MATHEMATICS OF COMPUTATION
VOLUME 64, NUMBER 209
JANUARY 1995, PAGES 305-321

IMPRIMITIVE NINTH-DEGREE NUMBER FIELDS WITH SMALL DISCRIMINANTS

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ABSTRACT. We present tables of ninth-degree nonprimitive (i.e., containing a cubic subfield) number fields. Each table corresponds to one signature, except for fields with signature (3,3), for which we give two different tables depending on the signature of the cubic subfield. Details related to the computation of the tables are given, as well as information about the CPU time used, the number of polynomials that we deal with, etc. For each field in the tables, we give its discriminant, the discriminant of its cubic subfields, the relative polynomial generating the field over one of its cubic subfields, the corresponding (irreducible) polynomial over $\mathbb{Q}$, and the Galois group of the Galois closure. Fields having interesting properties are studied in more detail, especially those associated with sextic number fields having a class group divisible by 3.

1. INTRODUCTION

The computational time required for the construction of tables of number fields, using the methods known at the present time, grows exponentially with the degree of the fields under consideration.

For this reason, extensive tables of number fields exist only for degrees up to six ([1, 2, 5, 6, 10, 11, 12, 14, 19, 21, 22, 23, 24, 29]).

Martinet's generalization [20] of the Hunter-Pohst theorem [26] has made it possible to study the relative extensions using the same methods as those used in the primitive case. As a result, many extensive tables have been constructed for sextic fields ([21, 22, 23, 24]), as well as the first minima for discriminants of totally imaginary [9] and totally real [27] octic fields.

All relative extensions known up to date are either relative extensions of a quadratic field (the task in this case is easy because the ground field is abelian), or are quadratic extensions of a number field (and in this case, class field theory can be used, avoiding most of the calculations [27]).

What will be studied here is the case of imprimitive ninth-degree number fields, i.e., relative cubic extensions of a cubic field. To do this, tables of cubic fields with a sufficient amount of arithmetic data are needed. The advantage of working with relative extensions is that the computation only involves relative cubic polynomials (i.e., having only 3 coefficients).

Recently, A.M. Odlyzko advised us that H. Fujita [15] discovered the first three imprimitive totally real number fields of degree nine.

Received by the editor March 30, 1993.
1991 *Mathematics Subject Classification*. Primary 11Y40.
Key words and phrases. Cubic fields, relative extensions, discriminant.

TABLE I

signature	subfield	minimal discriminant
$(9, 0)$	real	$1\,62403\,85609 = 49^3 \cdot 1\,38041$
$(7, 1)$	real	$-2668\,161671 = -49^3 \cdot 22679$
$(5, 2)$	real	$4678\,90073 = 49^3 \cdot 41 \cdot 97$
$(3, 3)_{\mathbb{R}}$	real	$-1645\,90951 = -49^3 \cdot 1399$
$(3, 3)_{\mathbb{C}}$	complex	$-1108\,52311 = -31^3 \cdot 61^2$
$(1, 4)$	complex	$322\,06049 = 23^3 \cdot 2647$

In this paper, six tables of imprimitive ninth-degree fields are presented. Each table corresponds to one signature, except for signature (3,3), for which two tables are given, one for each conjugacy class of the Frobenius at infinity: Table $(3, 3)_{\mathbb{R}}$ contains number fields having a real cubic subfield; table $(3, 3)_{\mathbb{C}}$ contains number fields having a complex cubic subfield; naturally, the intersection of these two tables is not empty.

Minimal discriminants of imprimitive ninth-degree fields are given for each signature in Table I.

In §2, Martinet's theorem mentioned previously is stated and some explanation concerning its use in the study of the relative extensions is also given. Special emphasis is placed on the algorithms used to obtain faster computations.

The description of the six tables is given in §3. The CPU time needed to compute the table and the number of polynomials studied are also specified.

Section 4 contains a more detailed analysis of several interesting fields found in the tables. In particular, the existence of some nonisomorphic fields having the same discriminant is proven as well as the relation between these fields and certain sextic fields with a class number divisible by 3 or 9.

Finally, in §5, the Galois group of the Galois closure of each field appearing in the tables is computed.

The complete tables are available on floppy disk (source TEX); contact the authors by e-mail.

2. RELATIVE CUBIC EXTENSIONS OF A CUBIC FIELD

For a number field L, its ring of integers is denoted by $\mathbb{Z}_L$ and its discriminant by d_L. Throughout the paper, K will denote an imprimitive ninth-degree number field. Because K is assumed imprimitive, it contains, at least, one cubic subfield denoted by k. So, K/k is a relative cubic extension, and we denote its relative ideal discriminant by δ.

According to the formula $d_K = (-1)^s |d_k|^3 N_{k/\mathbb{Q}}(\delta)$, where (r, s) is the signature of K and $N_{k/\mathbb{Q}}$ the norm of the extension $k/\mathbb{Q}$, the inequality $|d_k| \le |d_K|^{1/3}$ is valid for all the possible subfields k of K.

There are three $\mathbb{Q}$-embeddings of k into $\mathbb{C}$, denoted by σ_1, σ_2, σ_3; each of them can be extended in three different ways to give a $\mathbb{Q}$-embedding of K into $\mathbb{C}$; we denote by σ_i^j, $j = 1, 2, 3$, the $\mathbb{Q}$-embeddings of K in $\mathbb{C}$ whose restriction to k is σ_i, $i = 1, 2, 3$.

For each $\theta \in K$, the 'relative trace' Tr_i for K/k is defined as the trace of $\sigma_i^j(\theta) \in \sigma_i^j(K)$ relative to $\sigma_i(k)$ for $i = 1, 2, 3$. Thus, we have $Tr_i(\theta) = \sigma_i^1(\theta) + \sigma_i^2(\theta) + \sigma_i^3(\theta)$.

The following theorem is essential:

Theorem 1 (Martinet [20]). *There exists an integer $\theta \in \mathbb{Z}_K$, $\theta \notin k$, such that*

$$(*) \qquad \sum_{i=1}^{3}\sum_{j=1}^{3} |\sigma_i^j(\theta)|^2 \leq \frac{1}{3}\sum_{i=1}^{3} |Tr_i(\theta)|^2 + \left(\frac{64|d_K|}{81|d_k|}\right)^{1/6}.$$

Moreover, if θ satisfies $(*)$, the same is true for all integers having the form $\theta + a$ with $a \in \mathbb{Z}_k$ as well as for $-\theta$.

If the element $\theta \in \mathbb{Z}_K$ whose existence is asserted in Theorem 1 is not a primitive element of the extension $K/\mathbb{Q}$, then K must contain at least two different cubic subfields. Fields having this property can be easily computed directly from tables of cubic fields. Thus, it can be assumed in what follows that $K = \mathbb{Q}(\theta) = k(\theta)$, where θ is a root of a monic irreducible relative cubic polynomial $P(X) = X^3 + aX^2 + bX + c \in \mathbb{Z}_k[X]$.

Let us define

$$\sigma_i(P(X)) = P_i(X) = X^3 + \sigma_i(a)X^2 + \sigma_i(b)X + \sigma_i(c) = X^3 + a_iX^2 + b_iX + c_i$$

for each $i = 1, 2, 3$. Then $f(X) = P_1(X)P_2(X)P_3(X)$ is the minimal polynomial of θ over $\mathbb{Q}$.

For $1 \leq r \leq 9$, r odd, and $s = (9-r)/2$, let $B_r > 0$ be an upper bound depending on the signature (r, s) (the reasons for the choice of the value of these constants will be explained in the next section). Let us suppose that r is fixed, and denote B_r by B. To construct all the fields K of signature (r, s) satisfying $|d_K| \leq B$, it is necessary to consider every cubic field k having a discriminant d_k such that $|d_k| \leq B^{1/3}$. Moreover, the signature (r', s') of k must be compatible with the signature of K, i.e., we must have $s \geq 3s'$. For such a cubic field k, let $\{1, \alpha, \beta\}$ be an integral basis.

To compute a field K belonging to the table is equivalent to computing an irreducible polynomial $P(X) = X^3 + aX^2 + bX + c \in \mathbb{Z}_k[X]$ such that $K = k(\theta)$ and $P(\theta) = 0$. The coefficients a, b, c of this polynomial can be determined in the following manner.

Define $T_2(\theta) = \sum_{i=1}^{3}\sum_{j=1}^{3} |\sigma_i^j(\theta)|^2$. Martinet's theorem shows that every field K with $|d_K| \leq B$ contains an integer θ such that

$$T_2(\theta) \leq \frac{1}{3}\sum_{i=1}^{3} |a_i|^2 + C' \quad \text{with } C' = \left(\frac{64B}{81|d_k|}\right)^{1/6}.$$

The second part of the same theorem asserts that the first coefficient a of $P(X)$ can be chosen as $a = x_1 + x_2\alpha + x_3\beta$ with $x_1, x_2, x_3 \in \{0, 1, -1\}$ and satisfying $100x_1 + 10x_2 + x_3 \geq 0$; there are only 14 possible choices for this coefficient.

If the value of a has been fixed, the value of $T_2(\theta)$ is bounded by a real constant $C = C(a)$ depending only on the given value of a. The following lemma is very useful from a practical point of view.

Lemma 1. *For every field K containing a cubic subfield k there exists $a \in \mathbb{Z}_k$ such that*

(i) $P(\theta) = 0$ *and* $K = k(\theta)$;

(ii) $\frac{1}{3}\sum_{i=1}^{3}|a_i|^2 + C' = C$ *is minimum.*

Proof. From Theorem 1, the value of a can be chosen among 14 values. If θ is a root of $P(X)$ generating the extension K/k, then the integer $\theta + y$ with $y \in \mathbb{Z}_k$ is also a primitive element of K/k. Define $-a' = Tr_{K/k}(\theta + y) = -a + 3y$; because $\sum_{i=1}^{3}|\sigma_i(a')|^2$ is a positive definite quadratic form, there exists y minimizing it (the value of y can be obtained by a back-tracking method) and the value of a has to be replaced by this minimum and the value of C is $C = \frac{1}{3}\sum_{i=1}^{3}|a_i|^2 + C'$. □

Once a suitable value of a in $P(X)$ has been determined, there exists in K an integer for which $T_2(\theta) \le C$. Let us define

$$s_2(\theta) = -2b + a^2 = \sum_{j=1}^{3}\sigma^j(\theta)^2 = y_1 + y_2\alpha + y_3\beta$$

(where σ^j for $j = 1, 2, 3$ are the k-isomorphisms from K into $\mathbb{C}$), and

$$S_2(\theta) = \sum_{i=1}^{3}\sigma_i(s_2(\theta)) = \sum_{i=1}^{3}\sum_{j=1}^{3}\sigma_i^j(\theta)^2.$$

From the inequalities $|S_2(\theta)| \le \sum_{i=1}^{3}|\sigma_i(s_2(\theta))| \le T_2(\theta) \le C$ and

$$\sum_{i=1}^{3}|\sigma_i(s_2(\theta))|^2 \le \left(\sum_{i=1}^{3}|\sigma_i(s_2(\theta))|\right)^2 \le C^2\ ,$$

an upper bound is obtained for the values of a positive definite quadratic form in the variables y_1, y_2, and y_3 and among all the integral solutions in this bounded convex body, only those for which the variables y_i have the same parity as the x_i''s, where $a^2 = x_1' + x_2'\alpha + x_3'\beta$, have to be considered. Finally, one can easily deduce all the possible values for the coefficient b in $P(X)$.

Let us now assume that a and b have been fixed, and let $c = z_1 + z_2\alpha + z_3\beta$ be the expression of c in $\mathbb{Z}_k$. From the formula $|c| = \prod_{j=1}^{3}|\sigma^j(\theta)|$ and the inequality between arithmetic and geometric means we obtain

$$|c|^2 \le \frac{1}{27}\left(\sum_{j=1}^{3}|\sigma^j(\theta)|^2\right)^3,$$

and we can deduce the inequalities

$$\sum_{i=1}^{3}|\sigma_i(c)|^2 \le \frac{1}{27}\sum_{i=1}^{3}\left(\sum_{j=1}^{3}|\sigma_i^j(\theta)|^2\right)^3 \le \frac{1}{27}\left(\sum_{i=1}^{3}\sum_{j=1}^{3}|\sigma_i^j(\theta)|^2\right)^3 \le \frac{1}{27}C^3.$$

The expression is still a bounded positive definite quadratic form in the variables z_1, z_2, z_3, and it is possible to compute all the integral points inside the ellipsoid [28].

Finally, a polynomial $P(X) \in k[X]$ is obtained for which it is necessary to test if $P(X)$ defines a number field K having a suitable signature. If the cubic ground field is a complex one, it is enough to compute the sign of the discriminant of the polynomial $\sigma(P(X))$, where σ is the unique real embedding from k into $\mathbb{C}$. If the cubic ground field k is real, the sign of the discriminant has to be determined for each conjugate of $P(X)$.

The computation of the roots of the polynomials $P_1(X)$, $P_2(X)$, $P_3(X)$ was done using Cardano's formulæ, and at this point the inequality (*) of Theorem 1 was checked. When complex places exist, this test eliminates more than 99% of the polynomials having the appropriate signature. However, in the case of totally real fields K, this computation can be avoided because in this case $Tr_{K/\mathbb{Q}}(\theta^2)$ is a rational integer.

The next step consists in testing for the irreducibility of $f(X)$. The polynomial $f(X)$ is eliminated either because $P_1(X)$ belongs to $\mathbb{Z}[X]$ or because one of the roots of $f(X)$ is in $\mathbb{Z}$. The last condition is easy to verify by using the approximate values of the complex roots already computed. Finally, we search for the possible existence of an irreducible cubic factor of $f(X)$ in $\mathbb{Z}[X]$; this is done by factoring $f(X)$ over $\mathbb{Z}$, using the PARI system [4].

For the remaining irreducible polynomials having a suitable signature and satisfying the inequality (*) of Theorem 1, the discriminant of K is computed by using the version written by D. Ford [13] of the ROUND 2 algorithm of Pohst and Zassenhaus [8, p. 305, Algorithm 6.1.8] implemented in the PARI system.

At this stage, the constructed tables of number fields are complete (with the possible exception of the fields obtained as a compositum of two cubic ground fields) but some fields in the tables can be $\mathbb{Q}$-isomorphic. In fact, practice proves that for small discriminants this is often true. The following lemma is useful.

Let $P(X)$ and $Q(X)$ be two relative polynomials generating respectively the fields K and L with the same absolute discriminant. Denote by $\sigma_i(P)$ and $\sigma_i(Q)$ for $i = 1, 2, 3$ the conjugates of $P(X)$ and $Q(X)$, and by $\sigma_i^j(\theta)$ and $\sigma_i^j(\eta)$ for $j = 1, 2, 3$ their respective roots in $\mathbb{C}$.

Lemma 2 [3]. *The fields K and L are $\mathbb{Q}$-isomorphic if and only if there exist four permutations s, s_1, s_2, $s_3 \in S_3$ such that for all $h \in \mathbb{N}$ we have*

$$\sum_{j=1}^{3}(\sigma_i^j(\theta))^h \cdot \sigma_{s(i)}^{s_i(j)}(\eta) \in \sigma_i(\mathbb{Z}_k) \qquad \textit{for } i = 1, 2, 3 .$$

The most tedious case is the totally real one, where this test may require $6^4 = 1296$ trials; the easiest case corresponds to the signature (1,4), where at most 24 trials are necessary. Once a suitable permutation for the roots is obtained, it is easy to compute the equations with rational coefficients (the norm of the index $[\mathbb{Z}_K : \mathbb{Z}[\theta]]$ is the denominator of the coefficients) relating the roots of $Q(X)$ and the roots of $P(X)$.

Whenever such a $\mathbb{Q}$-isomorphism was not found, we searched for rational prime numbers having different decompositions as a product of prime ideals in K and L. The existence of different factorizations for a prime number in K and L is a sufficient (but not necessary) condition for these fields to be nonisomorphic. In all the cases considered, when the algorithm of Lemma 2 failed to prove that the fields were isomorphic, prime numbers having different factorizations in these fields were always found, thus providing proof for the isomorphism test.

What remains to be considered is simply the case where the integer $\theta \in K$, $\theta \notin k$, whose existence is asserted by Theorem 1, is not a primitive element for $K/\mathbb{Q}$. Because the field K can only contain cubic subfields, the field $k_1 = \mathbb{Q}(\theta)$ is a cubic extension of $\mathbb{Q}$ and $k \neq k_1$. If k_1 were a conjugate of k, the compositum kk_1 would be the Galois closure of k in an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$; but this is impossible because K does not contain any sextic subfield. This proves that the relative polynomial $P(X)$ having θ as a root is actually a polynomial with coefficients in $\mathbb{Z}$.

In this case, to find a primitive element of $K/\mathbb{Q}$, the following result due to J. Martinet (private communication) was used.

Theorem 2. *Every ninth-degree extension K/k_0 of a number field can only contain* $0, 1, 2$ *or* 4 *cubic extensions of* k_0.

Proof. It is sufficient to prove that a field K containing more than one cubic extension of k_0 must contain exactly 2 or 4 cubic subfields. Denote by k_1 and k_2 two cubic extensions of k_0 in a given algebraic closure $\overline{k_0}$; then we have $K = k_1k_2$, because the fields k_1 and k_2 cannot be k_0-isomorphic; let d_1 and d_2 be the respective discriminants of k_1/k_0 and k_2/k_0 (in k_0^*/k_0^{*2}); let N be the Galois closure of K/k_0 into $\overline{k_0}$, and G the Galois group of N/k_0. Exchanging, if necessary, k_1 and k_2, we find that the following cases cover all the different possibilities:

(A) d_1 and d_2 are perfect squares.
(B) d_1 is a perfect square and d_2 is not.
(C) d_1 and d_2 are not perfect squares but d_1/d_2 is.
(D) d_1 and d_2 are not perfect squares and d_1/d_2 is not.

The group G can then be identified with a subgroup of the direct product $S_3 \times S_3$. Specifically, in the case (A), G is isomorphic to $C_3 \times C_3$, and the stabilizer of K is the subgroup $\langle 1 \rangle$ of G; so, G has four normal subgroups of order 3, and K contains four cyclic cubic subfields.

In the case (B), G is isomorphic to $C_3 \times S_3$, and this group can be defined as $G = \langle \sigma_1, \sigma_2, \tau \rangle$ with the relations $\sigma_1^3 = \sigma_2^3 = \tau^2 = 1$, $\sigma_1\sigma_2 = \sigma_2\sigma_1$, $\sigma_1\tau = \tau\sigma_1^2$, $\sigma_2\tau = \tau\sigma_2$, where $\langle \tau \rangle$ is the stabilizer of K. Then G has two subgroups of order 6 containing τ, say $\langle \sigma_1, \tau \rangle$ which is normal, and $\langle \sigma_2, \tau \rangle$ which is not normal; therefore, K contains two nonisomorphic cubic subfields, one of them being cyclic, but not the other.

In the case (C), G is isomorphic to $\langle \sigma_1, \sigma_2, \tau \rangle$ of $S_3 \times S_3$ with the relations $\sigma_1^3 = \sigma_2^3 = \tau^2 = 1$, $\sigma_1\sigma_2 = \sigma_2\sigma_1$, $\sigma_1\tau = \tau\sigma_1^2$, $\sigma_2\tau = \tau\sigma_2^2$, where τ is the element in G of order 2 leaving K invariant; therefore, G has four subgroups isomorphic to S_3, containing τ, and they are not normal; so, K has four nonisomorphic cubic subfields which are noncyclic, associated with a unique common quadratic extension of k_0, which is the field fixed by the subgroup $\langle \sigma_1, \sigma_2 \rangle$ of order 9 and index 2.

Finally, in the case (D), the group G can be written in the form $\langle \sigma_1, \tau_1 \rangle \times \langle \sigma_2, \tau_2 \rangle$, where $\langle \sigma_i, \tau_i \rangle$ is isomorphic to S_3 for $i = 1, 2$. We may suppose that the stabilizer of K is the group $\langle \tau_1, \tau_2 \rangle$, isomorphic to V_4 (Klein group), and the only subgroups of G of index 3 containing $\langle \tau_1, \tau_2 \rangle$ are $\langle \sigma_1, \tau_1, \tau_2 \rangle$ and

$\langle\sigma_2, \tau_1, \tau_2\rangle$. Thus, K contains exactly two nonisomorphic cubic extensions of k_0; these fields are not Galois fields, and they are associated with different quadratic extensions of k_0.

Corollary. *Let $k_1 = \mathbb{Q}(\theta_1)$ and $k_2 = \mathbb{Q}(\theta_2)$ be two nonisomorphic cubic subfields of a ninth-degree field K over $\mathbb{Q}$. Then, among the numbers $\theta_1 + \theta_2$, $\theta_1 - \theta_2$, $\theta_1 + 2\theta_2$, one (at least) is a primitive element of $K/\mathbb{Q}$.*

3. Description of the tables and their computing time

First, limits for the different tables were fixed. In the case of the totally real fields, the bound was fixed to obtain a table which would include the compositum of the abelian cubic fields of discriminant 49 and 81. Consequently, the corresponding table contains all the imprimitive fields of ninth degree having a discriminant smaller than 6 30000 00000. There are exactly 27 number fields with this property, and all of them are characterized, up to an isomorphism, by their discriminant.

The size of the tables, for the other signatures, was fixed depending on an estimate of the CPU time required. The table corresponding to signature (7,1) contains the 23 imprimitive fields having a discriminant larger than $-70000\,00000$, and they are also characterized by their discriminant.

The table corresponding to signature (5,2) contains the 154 fields having a discriminant smaller than 50000 00000, and for 5 values of the discriminant there is a pair of nonisomorphic fields.

In the table corresponding to signature $(3, 3)_{\mathbb{R}}$ there are 223 fields having a discriminant larger than $-40000\,00000$ and there are 3 pairs and 3 quadruplets of nonisomorphic fields having the same discriminant.

For the same signature, but in table $(3, 3)_{\mathbb{C}}$, there are 200 fields having a discriminant larger than $-10000\,00000$, including 2 pairs of nonisomorphic fields having the same discriminant.

The intersection of tables $(3, 3)_{\mathbb{R}}$ and $(3, 3)_{\mathbb{C}}$ is not empty. These tables have 7 fields in common, corresponding to ninth-degree fields containing a real and a complex cubic subfield simultaneously.

Finally, for signature (1,4) the table contains the 485 fields having a discriminant smaller than 5000 00000, and it includes 9 pairs of nonisomorphic fields with the same discriminant.

The computation involved several phases. During the first phase we wanted to eliminate as many polynomials as possible either by using the signature (Sturm's algorithm) or by explicitly computing the value of $T_2(\theta)$ from the roots of the relative polynomials. This last procedure is called 'eliminated by trace' in Table II (next page). In the second phase we tested to see if the remaining polynomials were irreducible. We then computed the discriminant of the field, using Dedekind's criterion first, and when we were unable to decide whether or not the field had to be preserved, we used 'ROUND 2' to evaluate the discriminant of the field. In the third phase we considered all the polynomials of degree nine that are the cube of a cubic polynomial in $\mathbb{Z}[X]$, and we constructed a primitive element of the extension $K/\mathbb{Q}$ as shown in the corollary to Theorem 2. We observed that in every case the newly found polynomial generated a number field isomorphic to a field already found in the tables.

Finally, we determined whether or not isomorphisms exist between fields having the same discriminant.

All the computations were performed on work stations SPARC 1 or SPARC 2 and, except for the first phase, we used the PARI system.

In the next table precise indications are given concerning the CPU time used in computations as well as the number of polynomials treated. It appears that most of the CPU time used concerns phases I and II; for this reason, the type of the SPARC work station used is also indicated. All the computational time is expressed in minutes.

TABLE II

signature	(9,0)	(7,1)	(5,2)	$(3,3)_{\mathbb{R}}$	$(3,3)_{\mathbb{C}}$	(1,4)
subfields	real	real	real	real	complex	complex
tested polynomials	155 69964	12019 10861	8394 23372	6849 14826	5182 09760	2388 86137
eliminated by signature	154 87260	11837 70376	6452 51604	1023 53452	4702 04181	222 69560
eliminated by trace	*	180 78315	1939 58698	5219 12888	479 68716	2165 36090
remaining polynomials	72674	62170	2 13070	2 81848	26086	72404
time phase I	15 †	729 †	1742	4153	1273	4728
reducible polynomials	30116	17972	16533	4334	10767 ‡	7644 ‡
Dedekind's criterion	34500	35858	1 53190	2 16074	17640	47682
eliminated by round2	5448	6182	27094	36248	5260	14851
remaining fields	2002	2058	16252	24704	3176	9589
time phase II	698 †	993	7242	4709 †	927	1471 †
fields in the table	27	23	154	223	200	485
total time used	792	1722	8984	8863	2200	6262

†These computations were done on a SPARC 2 work station.

‡These computations were done during phase I.

4. DESCRIPTION OF SPECIFIC FIELDS IN THE TABLES

4.1. In the table corresponding to signature (9,0), the minimum discriminant corresponds to an extension of the cubic field $k_1 = \mathbb{Q}(\alpha_1)$ of discriminant 49. Here, α_1 is a zero of the polynomial $X^3 + X^2 - 2X - 1$, and the relative discriminant of the extension K/k_1 is the prime ideal $\mathfrak{p} = (138041, \alpha_1 + 113260)$ of k_1.

The field of discriminant $1\,69835\,63041 = 19^8$ is the maximal real subfield of the cyclotomic field $\mathbb{Q}(\zeta_{19})$.

In the same table, the field of discriminant $1\,75152\,30173 = 7^6 \cdot 53^3$ has two cubic subfields: k_1 and the cubic field of discriminant $2597 = 7^2 \cdot 53$.

The last field in the table is the abelian field of discriminant $6\,25235\,02209 = 3^{12} \cdot 7^6$; it is the compositum of the field k_1 and the cyclic field $k_2 = \mathbb{Q}(\alpha_2)$ of discriminant 81, where α_2 is a zero of $X^3 - 3X - 1$. This field also has as subfields the two nonisomorphic cubic fields of discriminant $3969 = 3^4 \cdot 7^2$.

For 5 fields in this table the norm of the relative discriminant is a square: $d_{K_1} = 1\,64403\,05941 = 37^2 \cdot 229^3$; $d_{K_2} = 1\,69835\,63041 = 19^8$; $d_{K_3} = 3\,13810\,59609 = 3^{22}$; $d_{K_4} = 3\,67630\,77169 = 7^6 \cdot 13^2 \cdot 43^2$ and $d_{K_5} = 5\,32866\,43921 = 7^6 \cdot 673^2$. These fields $K_1, \dots, K_5$ are subfields of ray class fields of their corresponding cubic subfields.

4.2. In the table corresponding to signature (7,1) the minimal discriminant is $-26681\,61671 = -7^6 \cdot 22679$, and the relative discriminant is the ideal $\mathfrak{p} = (22679, \alpha_1 - 28)$ of k_1.

4.3. The minimal discriminant for imprimitive nonic fields of signature (5,2) is $4678\,90073 = 7^6 \cdot 41 \cdot 97$, and the relative discriminant is the ideal $\mathfrak{q} = (41, \alpha_1 + 11)(97, \alpha_1 - 41)$ of k_1. We can now state the following result:

Proposition 1. *For imprimitive number fields of degree 9 and signature* $(5, 2)$ *having a discriminant in the interval* $4678\,90073 \le d \le 50000\,00000$ *there is a pair of nonisomorphic fields with the same discriminant for the following values of* d: $12090\,78773 = 7^6{\cdot}43{\cdot}239$, $30175\,79201 = 7^6{\cdot}13{\cdot}1973$, $31658\,16941 = 7^6 \cdot 71 \cdot 379$, $40914\,79273 = 7^6 \cdot 83 \cdot 419$ *and* $47945\,49697 = 7^6 \cdot 83 \cdot 491$.

Proof. A look at the table and a straightforward computation of the relative discriminants. □

Corollary. *There are pairs of nonisomorphic number fields of degree* 6 *and signature* $(2, 2)$ *with the same discriminant for the following values of the discriminant*: $246\,75077 = 7^4 \cdot 43 \cdot 239$, $615\,84249 = 7^4 \cdot 13 \cdot 1973$, $646\,0859 = 7^4{\cdot}71{\cdot}379$, $834\,99577 = 7^4{\cdot}83{\cdot}419$ *and* $978\,47953 = 7^4{\cdot}83{\cdot}491$. *All these sextic fields have a class number divisible by* 3.

Proof. The relative quadratic extensions L_1 and L_2 of k_1 attached by Galois theory to the extensions K_1/k_1 and K_2/k_1 have different relative discriminants over k_1. Denote by N_1 and N_2 the Galois closure of the respective extensions K_1/k_1 and K_2/k_1. The extensions N_1/L_1 and N_2/L_2 are relative cyclic extensions unramified at the finite places. Consequently, the class number of L_1 and L_2 must be divisible by 3. □

The same reasoning can by applied to the sextic fields L_i attached to the extensions K_i/k_1 for $i = 3, \dots, 10$.

4.4. In the table corresponding to signature $(3, 3)_{\mathbb{R}}$ the minimal discriminant is $-1645\,90951 = -7^6 \cdot 1399$ and the field having this discriminant is a relative extension of k_1 with relative discriminant $(1399, \alpha_1 + 347)$.

Proposition 2. *Among the imprimitive ninth-degree number fields of signature* $(3, 3)$ *having a real cubic subfield and a discriminant* d *belonging to the interval* $-40000\,00000 \le d \le -1645\,90951$, *there exists a pair of nonisomorphic fields with the same discriminant for the following values of* d: $-23156\,852676 = -3^9{\cdot}7^6$, $-28386\,35072 = -2^6{\cdot}7^6{\cdot}13{\cdot}29$ *and* $-33204\,07727 = -7^6{\cdot}13^2{\cdot}167$. *There also exists a quadruplet of nonisomorphic fields with the same discriminant for the values of* d: $-20455\,63163 = -7^6 \cdot 17387$, $-21410\,94151 = -7^6 \cdot 18199$ *and* $-23867\,45263 = -7^6 \cdot 20287$.

Proof. Only one of the fields of discriminant $-3^9 \cdot 7^6$ has an ideal of norm 43. In one of the fields of discriminant $-2^6 \cdot 7^6 \cdot 13 \cdot 29$ there is only one prime ideal over 37 but not in the other field. Similarly, there is only one prime ideal over 19 in one of the fields of discriminant $-7^6 \cdot 13^2 \cdot 167$ and 3 prime ideals over 19 in the other field.

For $d = -2045563163$ the decomposition of the primes 11,17, and 19 shows that the four fields having this discriminant are not isomorphic.

For $d = -2141094151$, consider the primes 17, 47, and 103.

For $d = -2386745263$, consider the primes 19, 31, and 37. □

Corollary. *The sextic number fields*

$$L_1 = \mathbb{Q}(\beta_1), \quad L_2 = \mathbb{Q}(\beta_2), \quad L_3 = \mathbb{Q}(\beta_3)$$

having the respective discriminants $d_{L_1} = -41746187 = -7^4 \cdot 17387$, $d_{L_2} = -43695799 = -7^4 \cdot 18199$, $d_{L_3} = -48709087 = -7^4 \cdot 20287$, *where* β_1, β_2, β_3 *are zeros of the polynomials*

$$R_1(X) = X^6 - 2X^5 + 20X^4 - 27X^3 + 140X^2 - 98X + 343,$$
$$R_2(X) = X^6 - X^5 + 19X^4 - 13X^3 + 133X^2 - 49X + 343,$$
$$R_3(X) = X^6 - X^5 + 20X^4 - 9X^3 + 142X^2 - 18X + 377,$$

have a class group isomorphic to $C_3 \times C_3$.

Proof. These sextic fields are relative quadratic extensions of the cubic field k_1 of discriminant 49 and are associated with the quadruplets of relative cubic extensions of k_1 having the same relative discriminant. The Galois closure for each field of degree nine in a quadruplet is an unramified relative cubic extension (even at infinite places) of one of these sextic number fields. According to class field theory, the existence of 4 unramified relative extensions proves that the class group of these sextic fields must have a 3-rank bigger than 1. The exact value of the class number of these fields was computed using the KANT [17] system and, in every case, the class group was found to be isomorphic to $C_3 \times C_3$. □

Some fields in this table also have a complex cubic subfield, and they appear in the table corresponding to signature $(3, 3)_{\mathbb{C}}$ when their discriminant belongs to the interval covered by that table. The complete list of these fields is given in Table III.

These fields are of particular interest because they give supplementary information about the class number of the complex cubic subfields. The table corresponding to signature $(3, 3)_{\mathbb{C}}$ has 7 fields in common with the table corresponding to signature $(3, 3)_{\mathbb{R}}$. For 6 of these fields, the norm of the relative discriminant is equal to 1. For 3 other fields in Table III having a discriminant beyond the limits of table $(3, 3)_{\mathbb{C}}$ the discriminant of the field of degree nine is the cube of the discriminant of the complex cubic subfield, and the relative extension has a discriminant of norm equal to 1. It is therefore clear that these complex cubic fields must have a class number divisible by 3.

TABLE III

d_k	$d_{k'}$	d_K	Factorization of d_K
49	-588	-2032 97472	$-2^6 \cdot 3^3 \cdot 7^6$
81	-648	-2720 97792	$-2^9 \cdot 3^{12}$
169	-676	-3089 15776	$-2^6 \cdot 13^6$
81	-891	-7073 47971	$-3^{12} \cdot 11^3$
49	-931	-8069 54491	$-7^6 \cdot 19^3$
81	-108	-9183 30048	$-2^6 \cdot 3^{15}$
49	-980	-9411 92000	$-2^6 \cdot 5^3 \cdot 7^6$
81	-243	-11622 61467	-3^{19}
361	-1083	-12702 38787	$-3^3 \cdot 19^6$
49	-23	-14314 35383	$-7^6 \cdot 23^3$
49	-1176	-16263 79776	$-2^9 \cdot 3^3 \cdot 7^6$
81	-135	-17936 13375	$-3^{15} \cdot 5^3$
49	-1323	-23156 85267	$-3^9 \cdot 7^6$
81	-324	-27549 90144	$-2^6 \cdot 3^{16}$
49	-31	-35048 81359	$-7^6 \cdot 31^3$
321	-107	-35391 49227	$-3^3 \cdot 107^4$

A comparison with Angell's table [1] of complex cubic fields of discriminant larger than $-(4 \cdot 10^9)^{1/3}$ and a class number divisible by 3 shows that only the three nonisomorphic fields of discriminant -1228 do not appear in our Table III. It is nonetheless easy to verify that the ninth-degree fields containing such a cubic subfield are outside the limits of table $(3, 3)_{\mathbb{C}}$. When the real cubic subfield k of K (with the notations in Table III) is abelian, K is a relative cyclic extension of the complex cubic field k'.

It can also be pointed out that between the two nonisomorphic fields of discriminant $-23156\,85267$ only one field contains a complex cubic subfield.

4.5. In the table corresponding to signature $(3, 3)_{\mathbb{C}}$ the minimal discriminant is $-1108\,52311 = -31^3 \cdot 61^2$, and this field is an extension of the complex cubic field k_4 of discriminant equal to -31. The relative discriminant of the extension K/k_4 is the ideal $(61, \alpha_4 - 24)^2$, where α_4 is a zero of the polynomial $X^3 + X - 1$ and $k_4 = \mathbb{Q}(\alpha_4)$.

Proposition 3. *Among the imprimitive number fields of degree* 9 *and signature* (3, 3) *containing a complex cubic subfield whose discriminant belongs to the interval* $-10000\,00000 \le d \le -1108\,52311$, *there exists a pair of nonisomorphic* *fields having the same discriminant for the values of* d: $-5235\,82511 = -23^4 \cdot 1871$ *and* $-8759\,38831 = -23^3 \cdot 71993$.

Proof. For the first pair, consider the prime 11, and for the second, consider the prime 5. □

Corollary. *There exists a pair of nonisomorphic sextic number fields with signature* $(2, 2)$ *and discriminants* $227\,64457 = 23^3 \cdot 1871$ *and* $380\,84297 = 23^2 \cdot 71993$. *These sextic fields have a class number divisible by* 3.

Proof. These sextic fields are relative quadratic extensions of the complex cubic field k_3 of discriminant -23 generated over $\mathbb{Q}$ by a zero α_3 of $X^3 + X^2 - 1$, and they are associated with the two pairs of ninth-degree fields whose discriminants were given in Proposition 3. It is easy to verify that the two sextic fields of discriminant $227\,64457$ are generated respectively by a zero of the polynomials $X^6 - 2X^5 - 22X^4 + 21X^3 + 114X^2 + 185X - 809$ and $X^6 - 29X^4 - 8X^3 + 262X^2 + 116X - 605$.

The generating polynomials for the sextic fields of discriminant $380\,84297$ are $X^6 - X^5 - 31X^4 + 25X^3 + 313X^2 - 197X - 1219$ and $X^6 - 2X^5 - 27X^4 + 50X^3 + 286X^2 - 292X - 1127$.

As previously established in similar proofs, the class number of these fields must be divisible by 3. □

4.6. The minimal discriminant for ninth-degree number fields of signature (1,4) containing a cubic subfield is $322\,06049 = 23^3 \cdot 2647$. The field K having this discriminant is a relative cubic extension of k_3, and the relative discriminant of K/k_3 is the prime ideal $(2647, \alpha_3 + 143)$.

This field as well as the next two fields in the table, of discriminants $338\,60761 = 11^2 \cdot 23^4$ and $350\,28793 = 23^3 \cdot 2879$, are Euclidean for the norm, and they were discovered by Leutbecher [18].

Proposition 4. *Among the imprimitive ninth-degree number fields of signature* $(1, 4)$ *whose discriminant belongs to the interval* $322\,06049 \le d \le 5000\,00000$, *there are* 9 *pairs of nonisomorphic fields having the same discriminant. The corresponding values for these discriminants are*: $949\,87769 = 23^3 \cdot 37 \cdot 211$, $1540\,22053 = 23^3 \cdot 12659$, $2297\,49461 = 23^4 \cdot 821$, $2468\,44096 = 2^6 \cdot 23^3 \cdot 317$, $3298\,45952 = 2^6 \cdot 31^3 \cdot 173$, $3515\,16797 = 23^3 \cdot 167 \cdot 173$, $3618\,83081 = 7^2 \cdot 23^3 \cdot 607$, $3698\,76800 = 2^6 \cdot 5^2 \cdot 19 \cdot 23^3$, *and* $4744\,81257 = 3 \cdot 31^3 \cdot 5309$.

Proof. For each pair of fields having the same discriminant, we can produce a prime number having a different decomposition in the two fields as a product of prime ideals. □

Corollary. *There exists a pair of nonisomorphic totally complex sextic fields with the same discriminant for the following values of* d: $-41\,29903 = -23^2 \cdot 37 \cdot 211$, $-66\,96611 = -23^2 \cdot 12659$, $-99\,89107 = -23^3 \cdot 821$, $-106\,40192 = -2^6 \cdot 31^2 \cdot 173$, $-107\,32352 = -2^6 \cdot 23^2 \cdot 317$, $-152\,83339 = -23^2 \cdot 167 \cdot 173$ *and* $-153\,05847 = -3 \cdot 31^2 \cdot 5309$. *The class number of all these sextic fields is divisible by* 3.

Proof. The proof is similar to that already established. □

Remark. The two sextic fields associated with the fields of degree 9 and discriminant 3618 83081 do not have the same discriminant. For one of these nonic fields the relative discriminant over k_3 is the ideal $(7, \alpha_3^2+4\alpha_3+5)(607, \alpha_3+184)$, and the absolute discriminant of the associated sextic field is $-157\,34047$. This field is generated over $\mathbb{Q}$ by a zero of the polynomial $X^6-2X^5+21X^4-45X^3+192X^2-188X+488$. The relative discriminant for the other nonic field having the same absolute discriminant is the ideal $(7, \alpha_3+4)^2(607, \alpha_3+184)$, and the relative quadratic extension over k_3 associated with it has an absolute discriminant equal to $-3\,21103$ and it is generated by a zero of the polynomial $X^6-3X^5+7X^4-9X^3+11X^2-7X+11$.

Similarly, for one of the nonic fields with discriminant 3698 76800, the relative discriminant over k_3 is the ideal $(2)^2(5, \alpha_3^2-\alpha_3+2)(19, \alpha_3+3)$, and the associated sextic field, of discriminant $-160\,81600$, is generated by a zero of $X^6-4X^4-65X^2+475$. For the other nonic field with the same discriminant, the relative discriminant over k_3 is the ideal $(2)^2(5, \alpha_3+2)^2(19, \alpha_3+3)$, and the associated sextic field, of discriminant -10051, is generated by a zero of $X^6-3X^5+4X^4-4X^3+3X^2-X+1$.

We conclude this section with the following theorem whose proof is clear:

Theorem 3. *Let K denote a number field of degree 9 appearing in the tables for which the relative discriminant δ of the extension K/k is not divisible by the square of any prime ideal of k. Let L denote the sextic field containing k. If $|d_L| = d_k^2 \cdot N_{k/\mathbb{Q}}(\delta)$, then the class number of L is divisible by 3.*

5. Galois groups

Let $K = \mathbb{Q}(\theta)$ be an imprimitive ninth-degree number field, and denote by Γ the Galois group of the Galois closure of $K/\mathbb{Q}$. The action of Γ on θ allows us to consider Γ as a transitive ninth-degree permutation group. When this is the case, we will say that K is of type Γ.

All the transitive groups up to degree 11 are known [7] and, up to conjugacy, there are 34 ninth-degree transitive groups. Among them, 23 are imprimitive, and 12 are imprimitive and even. In this section, we use the notations of [7] to designate these groups. For example, we denote by T_5^+ group T_5 in [7] and the upper sign $^+$ means that this group is even and, consequently, the discriminant of a field K of type T_5^+ is a square.

The following proposition is due to Y. Eichenlaub (private communication).

Proposition 5. *In each signature, the possible Galois groups for imprimitive ninth-degree number fields are*:

$(9, 0)$: $T_1^+, T_2^+, T_3^+, T_4, T_5^+, T_6^+, T_7^+, T_8, T_{10}^+, T_{11}^+, T_{12}, T_{13}, T_{17}^+, T_{18}, T_{20}, T_{21}^+, T_{22}, T_{24}, T_{25}^+, T_{28}, T_{29}, T_{30}^+, T_{31}$

$(7, 1)$: T_{28}, T_{31}

$(5, 2)$: $T_{25}^+, T_{28}, T_{29}, T_{30}^+, T_{31}$

$(3, 3)$: $T_4, T_8, T_{12}, T_{13}, T_{18}, T_{20}, T_{22}, T_{24}, T_{28}, T_{29}, T_{31}$

$(1, 4)$: $T_3^+, T_5^+, T_8, T_{10}^+, T_{11}^+, T_{18}, T_{21}^+, T_{24}, T_{30}^+, T_{31}$

To compute the Galois group for each ninth-degree polynomial in the tables, we use the classical relative resolvent method ([16] and [30]), based on the following property: Let us denote by G a transitive subgroup of degree n of S_n, and by H a subgroup of G. An H-polynomial for G is a polynomial $F(X_1, \dots, X_n) \in \mathbb{Z}[X_1, \dots, X_n]$ such that

$$H = \{\sigma \in G \mid F(X_{\sigma(1)}, \dots, X_{\sigma(n)}) = F(X_1, \dots, X_n)\}.$$

If $\theta_1, \dots, \theta_n$ denote the conjugates of θ in $\mathbb{C}$, then the G-resolvent related to H and associated with F is the polynomial

$$R_{(G,H,F)}(Y) = \prod_{\sigma \in A} (Y - F(\theta_{\sigma(1)}, \dots, \theta_{\sigma(n)})) ,$$

where A is a complete set of the right cosets of G modulo H (the right coset associated with $\pi \in G$ is πH).

Proposition 6. *Suppose that for a fixed order of the roots $(\theta_1, \dots, \theta_n)$ of the irreducible polynomial defining the extension $K/\mathbb{Q}$, the Galois group Γ is included in G. Then:*

(1) *$\sigma \in G$ and $F(\theta_{\sigma(1)}, \dots, \theta_{\sigma(n)}) \notin \mathbb{Z}$ implies $\sigma^{-1}\Gamma\sigma \not\subset H$,*

(2) *if $F(\theta_{\sigma(1)}, \dots, \theta_{\sigma(n)}) \in \mathbb{Z}$ is a simple root of $R_{(G,H,F)}$, then $\sigma^{-1}\Gamma\sigma \subset H$.*

The Galois group of the fields in the tables are, according to the signature:

$(9, 0)$: two fields of type $T_1^+ = C_9$, one of type $T_2^+ = C_3^2$, one of type $T_4 = C_3 \times S_3$, two of type T_{17}^+, one of type T_{20}, 19 of type T_{28} and one of type T_{31}.

$(7, 1)$: all the fields of type T_{28}.

$(5, 2)$: 148 fields of type T_{28} and 6 of type T_{31}.

$(3, 3)_{\mathbb{R}}$: 15 fields of type $T_4 = C_3 \times S_3$, one of type $T_8 = C_3^2 \rtimes C_3^2$, one of type T_{13}, two of type T_{18}, 3 of type T_{22}, 189 of type T_{28}, and 12 of type T_{31}.

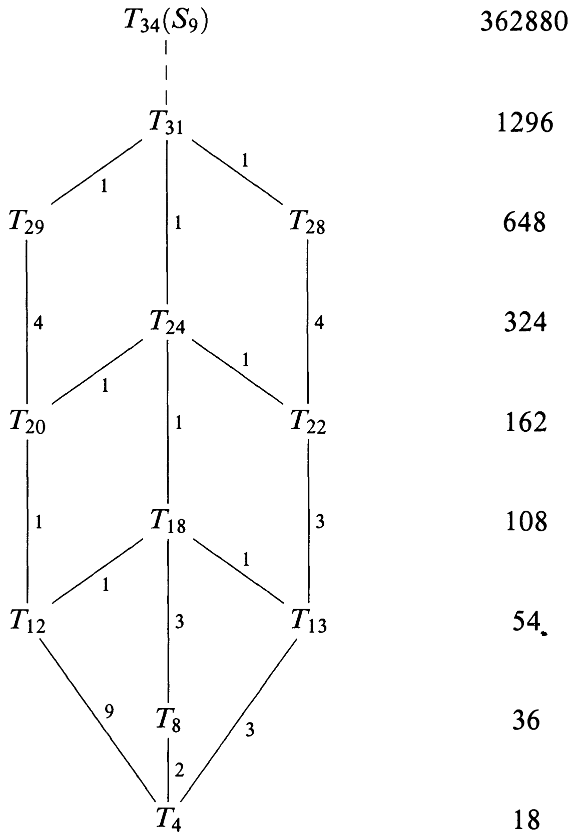
$(3, 3)_{\mathbb{C}}$: 7 fields of type $T_4 = C_3 \times S_3$, 5 of type T_{12}, 31 of type T_{20}, one of type T_{29}, and 156 of type T_{31}. The seven fields with type T_4 are those belonging to the tables $(3, 3)_{\mathbb{R}}$ and $(3, 3)_{\mathbb{C}}$.

$(1, 4)$: one field of type $T_8 = C_3^2 \rtimes C_3^2$, two of type T_{18}, one of type T_{24}, two of type T_{30}^+, and 479 of type T_{31}.

The next diagram gives the graph of all transitive imprimitive ninth-degree odd groups. In the right column the order of the groups is indicated, and a line joining two groups indicates an inclusion up to conjugacy by elements of S_9. The number on the line gives the number of conjugate copies pairwise distinct from the smallest group included in the biggest one.

The maximal transitive imprimitive ninth-degree group is T_{31}, of order 1296.

For each line in the graph of the transitive imprimitive even (resp. odd) groups, the relative G-polynomials are computed using the method of [16] (cf. [25]).



We also have:

Proposition 7. *For the following ninth-degree transitive imprimitive groups, the minimal discriminants are, according to the signature*:

$(9, 0)$: 1 69835 63041 *for the type* T_1^+, 6 25235 02209 *for the type* T_2^+, 1 75152 30173 *for the type* T_4, 3 67630 77169 *for the type* T_{17}^+, 1 64403 05941 *for the type* T_{20}, 1 62403 85609 *for the type* T_{28} *and* 5 30389 58912 *for the type* T_{31}.

$(7, 1)$: −26681 61671 *for the type* T_{28}.

$(5, 2)$: 4678 90073 *for the type* T_{28} *and* 12999 58592 *for the type* T_{31}.

$(3, 3)$: −2032 97472 *for the type* T_4, −35391 49227 *for the type* T_8, −3573 66875 *for the type* T_{12}, −23156 85267 *for the type* T_{13}, −11119 34656 *for the type* T_{18}, −1108 52311 *for the type* T_{20}, −12724 91584 *for the type* T_{22}, −1645 90951 *for the type* T_{28}, −5624 19575 *for the type* T_{29} *and* −1471 84199 *for the type* T_{31}.

$(1, 4)$: 3624 67097 *for the type* T_8, 2394 83061 *for the type* T_{18}, 4497 28821 *for the type* T_{24}, 338 60761 *for the type* T_{30}^+ *and* 322 06049 *for the type* T_{31}.

BIBLIOGRAPHY

1. I. O. Angell, *A table of complex cubic fields*, Bull. London Math. Soc. **5** (1973), 38–38.
2. ——, *A table of totally real cubic fields*, Math. Comp. **30** (1976), 184–187.
3. A.-M. Bergé, J. Martinet, and M. Olivier, *The computation of sextic fields with a quadratic subfield*, Math. Comp. **54** (1990), 869–884.
4. D. Bernardi, Ch. Batut, H. Cohen, and M. Olivier, *User's guide to PARI-GP version* 1.37, Publ. Université Bordeaux I (1992).
5. J. Buchmann and D. Ford, *On the computation of totally real quartic fields of small discriminant*, Math. Comp. **52** (1989), 161–174.
6. J. Buchmann, D. Ford, and M. Pohst, *Enumeration of quartic fields of small discriminant*, Math. Comp. **61** (1993), 873–879.
7. G. Butler and J. McKay, *The transitive groups of degree up to eleven*, Comm. Algebra **11** (1983), 863–911.
8. H. Cohen, *A course in computational algebraic number theory*, Springer-Verlag, Berlin, 1993.
9. F. Diaz y Diaz, *Petits discriminants des corps de nombres totalement imaginaires de degré* 8, J. Number Theory **25** (1987), 34–52.
10. ——, *A table of totally real quintic number fields*, Math. Comp. **56** (1991), 801–808.
11. ——, *Supplement to "A table of totally real quintic number fields"*, Math. Comp. **56** (1991), S1–S12.
12. V. Ennola and R. Turunen, *On totally real cubic fields*, Math. Comp. **44** (1985), 495–518.
13. D. J. Ford, *On the computation of the maximal order in a Dedekind domain*, Ph. D. Dissertation, Ohio State University, 1978.
14. D. Ford and M. Pohst, *The totally real* A_5 *extension of degree* 6 *with minimum discriminant*, Experimental Math. **1** (1992), 231–235.
15. H. Fujita, *The minimum discriminant of totally real algebraic number fields of degree* 9 *with cubic subfields*, Math. Comp. **60** (1993), 801–810.
16. K. Girstmair, *On invariant polynomials and their application in field theory*, Math. Comp. **48** (1987), 781–797.
17. J. Graf von Schmettow, *KANT—a tool for computations in algebraic number fields*, Computational Number Theory (A. Pethö, M. E. Pohst, H. C. Williams, and H. G. Zimmer, eds.), de Gruyter, Berlin and New York, 1989, pp. 321–330.
18. A. Leutbecher, *Euclidean fields having a large Lenstra constant*, Ann. Inst. Fourier **35** (1985), 83–106.
19. P. Llorente and A. V. Oneto, *On the real cubic fields*, Math. Comp. **39** (1982), 689–692.
20. J. Martinet, *Méthodes géométriques dans la recherche des petits discriminants*, Séminaire de Théorie des Nombres de Paris 1983/1984, Birkhäuser, Basel, 1985, pp. 147–179.
21. M. Olivier, *Tables de corps sextiques contenant un sous-corps quadratique* (I), Séminaire de Théorie des Nombres de Bordeaux 1 Sér. 2 (1989), 205–250.
22. ——, *Corps sextiques contenant un corps quadratique* (II), Séminaire de Théorie des Nombres de Bordeaux 2 Sér. 2 (1990), 49–102.
23. ——, *Corps sextiques contenant un corps cubique* (III), Séminaire de Théorie des Nombres de Bordeaux 3 Sér. 2 (1991), 201–245.
24. ——, *Corps sextiques primitifs* (IV), Séminaire de Théorie des Nombres de Bordeaux 3 Sér. 2 (1991), 381–404.
25. ——, *Computation of Galois groups for polynomials with degree up to eleven*, submitted to J. Algebra.
26. M. Pohst, *On the computation of number fields of small discriminant including the minimum discriminants of sixth degree fields*, J. Number Theory **14** (1982), 99–117.
27. M. Pohst, J. Martinet, and F. Diaz y Diaz, *The minimum discriminant of totally real octic fields*, J. Number Theory **36** (1990), 145–159.

28. M. Pohst and H. Zassenhaus, *Algorithmic algebraic number theory*, Cambridge Univ. Press, 1989.

29. A. Schwarz, M. Pohst, and F. Diaz y Diaz, *A table of quintic number fields*, Math. Comp. **63** (1994), 361–376.

30. R. P. Stauduhar, *The determination of Galois groups*, Math. Comp. **27** (1973), 981–996.

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