MATHEMATICS OF COMPUTATION
VOLUME 64, NUMBER 209
JANUARY 1995, PAGES 295-304

ON THE ABSOLUTE MAHLER MEASURE OF POLYNOMIALS HAVING ALL ZEROS IN A SECTOR

GEORGES RHIN AND CHRISTOPHER SMYTH

ABSTRACT. Let α be an algebraic integer of degree d, not 0 or a root of unity, all of whose conjugates α_i are confined to a sector $|\arg z| \le \theta$. We compute the greatest lower bound $c(\theta)$ of the absolute Mahler measure $(\prod_{i=1}^{d} \max(1, |\alpha_i|))^{1/d}$ of α, for θ belonging to nine subintervals of $[0, 2\pi/3]$. In particular, we show that $c(\pi/2) = 1.12933793$, from which it follows that any integer $\alpha \ne 1$ and $\alpha \ne e^{\pm i\pi/3}$ all of whose conjugates have positive real part has absolute Mahler measure at least $c(\pi/2)$. This value is achieved for α satisfying $\alpha + 1/\alpha = \beta_0^2$, where $\beta_0 = 1.3247\ldots$ is the smallest Pisot number (the real root of $\beta_0^3 = \beta_0 + 1$).

1. INTRODUCTION

Let $P(z) \ne z$ be a monic polynomial with integer coefficients, irreducible over the rationals, of degree $d \ge 1$, and having zeros $\alpha_1, \ldots, \alpha_d$. Its *relative* Mahler measure ("height") $M(P)$, given by

$$M(P) = \prod_{i=1}^{d} \max(1, |\alpha_i|),$$

is either 1 (if P is cyclotomic) or thought to be bounded away from 1 by an absolute constant (if P is not cyclotomic) [1, 2]. When the zeros of P are restricted to a closed set V which does not contain the whole unit circle, however, one can say much more. Then, from a result of Langevin [4] there is a constant $c_V > 1$ such that the *absolute* Mahler measure $\Omega(P) := M(P)^{1/d}$ for such P is either 1 or else satisfies

$$\Omega(P) \ge c_V.$$

The aim of this paper is to try to find the largest value for the constants c_V when V is the sector $\{z: |\arg z| \le \theta\}$, where $0 \le \theta < \pi$. We denote this best value by $c(\theta)$. It is clear that $c(\theta)$ is a nonincreasing function of θ, and, using the polynomials $z^{2k+1} - 2$ as $k \to \infty$, that $c(\theta) \to 1$ as $\theta \to \pi$. We succeed in finding $c(\theta)$ exactly for θ in nine intervals (see the Theorem below). We suspect that in fact $c(\theta)$ is a "staircase" function of θ, which is constant except for finitely many left discontinuities in any interval $[0, \Theta)$ for $\Theta < \pi$. [It is clear that $c(\theta)$ would then have infinitely many discontinuities on $[0, \pi)$.]

Received by the editor June 24, 1993 and, in revised form, December 6, 1993.
1991 *Mathematics Subject Classification.* Primary 11R04, 12D10.

TABLE 1. Intervals $[\theta_i, \theta_i']$ where $c(\theta)$ is known exactly. Here $c(\theta) = c(\theta_i) = \Omega(P)$ for $\theta \in [\theta_i, \theta_i']$, and b_i is a lower bound for $c(\theta_i-) - c(\theta_i)$. The polynomial P is read off from Table 3

i	c(θ)	θ_i	θ_i'	P	b_i
1	1.61803399	0.00000000	17.39	P2	
2	1.53922234	26.40874008	26.65	P7	0.00085
3	1.49363278	30.44014506	30.59	P8	0.00341
4	1.30305506	47.94143202	49.46	P9	0.00002
5	1.25926867	60.89019592	63.87	P12	0.00001
6	1.21060779	73.63161482	73.99	P14	0.00006
7	1.15461811	80.24103363	81.40	P17	0.00006
8	1.12933793	86.70851871	91.40	P18	0.00001
9	1.05542318	112.64711862	115.32	P21	0.00008

Our main result is the following:

Theorem. *There is a continuous, monotonically decreasing function $f(\theta)$, which is > 1 for $0 \le \theta \le 2\pi/3$, and a staircase function $g(\theta) > 1$ such that*

$$\min(f(\theta), g(\theta)) \le c(\theta) \le g(\theta) \qquad (0 \le \theta < \pi).$$

Table 1 shows nine intervals $[\theta_i, \theta_i']$ where $f(\theta) > g(\theta)$, so that $c(\theta) = g(\theta) = g(\theta_i)$ for θ in those intervals. Furthermore, $c(\theta)$ has a discontinuity at $\theta = \theta_i$ $(\theta > 0)$, a lower bound $b_i := f(\theta_i) - g(\theta_i)$ for $c(\theta_i-) - c(\theta_i)$ being shown in Table 1 also. We call such angles θ_i *critical* angles. The functions f and g are shown in Figure 1.

The function $f(\theta)$ is given by $f(\theta) := \max(f_1(\theta), f_2(\theta), \dots, f_9(\theta))$, where the $f_i(\theta)$ are defined as follows:

Let W_θ be the sector $\{|z| \le 1, |\arg z| \le \theta\}$. Then

$$f_i(\theta) = \left\{ \max_{z \in W_\theta} \left| z^{a_i} \prod_j P_{ij}(z)^{e_{ij}} \right| \right\}^{-1/(2a_i + \sum_j e_{ij} \deg P_{ij})} \tag{1}$$

where the a_i, and P_{ij}, and the e_{ij} are given by Table 2, using the polynomials of Table 3.

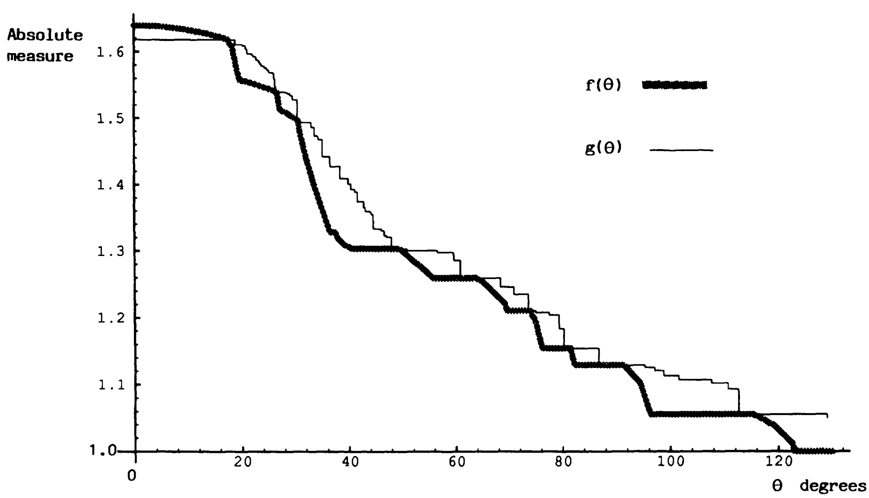

FIGURE 1. The functions $f(\theta)$ and $g(\theta)$. The nine intervals where $f(\theta) > g(\theta)$, and so where $c(\theta)$ is known exactly, are given in Table 1

TABLE 2. The auxiliary function $A_i(z) = z^{a_i} \prod_j P_{ij}(z)^{e_{ij}}$ used to compute $f_i(\theta)$ $(i = 1, \ldots, 9)$. (See equation (1).)

i	θ'ᵢ	Polynomials P_{ij}	Exponents e_{ij}	a_i
1	17.39	P1 P2 P3 P4 P5	21021 05609 00135 00264 00053	20831
2	26.65	P1 P6 P7	26358 00726 00255	19499
3	30.59	P1 P8	30077 00387	18762
4	49.46	P1 P9 P10 P11	19000 00964 00642 13732	11807
5	63.87	P1 P11 P12 P13 P14	10218 18924 00572 00369 00989	13958
6	73.99	P1 P11 P14 P15 P16	07363 26215 00525 00436 00033	12974
7	81.40	P1 P11 P17 P18 P19	04785 23747 02185 00299 02617	09215
8	91.40	P1 P11 P18 P19 P20	06679 13137 02400 09200 00808	12168
9	115.32	P1 P11 P19 P20 P21	03973 05717 05892 06225 01039	11251
		P22 P23	04497 00688	

TABLE 3. Reciprocal polynomials used in Tables 1 and 2. Here, $d = \deg P$, and $\varphi(P) = \max\{|\arg z| : P(z) = 0\}$

P	Ω(P)	φ(P)	d	Highest half of coefficients of P
P1	1.000000	0.000000	2	1 -2
P2	1.618034	0.000000	2	1 -3
P3	1.610559	18.863480	8	1 -12 58 -143 193
P4	1.611995	20.717188	12	1 -18 141 -628 1756 -3219 3935
P5	1.634404	17.665834	16	1 -25 281 -1873 8238 -25211 55246 -88031 102749
P6	1.547928	26.301669	10	1 -14 85 -287 585 -739
P7	1.539222	26.408740	4	1 -5 9
P8	1.493633	30.440145	6	1 -8 26 -37
P9	1.303055	47.941432	6	1 -5 13 -17
P10	1.300734	50.830684	8	1 -7 26 -53 67
P11	1.000000	60.000000	2	1 -1
P12	1.259269	60.890196	6	1 -4 10 -13
P13	1.245865	68.365783	12	1 -7 30 -85 175 268 309
P14	1.210608	73.631615	6	1 -3 7 -9
P15	1.208398	74.983796	8	1 -4 12 -21 25
P16	1.238359	73.295530	8	1 -4 13 -23 28
P17	1.154618	80.241034	8	1 -3 8 -13 15
P18	1.129338	86.708519	6	1 -2 4 -5
P19	1.000000	90.000000	2	1 0
P20	1.000000	108.000000	4	1 -1 1
P21	1.055423	112.647119	8	1 -1 2 -3 3
P22	1.000000	120.000000	2	1 1
P23	1.000000	128.571429	6	1 -1 1 -1

No attempt has been made to get good lower bounds b_i in Table 1—their significance lies in the *existence* of the discontinuity.

The function $g(\theta)$ is the decreasing staircase having left discontinuities at the angles given (in degrees) in Table 4 (next page). The corresponding absolute measure is the new smaller value of $g(\theta)$. There is no mystery about the function $g(\theta)$: it is simply the smallest value of $\Omega(P)$ that we could find, for P having all its zeros in $|\arg z| \leq \theta$.

Alternative representations of the polynomials of Table 4, in terms of polynomials with small coefficients, are given in Table 5 (see p. 299).

As an immediate consequence of the Theorem we have

Corollary. *Suppose that* P *is a monic irreducible polynomial with integer coefficients such that all its zeros have positive real part. Then either* $P(z) = z - 1$ *or* $z^2 - z + 1$ *or* $\Omega(P) \geq 1.12933793$. *This constant is best possible, as it is* $\Omega(z^6 - 2z^5 + 4z^4 - 5z^3 + 4z^2 - 2z + 1)$ (*see polynomial* 80 *of Table* 4). [*Note that a zero* α *of polynomial* 80 *satisfies* $\alpha + \alpha^{-1} = \beta_0^2$, *where* $\beta_0 = 1.3247\ldots$ *is the smallest Pisot number* (*satisfying* $\beta_0^3 - \beta_0 - 1 = 0$).]

TABLE 4. The polynomials having the smallest known absolute measure $\Omega(P)$ among P having all zeros in $|\arg z| \leq \varphi(P)$. (All P shown are reciprocal and $d = \deg P$.) Only those marked with an asterisk $(*)$ have been *proved* to have the minimum measure for that angle (see Table 1, and the Theorem)

		Ω(P)	φ(P)	d	Highest half of coefficients of P
*	1	1.61803399	0.00000000	2	1 -3
	2	1.61055890	18.86348024	8	1 -12 58 -143 193
	3	1.61042390	20.11828489	22	1 -34 535 -5172 34392 -166922 612521 -1737494 3864978 -6804357 9534443 -10665781
	4	1.60951906	20.18105687	14	1 -21 196 -1070 3789 -9141 15394 -18297
	5	1.60875146	20.21826375	12	1 -18 141 -628 1755 -3214 3927
	6	1.60602484	20.72545242	18	1 -27 332 -2464 12330 -44025 115826 -228798 342996 -392343
	7	1.60343922	20.78090097	12	1 -18 140 -618 1714 -3124 3811
	8	1.60252700	21.02657367	18	1 -27 334 -2503 12671 -45784 121768 -242624 365746 -419163
	9	1.59675950	21.09398100	14	1 -21 197 -1084 3873 -9419 15949 -18991
	10	1.59641464	21.33780558	20	1 -30 415 -3509 20274 -84800 265539 -635379 1176446 -1698152 1918389
	11	1.59490694	22.04919262	18	1 -27 333 -2483 12493 -44854 118597 -235195 353466 -404661
	12	1.59371757	22.15703712	18	1 -27 333 -2483 12492 -44841 118524 -234965 353020 -404107
	13	1.59248481	22.23364019	16	1 -24 260 -1679 7196 -21588 46646 -73704 85783
	14	1.59238987	22.60546073	20	1 -30 414 -3487 20055 -83494 260329 -620669 1146102 -1651585 1864727
	15	1.59195280	22.66102759	18	1 -27 333 -2483 12491 -44827 118441 -234694 352484 -403437
	16	1.58773857	22.77269643	14	1 -21 196 -1070 3790 -9147 15408 -18313
	17	1.58664156	23.04197694	14	1 -21 196 -1070 3789 -9140 15389 -18287
	18	1.58399074	23.32992728	10	1 -15 96 -337 703 -895
	19	1.58126124	23.56198519	10	1 -15 96 -336 699 -889
	20	1.57845870	23.80772861	10	1 -15 96 -335 695 -883
	21	1.57605659	24.36229590	12	1 -18 140 -620 1726 -3153 3847
	22	1.57502475	24.70049547	20	1 -29 389 -3199 18031 -73808 226950 -535188 980420 -1406001 1584867
	23	1.57337576	24.77142387	18	1 -26 310 -2244 11008 -38702 100653 -197235 294273 -336075
	24	1.57202659	24.93081408	12	1 -18 140 -619 1718 -3130 3815
	25	1.56822726	25.19319309	14	1 -20 180 -957 3327 -7930 13262 -15725
	26	1.56745492	25.99466170	16	1 -23 240 -1501 6265 -18406 39169 -61315 71139
	27	1.56085652	26.04815162	14	1 -20 179 -944 3254 -7700 12816 -15171
	28	1.55866420	26.14910977	14	1 -20 179 -943 3245 -7665 12741 -15075
	29	1.54792824	26.30166936	10	1 -14 85 -287 585 -739
*	30	1.53922234	26.40874008	4	1 -5 9
	31	1.53856371	28.26134817	8	1 -12 56 -135 179
	32	1.53690935	28.61161388	10	1 -14 85 -284 573 -721
	33	1.53634186	28.94624095	10	1 -14 84 -280 565 -711
	34	1.53417105	29.05559709	10	1 -16 104 -363 740 -933
	35	1.53294906	29.39289999	10	1 -14 85 -283 569 -715
	36	1.52803640	29.53677292	6	1 -9 30 -43
	37	1.52783633	29.97043209	8	1 -13 69 -168 223
	38	1.52317075	30.38499324	10	1 -14 83 -272 542 -679
*	39	1.49363278	30.44014506	6	1 -8 26 -37
	40	1.48712931	33.01882862	10	1 -14 85 -283 561 -701
	41	1.48613506	33.15594485	10	1 -14 83 -272 535 -667
	42	1.47303689	33.66516788	10	1 -14 82 -266 520 -647
	43	1.46831860	34.27948888	10	1 -12 65 -201 388 -481
	44	1.44220546	35.06754357	8	1 -11 51 -116 151
	45	1.42763753	36.51735901	6	1 -7 21 -29
	46	1.40892986	38.40495951	8	1 -10 43 -94 121
	47	1.40098750	39.89520338	12	1 -16 114 -441 1080 -1804 2133
	48	1.39293273	40.41069275	10	1 -12 68 -206 385 -471
	49	1.39197902	41.05028480	14	1 -20 156 -704 2094 -4399 6778 -7813
	50	1.38886516	41.11332694	12	1 -14 93 -350 849 -1414 1671
	51	1.38672830	41.61657428	14	1 -18 151 -709 2153 -4568 7071 -8161
	52	1.37405008	41.64379330	6	1 -6 17 -23
	53	1.36544255	42.81604800	14	1 -15 110 -482 1413 -2944 4518 -5201
	54	1.35958794	43.08523420	12	1 -12 70 -246 577 -946 1113
	55	1.35868017	43.93101605	14	1 -15 108 -467 1359 -2821 4322 -4973

TABLE 4 (*continued*)

	Ω(P)	φ(P)	d	Highest half of coefficients of P
56	1.35500148	44.18553691	10	1 -14 67 -182 319 -383
57	1.35359802	44.49883563	14	1 -14 97 -411 1184 -2446 3740 -4301
58	1.34524128	44.52804944	16	1 -17 138 -679 2283 -5594 10381 -14923 16821
59	1.33550191	44.55539531	8	1 -8 30 -61 77
60	1.33312797	44.62336482	12	1 -12 70 -242 557 -901 1055
61	1.33053856	46.06100005	14	1 -14 95 -392 1102 -2234 3376 -3867
62	1.33025466	46.31840149	10	1 -10 49 -136 243 -293
63	1.33007903	46.38397660	16	1 -16 126 -611 2041 -4988 9249 -13294 14985
64	1.32321829	46.60342750	10	1 -9 41 -110 194 -233
65	1.32093425	47.05677939	14	1 -13 85 -345 964 -1951 2948 -3377
* 66	1.30305506	47.94143202	6	1 -5 13 -17
67	1.30073415	50.83068415	8	1 -7 26 -53 67
68	1.29784756	56.57775241	12	1 -10 52 -167 369 -585 681
69	1.29767066	58.90632647	14	1 -11 66 -255 694 -1386 2082 -2381
70	1.29685473	59.42749399	8	1 -6 21 -42 53
71	1.28550928	59.64526005	12	1 -9 46 -147 324 -513 597
* 72	1.25926867	60.89019592	6	1 -4 10 -13
73	1.24586457	68.36578307	12	1 -7 30 -85 175 -268 309
74	1.23544344	70.85860808	10	1 -5 17 -37 59 -69
* 75	1.21060779	73.63161482	6	1 -3 7 -9
76	1.20839808	74.98379635	8	1 -4 12 -21 25
77	1.20421642	77.48168986	10	1 -5 18 -42 68 -79
78	1.18341933	79.38262510	10	1 -4 13 -26 39 -45
* 79	1.15461811	80.24103363	8	1 -3 8 -13 15
* 80	1.12933793	86.70851871	6	1 -2 4 -5
81	1.12563914	95.33291972	8	1 -2 5 -8 9
82	1.12081684	97.13490108	10	1 -3 8 -16 24 -27
83	1.11297184	98.75579012	10	1 -3 8 -15 21 -23
84	1.10689963	101.56299913	6	1 -1 2 -3
85	1.10191797	107.61406915	8	1 -1 3 -4 5
86	1.09296553	110.68926206	12	1 -1 4 -6 8 -11 11
* 87	1.05542318	112.64711862	8	1 -1 2 -3 3
88	1.05483984	128.99706301	10	1 -1 1 -2 2 -3
89	1.04927511	129.11512456	10	1 -1 1 -2 3 -3
90	1.04001156	131.10299832	10	1 -1 2 -3 3 -3
91	1.03901563	131.32718720	14	1 0 1 -2 2 -3 3 -3
92	1.03774032	136.29274959	14	1 0 0 -2 2 -1 2 -3
93	1.03410559	136.74259166	10	1 -1 2 -2 2 -3
94	1.02030622	137.10280567	12	1 0 1 -1 1 -2 1
95	1.01693963	143.15802667	14	1 0 0 -1 1 -1 1 -1
96	1.01659238	151.30902379	16	1 0 1 -1 1 -1 1 -2 1
97	1.01024571	152.17719835	18	1 1 1 0 0 -1 0 -1 0 -1
98	1.00992571	155.35339189	20	1 1 0 -1 0 0 0 -1 0 0 1
99	1.00897002	164.40870993	26	1 1 0 0 0 -1 0 0 0 0 0 0 0 -1
100	1.00854510	167.55942060	26	1 0 1 0 1 0 0 0 0 -1 0 -1 0 -1
101	1.00851313	168.05867296	22	1 0 1 0 0 -1 -1 -1 0 0 1 1
102	1.00787053	168.36676004	26	1 0 0 -1 0 0 0 0 0 0 1 0 0 -1
103	1.00571887	175.21007112	44	1 -1 0 0 0 0 0 -1 0 0 0 -1 0 0 0 0 0 0 0 1 0 0 1

TABLE 5. The small-coefficient polynomials Q corresponding to polynomials 1 to 87 of Table 4 (see equation (2)). Polynomials 88 onwards already have small coefficients (see §3)

	Ω(P)	φ	d	k	Coefficients of Q
1	1.618034	0.000000	2	2	1 -1
2	1.610559	18.863480	8	3	1 0 0 1 1
3	1.610424	20.118285	22	3	1 -1 -1 1 -1 -1 0 0 0 1 1 1
4	1.609519	20.181057	14	3	1 0 0 1 0 -1 -1 -1
5	1.608751	20.218264	12	3	1 0 0 2 3 2 1
6	1.606025	20.725452	18	3	1 0 -1 -1 -2 -3 -3 -3 -2 -1
7	1.603439	20.780901	12	3	1 0 -1 0 2 2 1
8	1.602527	21.026574	18	3	1 0 1 2 1 2 2 2 1 1
9	1.596760	21.093981	14	3	1 0 1 2 1 2 1 1
10	1.596415	21.337806	20	3	1 0 0 1 0 0 0 0 -1 -1 -1
11	1.594907	22.049193	18	3	1 0 0 1 1 2 2 3 2 1
12	1.593718	22.157037	18	3	1 0 0 1 0 0 0 1 1 1
13	1.592485	22.233640	16	3	1 0 0 1 1 1 -1 -1 -1
14	1.592390	22.605461	20	3	1 0 -1 -1 -1 0 -1 -2 -3 -2 -1
15	1.591953	22.661028	18	3	1 0 0 1 -1 -1 0 0 0 1
16	1.587739	22.772696	14	3	1 0 0 1 1 2 1 1

TABLE 5 (*continued*)

	Ω(P)	φ	d	k	Coefficients of Q										
17	1.586642	23.041977	14	3	1	0	0	1	0	0	0	1			
18	1.583991	23.329927	10	3	1	0	1	2	0	1					
19	1.581261	23.561985	10	3	1	0	1	3	2	2					
20	1.578459	23.807729	10	3	1	0	1	4	4	3					
21	1.576057	24.362296	12	3	1	0	-1	-2	-4	-3	-2				
22	1.575025	24.700495	20	3	1	1	1	2	2	2	2	2	1	0	-1
23	1.573376	24.771424	18	3	1	1	1	1	0	0	0	1	1	1	
24	1.572027	24.930814	12	3	1	0	-1	-1	-3	-4	-3				
25	1.568227	25.193193	14	3	1	1	2	3	2	2	1	1			
26	1.567455	25.994662	16	3	1	1	1	1	0	0	-1	-1	-1		
27	1.560857	26.048152	14	3	1	1	1	1	0	0	0	1			
28	1.558664	26.149110	14	3	1	1	1	2	3	4	3	2			
29	1.547928	26.301669	10	3	1	1	2	3	2	2					
30	1.539222	26.408740	4	2	1	-1	1								
31	1.538564	28.261348	8	3	1	0	-2	-3	-3						
32	1.536909	28.611614	10	2	1	-4	8	-4	3	1					
33	1.536342	28.946241	10	3	1	1	1	1	0	1					
34	1.534171	29.055597	10	2	1	-6	11	-9	-7	-1					
35	1.532949	29.392900	10	2	1	-4	8	-3	3	1					
36	1.528036	29.536773	6	3	1	0	0	2							
37	1.527836	29.970432	8	2	1	-5	11	7	1						
38	1.523171	30.384993	10	3	1	1	0	0	1	2					
39	1.493633	30.440145	6	2	1	-2	3	1							
40	1.487129	33.018829	10	3	1	1	2	7	2	-4					
41	1.486135	33.155945	10	2	1	-4	6	-4	-5	-1					
42	1.473037	33.665168	10	2	1	-4	5	-4	-5	-1					
43	1.468319	34.279489	10	2	1	-2	4	-1	2	1					
44	1.442205	35.067544	8	2	1	-3	5	5	1						
45	1.427638	36.517359	6	2	1	-1	2	1							
46	1.408930	38.404960	8	(5)	1	0	-3	0	3						
47	1.400987	39.895203	12	(5)	1	1	-1	-3	-2	2	3				
48	1.392933	40.410693	10	(5)	1	1	-2	-3	1	3					
49	1.391979	41.050285	14	(5)	1	0	-2	-1	3	3	-2	-3			
50	1.388865	41.113327	12	(5)	1	0	-1	0	-2	0	3				
51	1.386728	41.616574	14	(5)	1	1	-1	-1	-2	-2	2	3			
52	1.374050	41.643793	6	2	1	0	2	1							
53	1.365443	42.816048	14	(5)	1	-1	0	1	-1	-2	1	2			
54	1.359588	43.085234	12	(5)	1	-2	1	2	-3	0	2				
55	1.358680	43.931016	14	(5)	1	-1	-1	2	0	-3	1	2			
56	1.355001	44.185537	10	(5)	1	0	0	0	-1	-1					
57	1.353598	44.498836	14	(5)	1	-1	-1	2	0	-2	0	2			
58	1.345241	44.528049	16	(5)	1	-1	0	0	0	-1	0	1	1		
59	1.335502	44.555395	8	(5)	1	-1	-1	1	1						
60	1.333128	44.623365	12	(5)	1	-1	0	0	-1	1	1				
61	1.330539	46.061000	14	(5)	1	-1	-1	2	-1	-1	1	1			
62	1.330255	46.318401	10	(5)	1	0	-2	0	1	1					
63	1.330079	46.383977	16	(5)	1	-1	0	0	0	0	-1	1	1		
64	1.323218	46.603428	10	(5)	1	-1	0	0	0	1					
65	1.320934	47.056779	14	(5)	1	-1	-1	2	0	-2	1	1			
66	1.303055	47.941432	6	(5)	1	-1	0	1							
67	1.300734	50.830684	8	(5)	1	0	-2	1	1						
68	1.297848	56.577752	12	(5)	1	0	-3	3	0	-1	1				
69	1.297671	58.906326	14	(5)	1	0	-2	1	2	-2	0	1			
70	1.296855	59.427494	8	(5)	1	0	-1	0	1						
71	1.285509	59.645260	12	(5)	1	1	-3	0	3	-2	1				
72	1.259269	60.890196	6	(5)	1	0	-1	1							
73	1.245865	68.365783	12	1	1	-1	4	-4	3	-3	1				
74	1.235443	70.858608	10	1	1	0	2	-1	0	-1					
75	1.210608	73.631615	6	1	1	0	1	-1							
76	1.208398	74.983796	8	1	1	0	2	-1	-1						
77	1.204216	77.481690	10	1	1	0	3	-3	-1	1					
78	1.183419	79.382625	10	1	1	1	2	0	-2	-1					
79	1.154618	80.241034	8	1	1	1	1	-1	-1						
80	1.129338	86.708519	6	1	1	1	0	-1							
81	1.125639	95.332920	8	0	1	-2	1	-2	1						
82	1.120817	97.134901	10	1	1	2	1	-3	-1	1					
83	1.112972	98.755790	10	1	1	2	1	-2	-2	1					
84	1.106900	101.562999	6	0	1	-1	-1	-1							
85	1.101918	107.614069	8	0	1	-1	-1	-1	1						
86	1.092966	110.689262	12	0	1	-1	-2	-1	1	2	1				
87	1.055423	112.647119	8	0	1	-1	-2	0	1						

2. Earlier work

The paper of Langevin [4] forms the basis of this investigation. He also showed that $c(\pi/2) > 1.08$ in [5, p. 63]. Earlier, Schinzel [10] had obtained $c(0) = \frac{1}{2}(1+\sqrt{5}) = 1.61803399$.

The spectrum $\mathrm{Spec}(\theta) = \{\Omega(P) : P$ has all zeros in $|\arg z| \leq \theta\}$ is also of interest. In [11] the second author studied $\mathrm{Spec}(0)$. Mignotte [7], in an interesting application of a well-known result of Erdős and Turán on the uniformity of distribution of the arguments of zeros of certain sets of polynomials, showed that, for $\delta > 0$ the smallest limit point of $\mathrm{Spec}(\pi - \delta)$ is at least $1 + c\delta^3$, for an effective positive constant c.

3. Proof of the Theorem: Outline and search

The proof of the Theorem can be regarded simply as finding the functions f and g and proving that they have the values and properties claimed for them in the Theorem. The function g is found by a search, which we will outline shortly. The function f is obtained by semi-infinite linear programming, using the polynomials found in the search. This is described in §4.

A necessary condition for the exact evaluation of $c(\theta)$ by our method is to actually find the polynomial P with all zeros in $|\arg z| \leq \theta$ for which $\Omega(P)$ is in fact minimal for that sector. In any event, even if the smallest $\Omega(P)$ we find is *not* minimal, it clearly gives an upper bound for $c(\theta)$. The list of Table 4 and the corresponding staircase function $g(\theta)$ are the result of our search for such smallest $\Omega(P)$, for varying θ.

Our search for polynomials P, with small $\Omega(P)$ and zeros in a given sector, started with exhaustive searches for polynomials of degrees 3 and 4. For degrees 5 and 6, ad hoc searches were made, from which it became clear that the best polynomials were usually reciprocal. Further nonexhaustive searches were then made for good reciprocal polynomials of degrees 8 and 10. As a result of this extensive preliminary work, it became clear that the good polynomials were not only reciprocal, but also of one of six special types:

$$
(2)\quad
\begin{array}{lll}
z^n Q(z + z^{-1} - k) & (k = 3, 2, 1, 0) & \text{(Types } 1, 2, 3, 4) \\
z^n S(z + z^{-1} - 2), & & \text{(Type 5)} \\
 & \text{where } S(x) = Q(1)x^n Q(1 + 1/x), \text{ and} & \\
z^n (Q(z) + Q(1/z)) & & \text{(Type 6)}.
\end{array}
$$

Here, Q is a degree-n monic polynomial with small coefficients, with also $Q(1) = \pm 1$ for the fifth type. The reason for polynomials of these types giving good polynomials appears mysterious, however.

A systematic search was therefore conducted, using small-coefficient Q of degree up to 11, for polynomials of the six types. The range of coefficients of Q searched over varied with degree and polynomial type. This is how most of the polynomials $P(z)$ in Table 4 were obtained. The corresponding small-coefficient polynomials Q are shown in Table 5. This table excludes polynomials of the sixth type, since, for these polynomials, the coefficients of Q are the same as the highest half coefficients of P, so that P itself has small coefficients.

The polynomials P of the sixth type with large angle $\varphi(P) = \{\max|\arg z| : P(z) = 0\}$ were taken from Boyd's tables [3]. It should be recalled, however, that his tables are the result of a search for polynomials of small *relative* measure, and so are unlikely to be the polynomials P of smallest *absolute* Mahler measure for the corresponding $|\arg z| \le \varphi(P)$. Indeed, we do not expect that all of the unstarred polynomials P in Table 4 have minimal $\Omega(P)$ for their corresponding angle $\varphi(P)$. Rather, we publish the table in order to provide a target for any other enthusiasts to aim at!

We note in passing that Lehmer's polynomial $L(z) = z^{10}+z^9-z^7-z^6-z^5-z^4-z^3+z+1$ [6, 1, 2], although having the smallest known relative measure > 1, does not have the smallest absolute measure for its sector ($\Omega(L) = 1.016368$, zeros in $|\arg z| \le 160.61°$), being beaten by polynomial 97 of Table 4.

4. Proof of Theorem: Computation of the function $f(\theta)$

Langevin's proof [4] of his $\Omega(P) \ge c_V$ result, mentioned in §1, has three basic ingredients:

(i) The observation that the set $V_1 = V \cap \{z \in \mathbb{C} : |z| \le 1\}$ has transfinite diameter less than 1.

(ii) A result of Kakeya to the effect that for any set W of transfinite diameter less than 1 and symmetric about the real axis there is a nonzero polynomial A with integer coefficients such that $\mathrm{Sup}_{z\in W} |A(z)| < 1$.

(iii) Deduction of $\Omega(P) \ge c_V$ from (i) and (ii) using $W := \{z : z \in V \text{ and } \overline{z} \in V\}$.

For the computation of $f(\theta) = \max_{i=1}^{9} f_i(\theta)$, we use, for each f_i, an auxiliary polynomial A as in (ii). We choose such A of the form $z^a R(z)$, where a is a positive integer and R is a reciprocal polynomial of degree r with integer coefficients. To A is then associated the function

$$m(\theta) = \sup_{z\in W_\theta} |A(z)|^{1/(2a+r)}.$$

Then Langevin's argument of (iii) above, which we now reproduce, shows that

$$(3) \qquad \Omega(P) \ge 1/m(\theta) \quad \text{if } \gcd(P, A) = 1$$

for P irreducible, of degree d, with integer coefficients. For, if $\alpha_1, \dots, \alpha_d$ are the zeros of P, then, since $R(z) = z^r R(z^{-1})$, we have

$$\begin{aligned} 1 &\le \left|\prod_{i=1}^{d} \alpha_i^a R(\alpha_i)\right| = \prod_{|\alpha_i|\le 1} |\alpha_i^a R(\alpha_i)| \cdot \prod_{|\alpha_i|>1} |\alpha_i^{a+r} R(\alpha_i^{-1})| \\ &= \prod_{|\alpha_i|\le 1} |\alpha_i^a R(\alpha_i)| \cdot \prod_{|\alpha_i|>1} |(\alpha_i^{-1})^a R(\alpha_i^{-1})| \cdot \prod_{|\alpha_i|>1} \alpha_i^{2a+r} \\ &\le m(\theta)^{(2a+r)d} M(P)^{2a+r} \end{aligned}$$

whence $\Omega(P) \ge 1/m(\theta)$.

Each $f_i(\theta)$ was then defined, as in equation (1), to be the function $1/m(\theta)$ corresponding to an auxiliary function A chosen so that $f(\theta_i) > g(\theta_i)$, and so that the length of the interval $[\theta_i, \theta_i']$ over which $f(\theta) > g(\theta)$ was as long as we could find. Thus, if $g(\theta_i) = \Omega(P_*)$ (Table 4), then $\Omega(P_*) < f_i(\theta_i)$. From equation (3) it follows that P_* is a factor of A and that, among polynomials with all conjugates in $|\arg z| \le \theta_i$, only factors of A can have absolute measure less than $f_i(\theta_i)$. Now P_* does indeed divide A, and in fact has the smallest absolute measure among factors of A of measure > 1. It follows that $\Omega(P_*)$ is the smallest value of the absolute measure for polynomials having all zeros in

$|\arg z| \leq \theta$ for $\theta \in [\theta_i, \theta_i']$. Hence, $c(\theta) = \Omega(P_*)$ for these θ. Furthermore, $\Omega(P) \geq f(\theta_i) = g(\theta_i) + b_i = c(\theta_i) + b_i$ for any P having all its roots in the sector $|\arg z| < \theta_i$, i.e.,

$$c(\theta_i -) - c(\theta_i) \geq b_i.$$

The polynomial A is taken to be of the form

$$A(z) = z^a R(z) = z^a P_1(z)^{e_1} \cdots P_k(z)^{e_k},$$

where the polynomials P_j are chosen either to be cyclotomic or to have both absolute measure close to $g(\theta_i)$ and all zeros in $|\arg z| \leq \theta_i + \varepsilon$, where ε is small (not more than a few degrees). Table 3 shows the actual polynomials chosen.

It was for finding the best choice of exponents a, $e_1, \ldots, e_k$ that semi-infinite linear programming was needed. This was used in a similar way to our earlier papers ([8; 9; 11, II; 12]; see [11, II] for a brief outline of the method). However, in this case the computation was more complicated, since the region over which optimization was taking place was (the boundary of) a sector instead of a real interval, as previously. Table 2 gives the final exponents obtained.

5. Improving the function f

For simplicity of presentation (and so, at least in principle, checking by the reader!) of our results, we have given f as the maximum of only nine functions f_i, each chosen, as described above, to be large around the corresponding critical angle θ_i. In fact, we tried many other auxiliary functions A which we chose so that the corresponding function would be large at other angles θ. In no case, however, was $f(\theta) > g(\theta)$, so that $c(\theta)$ could not be evaluated exactly over any more intervals. We would, however, obtain a better lower bound $f^+(\theta)$ for $c(\theta)$ than that given by $f(\theta) = \max_{i=1}^9 f_i(\theta)$. For example, Table 6 shows two values of θ where $c(\theta)$ was "nearly" evaluated exactly. Further computation is needed to determine whether the failure of the method for these θ was due to a suboptional choice of A, or to the fact that the polynomial P with truly smallest $\Omega(P)$ for that θ had not been found.

TABLE 6. Two examples where an improved auxiliary function $A(z) = z^a \prod_j P_j(z)^{e_j}$ is used to compute $f^+(\theta)$ and hence obtain narrow bounds $f^+(\theta) \leq c(\theta) \leq g(\theta)$ for $c(\theta)$

θ	$f^+(\theta)$	$g(\theta)$	Polynomials P_j	Exponents e_j	a
18.8635	1.606109	1.61055890	P1 P2 P3 P4 P5	21588 05779 00188 00188 00030	20513
50.9000	1.299013	1.30074315	P1 P9 P10 P11	20083 00752 00228 14495	12253

Acknowledgment

Much of this work was done while the second author was visiting the Université de Metz. He wishes to thank the University in general, and the first author in particular, for their hospitality during his visits.

Bibliography

1. D. W. Boyd, *Variations on a theme of Kronecker*, Canad. Math. Bull. **21** (1978), 129–133.
2. ——, *Speculations concerning the range of Mahler's measure*, Canad. Math. Bull. **24** (4) (1981), 453–469.
3. ——, *Reciprocal polynomials having small measure.* I, Math. Comp. **35** (1980), 1361–1377; II, Math. Comp. **53** (1989), 353–357, S1–S5.
4. M. Langevin, *Minorations de la maison et de la mesure de Mahler de certains entiers algébriques*, C. R. Acad. Sci. Paris **303** (1986), 523–526.
5. ——, *Calculs explicites de constantes de Lehmer*, Univ. de Paris-Sud groupe de travail en théorie analytique et élémentaire des nombres 1986–87, vol. 88-01, pp. 52–68.

→ D. H. Lehmer, *Factorization of certain cyclotomic functions*, Ann. of Math. (2) **34** (1933), 461–479.

7. M. Mignotte, *Sur un théorème de M. Langevin*, Acta Arith. **54** (1989), 81–86.
8. G. Rhin, *Mesures d'irrationalité de log* 2, Seminaire de Théorie des Nombres de Bordeaux 1984/5, Université de Boreaux I.
9. ——, *Approximants de Padé et mesures effectives d'irrationalité*, Seminaire de Théorie des Nombres (May 1986), Progr. Math., vol. 71, Birkhäuser, Boston, 1987, pp. 155–164.
10. A. Schinzel, *On the product of the conjugates outside the unit circle of an algebraic number*, Acta Arith. **24** (1973), 385–399; Addendum **26** (1975), 329–331.
11. C. J. Smyth, *On the measure of totally real algebraic integers.* I, J. Austral. Math. Soc. Ser. A **30** (1980), 137–149; II, Math. Comp. **37** (1981), 205–208.
12. ——, *The mean values of algebraic integers*, Math. Comp. **42** (1984), 663–681.

URA CNRS NO. 399, DÉPARTEMENT DE MATHÉMATIQUE ET INFORMATIQUE, FACULTÉ DES SCIENCES, UNIVERSITÉ DE METZ, ILE DU SAULCY, F-57045 METZ CEDEX 1, FRANCE

E-mail address: `rhin@poncelet.univ-metz.fr`

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF EDINBURGH, JAMES CLERK MAXWELL BUILDING, KING'S BUILDINGS, MAYFIELD ROAD, EDINBURGH EH9 3JZ, SCOTLAND, UNITED KINGDOM

E-mail address: `chris@maths.ed.ac.uk`