m)w} w q_m(w) \right]^n dw. \tag{4.4}$$

From (4.3) it is clear that $S_N(z)$ is zero-free in the region defined by

$$\left| \int_0^{z/N} \left[e^{-(1+m)w} w q_m(w) \right]^n dw \right| < B_n. \tag{4.5}$$

The rest of the proof will be divided into three steps. In Step 1 we introduce an auxiliary function $F(z)$ and establish some basic properties. In Step 2 we define for every m a polynomial $q_m(t)$ and a function $F_m(z)$. We show that $F_m(z)$ tends to $F(z)$ as $m \to \infty$. Using these results we will show in Step 3 that for m large enough (say $m \geq m_0$) the inequality (4.5) holds for every $n \in \mathbf{N}$ and for every $|z| < RN = R(1+m)n$.

Then the theorem follows with $c_0 = 1 + m_0$.

Step 1. Take $r = 2R^2$ so that $0 < r < R < 1/2$ and let ρ be the measure on the circle $\xi = re^{i\theta}$ given by

$$d\rho(\xi) = \frac{1 - \cos\theta}{2\pi} d\theta.$$

The moments of ρ are easily computed:

$$\int \xi^k d\rho(\xi) = \begin{cases} 1 & \text{for } k = 0, \\ -r/2 & \text{for } k = 1, \\ 0 & \text{for } k \geq 2. \end{cases}$$

Define for $|z| > r$,

$$F(z) = -\text{Re } z + \int \log|z - \xi| \, d\rho(\xi). \tag{4.6}$$

Since for $|z| > r$,

$$\begin{aligned}\int \log|z-\xi|\,d\rho(\xi) &= \log|z| + \mathrm{Re}\int \log\left(1-\frac{\xi}{z}\right)d\rho(\xi)\\ &= \log|z| - \mathrm{Re}\sum_{k=1}^{\infty}\frac{1}{kz^k}\int \xi^k d\rho(\xi)\\ &= \log|z| + \frac{r}{2}\mathrm{Re}\,\frac{1}{z},\end{aligned}$$

we have

$$F(z) = -\mathrm{Re}\,z + \log|z| + \frac{r}{2}\mathrm{Re}\,\frac{1}{z}. \tag{4.7}$$

We need two properties of $F(z)$.

A: *$F(z)$ is constant on the circle $|z| = R$.*

Indeed, since $\mathrm{Re}\,(1/z) = \mathrm{Re}\,z/|z|^2$, we have for $|z| = R$,

$$F(z) = \log R + \mathrm{Re}\,z\left[-1+\frac{r}{2R^2}\right] = \log R.$$

B: *$F(x)$ is strictly increasing on the interval $(r,\, 1/2+\epsilon)$, where*

$$\epsilon := (1/4 - R^2)^{1/2} > 0. \tag{4.8}$$

Indeed, using (4.7), we compute for $z = x > r$,

$$F'(x) = -1 + \frac{1}{x} - \frac{r}{2x^2} = \frac{-(x-1/2)^2+\epsilon^2}{x^2}$$

and property B follows.

From properties A and B we obtain (recall $r < R < 1/2$)

$$\max_{|z|=R} F(z) < F(1/2) \tag{4.9}$$

and for some $\delta > 0$,

$$F(1/2) + \delta < F(x) \qquad \text{for all } x \in (1/2+\epsilon/2,\, 1/2+\epsilon). \tag{4.10}$$

In the rest of the proof, ϵ as defined in (4.8) and δ satisfying (4.10) will be fixed.

Step 2. For every m, take m points $\xi_{1,m}, \dots, \xi_{m,m}$ on the circle $|\xi| = r$ as follows. We let $\xi_{j,m} = re^{i\theta_{j,m}}$, where

$$\int_0^{\theta_{1,m}} \frac{1-\cos\theta}{2\pi}d\theta = \frac{1}{2m}, \qquad \int_{\theta_{j,m}}^{\theta_{j+1,m}} \frac{1-\cos\theta}{2\pi}d\theta = \frac{1}{m}, \quad j = 1, \dots, m-1.$$

In this way, we have $\xi_{m+1-j} = \bar{\xi}_j$ and no $\xi_{j,m}$ is real and positive. We also define $\xi_{0,m} = 0$. Put

$$q_m(z) = \prod_{j=1}^{m}(z - \xi_{j,m}).$$

Then $q_m(z)$ is a monic polynomial of degree m with real coefficients and $q_m(z) > 0$ for z real and positive. Let ρ_m be the normalized counting measure of the points $\xi_{0,m}, \xi_{1,m}, \dots, \xi_{m,m}$:

$$\rho_m = \frac{1}{m+1} \sum_{j=0}^{m} \delta_{\xi_{j,m}}.$$

The measures ρ_m converge to ρ in the weak *-topology for convergence of measures. Write

$$\begin{aligned} F_m(z) &= -\mathrm{Re}\, z + \frac{1}{m+1} \log |z q_m(z)| \\ &= -\mathrm{Re}\, z + \frac{1}{m+1} \sum_{j=0}^{m} \log |z - \xi_{j,m}| \qquad (4.11) \\ &= -\mathrm{Re}\, z + \int \log |z - \xi| d\rho_m(\xi). \end{aligned}$$

The function $F_m(z)$ is subharmonic on $\mathbf{C}$ and is harmonic for $z \neq \xi_{j,m}$, $j = 0, \dots, m$, so in particular, $F_m(z)$ is harmonic for $|z| > r$.

Comparing (4.6) and (4.11), we have that

$$\lim_{m \to \infty} F_m(z) = F(z) \tag{4.12}$$

pointwise for $|z| > r$. As the points $\xi_{j,m}$, $j = 0, \dots, m$, have absolute values $\leq r$, it easily follows from (4.11) that the functions $F_m(z)$ are uniformly bounded on compact subsets of $|z| > r$. Since the functions $F_m(z)$ are harmonic for $|z| > r$, this implies that they form a normal family (see, e.g., [5, Theorem 2.18]). It follows that the limit (4.12) is uniform on every compact subset of $|z| > r$.

Then by (4.9) and (4.10) we have for all m sufficiently large,

$$\max_{|z|=R} F_m(z) < F(1/2), \tag{4.13}$$

and

$$F(1/2) + \delta < F_m(x) \qquad \text{for all } x \in (1/2 + \epsilon/2,\ 1/2 + \epsilon). \tag{4.14}$$

Since $F_m(z)$ is subharmonic on $\mathbf{C}$, (4.13) also gives

$$\max_{|z| \leq R} F_m(z) < F(1/2). \tag{4.15}$$

Step 3. We take m such that (4.14), (4.15) hold and such that

$$e^{(1+m)\delta} \epsilon/2 \geq R. \tag{4.16}$$

For a given $n \in \mathbf{N}$ we write $N = (1+m)n$ and we are going to prove (4.5) for $|z| < RN$.

Note that by the definition (4.11)

$$\left|e^{-(1+m)w} w q_m(w)\right| = e^{(1+m)F_m(w)}. \tag{4.17}$$

Thus, if $|z| < RN$, then by (4.15) and (4.17),

$$\left|\int_0^{z/N} \left[e^{-(1+m)w} w q_m(w)\right]^n dw\right| \le Re^{NF(1/2)}. \tag{4.18}$$

Also by (4.4), (4.14), (4.17) and the fact that $q_m(w) > 0$ for $w > 0$, we have

$$B_n \ge \int_{1/2+\epsilon/2}^{1/2+\epsilon} e^{NF_m(w)} dw \ge e^{N(F(1/2)+\delta)}\epsilon/2. \tag{4.19}$$

From (4.18) and (4.19) we see that (4.5) holds for every $|z| < RN$ if $e^{N\delta}\epsilon/2 \ge R$. Since $N = (1+m)n$, this follows from (4.16). □

Corollary 7. *For every $a \in (-1, 1)$ there exist constants C_1, C_2 depending on a, such that, for every n, the minimal number N of nodes in a Chebyshev-type quadrature formula of degree n for $w_a(t)$ satisfies the inequalities*

$$C_1 n \le N \le C_2 n.$$

Proof. The upper bound follows easily from Theorem 6 and Corollary 3.

For the lower bound we take $C_1 = 1/2$. It is a general result that for any quadrature formula of degree $\ge n$ with arbitrary weights one needs more than $n/2$ nodes. □

5. Chebyshev-type quadrature on the torus

Fix $0 < a < 1$ and let T_a be the torus embedded in $\mathbf{R}^3$ with parametrization

$$\begin{aligned} x &= \cos\phi(1 + a\cos\psi), \\ y &= \sin\phi(1 + a\cos\psi), \qquad 0 \le \phi < 2\pi, \quad 0 \le \psi < 2\pi. \\ z &= a\sin\psi, \end{aligned}$$

The surface element is $a(1 + a\cos\psi)d\phi d\psi = d\sigma$ and the surface area is $4\pi^2 a$. A Chebyshev-type quadrature formula for T_a is a formula of the form

$$\frac{1}{4\pi^2 a}\iint_{T_a} f(x, y, z)\, d\sigma \approx \frac{1}{N}\sum_{i=1}^{N} f(x_i, y_i, z_i) \tag{5.1}$$

with $(x_i, y_i, z_i) \in T_a$. The degree of (5.1) is the maximal n such that equality holds for all polynomials in three variables $f(x, y, z)$ of total degree $\le n$.

Multidimensional Chebyshev-type quadrature formulas for various other regions were given by Korevaar and Meyers [6].

Theorem 8. *Let $0 < a < 1$. There exist constants $C_1, C_2 > 0$ (depending on a) such that the minimal number N of nodes needed for Chebyshev-type quadrature on T_a of degree $\ge n$ satisfies the inequalities*

$$C_1 n^2 \le N \le C_2 n^2.$$

Proof. The lower bound follows from a result on general quadrature formulas (i.e., not necessarily with equal weights) for 2-dimensional domains. Let

$$\frac{1}{4\pi^2 a}\iint_{T_a} f(x, y, z)\, d\sigma \approx \sum_{i=1}^{N} \lambda_i f(x_i, y_i, z_i)$$

be a quadrature formula of degree n with weights λ_i. Then for polynomials $g(x, y)$ of two variables of degree $\leq n$, we have

$$\iint_{A_a} g(x, y) w(x, y)\, dx dy = \sum_{i=1}^{N} \lambda_i g(x_i, y_i),$$

where A_a is the annulus

$$A_a := \{(x, y) \mid 1 - a \leq \sqrt{x^2 + y^2} \leq 1 + a\},$$

and $w(x, y)$ is a positive weight function on A_a. A result of Stroud [12, Theorem 3.15-1] shows that the number of nodes satisfies $N \geq n^2/8$.

For the upper bound, we first consider polynomials $f(x, y, z)$ of degree $\leq n$ which are even in y and z. For such polynomials we have

$$\begin{aligned}\frac{1}{4\pi^2 a}\iint_{T_a} f(x, y, z)\, d\sigma &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} F(\phi, \psi)(1 + a\cos\psi)\, d\phi d\psi \\ &= \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi} F(\phi, \psi)(1 + a\cos\psi)\, d\phi d\psi,\end{aligned}$$

where we have written

$$F(\phi, \psi) = f(\cos\phi(1 + a\cos\psi), \sin\phi(1 + a\cos\psi), a\sin\psi).$$

There is a polynomial $p(s, t)$ of degree $\leq 2n$ such that

$$p(\cos\phi, \cos\psi) = F(\phi, \psi),$$

and the substitutions $\cos\phi = s$, $\cos\psi = t$ give

$$\frac{1}{4\pi^2 a}\iint_{T_a} f(x, y, z)\, d\sigma = \int_{-1}^{1}\int_{-1}^{1} p(s, t)\frac{ds}{\pi\sqrt{1 - s^2}}\frac{(1 + at)dt}{\pi\sqrt{1 - t^2}}.$$

According to Corollary 7 there exist a constant $C > 0$, not depending on n, and $N_1 \leq 2Cn$ points $t_1, \ldots, t_{N_1}$ which are the nodes of a Chebyshev-type quadrature formula for $w_{-a}(t)$ of degree $2n$. There also exist $n + 1$ points $s_1, \ldots, s_{n+1}$ which are the nodes of a Chebyshev-type quadrature for $w_0(t)$ of degree $2n$. (Simply take the nodes of the $(n + 1)$-point Gauss-Chebyshev quadrature formula.) Then it is easy to see that

$$\int_{-1}^{1}\int_{-1}^{1} p(s, t)\frac{ds}{\pi\sqrt{1 - s^2}}\frac{(1 + at)dt}{\pi\sqrt{1 - t^2}} = \frac{1}{N_1(n + 1)}\sum_{i=1}^{n+1}\sum_{j=1}^{N_1} p(s_i, t_j)$$

holds for every polynomial $p(s, t)$ of degree $\leq 2n$. Take $\phi_i = \arccos s_i$, $\psi_j = \arccos t_j$, and

$$x_{ij} = \cos\phi_i(1 + a\cos\psi_j), \quad y_{ij} = \sin\phi_i(1 + a\cos\psi_j), \quad z_{ij} = a\sin\psi_j.$$

Then

$$\frac{1}{4\pi^2 a}\iint_{T_a} f(x, y, z)\,d\sigma = \frac{1}{N_1(n+1)}\sum_{i=1}^{n+1}\sum_{j=1}^{N_1} f(x_{ij}, y_{ij}, z_{ij})$$

holds for all polynomials $f(x, y, z)$ of degree $\leq n$ which are even in y and z. By the symmetry in y and z, it then follows that the $4N_1(n+1)$ points

$$(x_{ij}, \pm y_{ij}, \pm z_{ij}), \qquad i = 1, \dots, n+1, \quad j = 1, \dots, N_1,$$

are the nodes of a Chebyshev-type quadrature formula of size $\leq 8Cn(n+1)$ and degree n. □

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