MATHEMATICS OF COMPUTATION
VOLUME 64, NUMBER 209
JANUARY 1995, PAGES 427-438

DENSITY BOUNDS FOR THE $3x + 1$ PROBLEM. II. KRASIKOV INEQUALITIES

DAVID APPLEGATE AND JEFFREY C. LAGARIAS

Dedicated to the memory of D. H. Lehmer

ABSTRACT. The $3x + 1$ function $T(x)$ takes the values $(3x + 1)/2$ if x is odd and $x/2$ if x is even. Let a be any integer with $a \not\equiv 0 \pmod 3$. If $\pi_a(x)$ counts the number of n with $|n| \le x$ which eventually reach a under iteration by T, then for all sufficiently large x, $\pi_a(x) \ge x^{.81}$. The proof is based on solving nonlinear programming problems constructed using difference inequalities of Krasikov.

1. INTRODUCTION

The $3x+1$ problem concerns the iteration of the function $T: \mathbb{Z} \to \mathbb{Z}$ defined by

$$(1.1) \qquad T(x) = \begin{cases} \frac{3x+1}{2} & \text{if } x \equiv 1 \pmod 2, \\ \frac{x}{2} & \text{if } x \equiv 0 \pmod 2. \end{cases}$$

The $3x+1$ conjecture asserts that, for all $n \ge 1$, some iterate $T^{(k)}(n) = 1$ (see [4]). This paper studies, for $a \in \mathbb{Z}$, the function

$$(1.2) \qquad \pi_a(x) = \#\{n: |n| \le x \text{ and some } T^{(k)}(n) = a, \ k \ge 0\}.$$

It is well known that the growth of $\pi_a(x)$ depends on the residue class of $a \pmod 3$. If $a \equiv 0 \pmod 3$, then the preimages of a under T are exactly $\{2^k a: k \ge 0\}$; hence $\pi_a(x)$ grows logarithmically with x. Consequently, we assume $a \not\equiv 0 \pmod 3$, and our object is to prove lower bounds of the form

$$(1.3) \qquad \pi_a(x) \ge x^\gamma \quad \text{for } x \ge x_0(a),$$

for some constant $\gamma > 0$. In part I we showed one can take $\gamma = .65$ using an approach initiated by Crandall [2] and extended by Sander [5].

In this paper we derive bounds (1.3) using systems of difference inequalities found by Krasikov [3]. For each $k \ge 2$ there is a system $\mathcal{I}_k$ of such inequalities; Krasikov [3] used $\mathcal{I}_2$ to obtain $\gamma = .43$ in (1.3), and Wirsching [6] used $\mathcal{I}_3$ to obtain $\gamma = .48$. We extract information from the inequalities $\mathcal{I}_k$, by constructing families of auxiliary linear programs whose entries depend (nonlinearly) on a parameter $\lambda := 2^\gamma$. These linear programs have the property that a nonzero feasible solution yields a proof of (1.3) for its associated value of γ. In this fashion, using a well-chosen linear program derived from $\mathcal{I}_9$, we obtain by a computer-assisted proof the following result.

Received by the editor May 3, 1993 and, in revised form, December 17, 1993.
1991 *Mathematics Subject Classification.* Primary 11B37, 26A18; Secondary 39B12, 58F08.

Theorem 1.1. *For each* $a \not\equiv 0 \pmod 3$, *there is a positive constant* c_a *such that*

$$\pi_a(x) \geq c_a x^{.81} \quad \textit{for all } x \geq a. \tag{1.4}$$

The proof of Theorem 1.1 consists of writing down the linear program and an explicit nonzero feasible solution. This proof is too long to write down conveniently, as the linear program has $\frac{1}{2}(3^9 - 1)$ variables. In §3 we indicate how the linear program is obtained.

The Krasikov inequality approach for bounding γ in (1.3) appears superior to the tree-search approach studied in part I. The weakness of the tree-search approach is that it does not make full use of the fact that the leaves of the trees are somewhat well distributed in congruence classes $\pmod{3^k}$, so that the worst-case behavior assumed in the estimate of Theorem 2.1 of [1] cannot occur. Krasikov inequalities capture this "mixing" effect to some degree, even while searching to a much smaller depth k. On the other hand, the Krasikov inequality approach cannot give bounds for the quantities $n_k(a)$ studied in part I, nor does it seem adaptable to obtain any sort of upper bound estimates.

In §4 we discuss Krasikov's conjecture that, for any $\varepsilon > 0$, a bound of the form

$$\pi_a(x) \geq x^{1-\varepsilon} \quad \text{for } x \geq x_0(a, \varepsilon), \tag{1.5}$$

is implied by the inequalities $\mathscr{I}_k$, for sufficiently large k. The numerical evidence strongly suggests that this is true. We indicate obstacles to obtaining a rigorous proof of (1.5).

2. Krasikov-based lower bounds

Krasikov [3] developed a set of difference inequalities for counting the number of $3x+1$ iterates below a given bound. Define

(2.1)
$$\pi_a^*(x) := \#\{n \colon |n| \leq x, \text{ some } T^{(j)}(n) = a, \text{ all } |T^{(i)}(n)| \leq x \text{ for } 0 \leq i \leq j\}.$$

Note that $\pi_a^*(x) \leq \pi_a(x)$. For each residue class $m \pmod{3^k}$ with $m \not\equiv 0 \pmod 3$, Krasikov defines the function

$$\phi_k^m(y) := \inf\{\pi_a^*(2^y a) \colon a \equiv m \pmod{3^k} \text{ and } a \text{ is not in a cycle}\}. \tag{2.2}$$

This is well defined because there always exists some $a \equiv m \pmod{3^k}$ not in a cycle, namely $a = 2^l$ for suitable[1] $l \geq 3$, because 2 is a primitive root $\pmod{3^k}$ for all $k \geq 1$. The definition immediately implies that

$$\phi_{k-1}^m(y) = \min\{\phi_k^m(y), \phi_k^{m+3^{k-1}}(y), \phi_k^{m+2\cdot 3^{k-1}}(y)\}, \tag{2.3}$$

and also that

$$\phi_k^m(y) \text{ is a nondecreasing function of } y, \tag{2.4a}$$

$$\phi_m^k(y) \geq 1 \quad \text{for } y \geq 0. \tag{2.4b}$$

[1]The infimum in (2.2) is actually attained by some $a = 2^l$. The infimum is attained because $\phi_k^m(y)$ is integer-valued, so let a_0 be the minimal choice of $a \equiv m \pmod{3^k}$ attaining it, and let j be maximal with $T^{(j)}(n) = a_0$, for any n counted in $\pi_{a_0}^*(2^y a_0)$. It suffices to choose $2^l \equiv a_0 \pmod{3^{j+k}}$.

It is easy to see that

$$\phi_k^m(y) = \phi_k^{2m}(y-1), \quad \text{if } m \equiv 1 \pmod{3}, \tag{2.5}$$

and this relation can be used to express information purely in terms of $\phi_k^m(y)$ with $m \equiv 2 \pmod{3}$. Krasikov's inequalities[2] are as follows.

Proposition 2.1. *Set $\alpha = \log_2 3 \doteq 1.585$. Then, for all $k \geq 2$,*

(2.6a) $\phi_k^m(y) \geq \phi_k^{4m}(y-2) + \phi_{k-1}^{(4m-2)/3}(y+\alpha-2)$ *if* $m \equiv 2 \pmod{9}$,

(2.6b) $\phi_k^m(y) \geq \phi_k^{4m}(y-2)$ *if* $m \equiv 5 \pmod{9}$,

(2.6c) $\phi_k^m(y) \geq \phi_k^{4m}(y-2) + \phi_{k-1}^{(2m-1)/3}(y+\alpha-1)$ *if* $m \equiv 8 \pmod{9}$.

Proof. Let $(T^*)^{-1}$ denote the inverse operator to T on the domain $\{n : n \equiv 1$ or $2 \pmod{3}\}$, which is

$$(T^*)^{-1}(n) = \begin{cases} \{2n\} & \text{if } n \equiv 1, 4, 5, \text{ or } 7 \pmod{9}, \\ \{2n, \frac{2n-1}{3}\} & \text{if } n \equiv 2 \text{ or } 8 \pmod{9}. \end{cases}$$

The inequalities essentially encode $(T^*)^{-1}$ iterated the minimal number of times necessary to give images in the set $\{n : n \equiv 2 \pmod{3}\}$, which is

$$(T^{**})^{-1}(n) = \begin{cases} \{4n, \frac{4n-2}{3}\} & \text{if } n \equiv 2 \pmod{9}, \\ \{4n\} & \text{if } n \equiv 5 \pmod{9}, \\ \{4n, \frac{2n-1}{3}\} & \text{if } n \equiv 8 \pmod{9}. \end{cases}$$

For more details, see [3, Lemma 4]. □

For convenience in what follows, we use the abbreviation

$$[3^k] := \{m \pmod{3^k} : m \equiv 2 \pmod{3}\}. \tag{2.7}$$

Let $\mathcal{I}_k$ denote the system of inequalities (2.6) for $\{\phi_k^m(y) : m \in [3^k]\}$. We want to use these difference inequalities to get lower bounds for the $\phi_k^m(y)$. These inequalities relate the functions ϕ_k^m at a value y to $\phi_{k'}^{m'}$ at other values y', some of which are *retarded values* $y' < y$, while others are *advanced values* $y' > y$. We cannot immediately extract lower bounds, owing to the presence of advanced values. We can get lower bounds directly from sets of inequalities containing no advanced values. Property (2.4) allows us to obtain weaker inequalities containing only retarded values, by replacing each $y' \geq y$ with the value $y - \mu$ for some $\mu > 0$. We call this replacement operation *μ-truncation.*

Next, note that the right sides of the inequalities (2.6a)–(2.6c) involve only ϕ_j^n with $n \equiv 2 \pmod{3}$. We can obtain new inequalities by replacing any term $\phi_j^n(y')$ appearing on the right side of such an inequality by substituting the Krasikov inequality (2.6) for $\phi_j^n(y')$. We call this procedure *splitting* the term. Splitting operations can be applied repeatedly, in many possible orders.

We consider the following general method to obtain a set of inequalities starting from the 3^{k-1} inequalities $\mathcal{I}_k$. Perform some finite sequence of splittings of terms for each of these inequalities, and after this, μ-truncate each inequality to obtain a system of 3^{k-1} inequalities of the form

$$\phi_k^m(y) \geq \sum_{i \in I_m} \phi_{k_i}^{m_i}(y - \alpha_i), \quad \text{all } m \in [3^k]. \tag{2.8}$$

[2] Krasikov actually proves the slightly stronger bound $\phi_k^m(y) \geq \phi_k^{4m}(y-2) + [y+\alpha]$, if $m \equiv 5 \pmod{9}$, but we will not make use of this.

Here, each I_m is a different finite indexing set, and in this system of inequalities all arguments are strictly retarded, i.e., all $\alpha_i > 0$. As an example of this method applied to a single inequality, start with

$$\phi_3^{26}(y) \geq \phi_3^{23}(y-2) + \phi_2^{8}(y+\alpha-1),$$

split the last term to obtain

$$\phi_3^{26}(y) \geq \phi_3^{23}(y-2) + \phi_2^{5}(y+\alpha-3) + \phi_1^{2}(y+2\alpha-2),$$

then split the second term to obtain

$$\phi_3^{26}(y) \geq \phi_3^{23}(y-2) + \phi_2^{2}(y+\alpha-5) + \phi_1^{2}(y+2\alpha-2),$$

then μ-truncate to get

$$\phi_3^{26}(y) \geq \phi_3^{23}(y-2) + \phi_2^{2}(y+\alpha-5) + \phi_1^{2}(y-\mu).$$

For a fixed k, one can obtain infinitely many different systems (2.8) by this method. It is important that $\mu > 0$, to apply Theorem 2.1 below.

Let $\mathscr{L}_\mu$ denote a system of 3^{k-1} inequalities (2.8) obtained by this method, where μ indicates the value of the μ-truncation parameter. Any such system potentially yields exponential lower bounds for all $\phi_k^m(y)$, of the form

$$\phi_k^m(y) \geq a c_k^m \lambda^y, \quad \text{all } y > 0,$$

where $a > 0$ and $\lambda > 1$ is fixed, by associating to it a linear program (L_λ) given by

$$(\mathrm{L}_\lambda) \begin{cases} \text{maximize } c_1^2 & \\ c_k^m \leq \sum_{i \in I_m} c_{k_i}^{m_i} \lambda^{-\alpha_i}, & \text{all } m \in [3^k], \quad (2.9\text{a}) \\ c_j^n \leq c_{j+1}^{n+l\cdot 3^j}, \quad l = 0, 1, 2, & \text{all } n \in [3^j], \quad 1 \leq j \leq k-1, \quad (2.9\text{b}) \\ c_j^n \geq 0, & \text{all } n \in [3^j], \quad 1 \leq j \leq k, \quad (2.9\text{c}) \\ c_1^2 \leq 1. & (2.9\text{d}) \end{cases}$$

The key ingredients in this linear program are (2.9a) and (2.9b), which encode a reversing of the inequalities (2.8) and the inequalities (2.3), respectively.

Theorem 2.1. *Suppose that the linear program* (L_λ) *associated with a system of inequalities* (2.8) *has a feasible solution with* $c_1^2 > 0$. *Then* $c_j^n > 0$ *for all* $n \in [3^j]$, $1 \leq j \leq k$, *and there exists a positive constant* a *such that*

$$\phi_j^n(y) \geq a c_j^n \lambda^y, \quad \textit{all } y > 0, \tag{2.10}$$

for all $n \in [3^j]$, $1 \leq j \leq k$.

Proof. Let $\tilde{\mu} = \min\{\alpha_i : i \text{ in some } I_m\}$, and note that necessarily $\tilde{\mu} > 0$ because μ-truncation was used. We prove, by induction on the integer l, that (2.10) holds for all $y \in [0, l\tilde{\mu}]$. To handle the base case, define the integer M by

$$(M-1)\tilde{\mu} \leq \max\{\alpha_i : i \text{ in some } I_m\} < M\tilde{\mu}.$$

Since $\phi_j^n(y) \geq 1$, if we choose $a > 0$ small enough, then (2.10) will hold for all $y \in [0, M\tilde{\mu}]$. For the induction step, suppose $l \geq M$ and that (2.10) holds on $[0, l\tilde{\mu}]$. If $y \in (l\tilde{\mu}, (l+1)\tilde{\mu}]$, then all $y - \alpha_i \in [0, l\tilde{\mu}]$, and the induction hypothesis and (2.9a) give

$$\begin{aligned}
\phi_k^m(y) &\geq \sum_{i \in I_m} \phi_{k_i}^{m_i}(y - \alpha_i) \\
&\geq \sum_{i \in I_m} a c_{k_i}^{m_i} \lambda^{y-\alpha_i} = a\lambda^y \left(\sum_{i \in I_m} c_{k_i}^{m_i} \lambda^{-\alpha_i} \right) \\
&\geq a c_k^m \lambda^y, \quad \text{all } m \in [3^k].
\end{aligned}$$

It remains to treat $\phi_j^n(y)$ having $1 \leq j < k$. We proceed by a second, downward induction on j, the base case $j = k$ being proved. Now suppose case $j+1$ is proved; then

$$\begin{aligned}
\phi_j^n(y) &= \min(\phi_{j+1}^n(y), \phi_{j+1}^{n+3^j}(y), \phi_{j+1}^{n+2\cdot 3^j}(y)) \\
&\geq \min(a c_{j+1}^n \lambda^y, a c_{j+1}^{n+3^j} \lambda^y, a c_{j+1}^{n+2\cdot 3^j} \lambda^y) \\
&= a\lambda^y \min(c_{j+1}^n, c_{j+1}^{n+3^j}, c_{j+1}^{n+2\cdot 3^j}) \\
&\geq a c_j^n \lambda^y, \quad \text{all } n \in [3^j],
\end{aligned}$$

using the induction hypothesis and (2.9b). This completes the second induction which in turn completes the first induction. Finally, $c_1^2 > 0$ implies that all $c_j^n > 0$, on using (2.9b). □

We now have two problems: first, for a given system $\mathscr{L}_\mu$ to maximize the allowable value of λ, and second, to find that system $\mathscr{L}_\mu$ maximizing this quantity. We consider these in order.

For any fixed system $\mathscr{L}_\mu$ given by (2.8), if it has a solution with $c_1^2 > 0$, it has one with $c_1^2 = 1$ by rescaling the variables. Hence the problem of finding the maximal λ attainable using Theorem 2.1 is just the nonlinear programming problem

$$\text{(N)} \quad \begin{cases} \text{maximize } \lambda \\ \quad (\mathrm{L}_\lambda) \text{ has a feasible solution with } c_1^2 = 1. \end{cases}$$

Let $\lambda^*(\mathscr{L}_\mu)$ denote the optimal value of (N); note that this value is attained. We let $\mu \to 0$ and consider the limiting system obtained with $\mu = 0$, since one has

$$\lim_{\mu \to 0^+} \lambda^*(\mathscr{L}_\mu) = \lambda^*(\mathscr{L}_0).$$

However, for the limiting system $\mathscr{L}_0$ we can only conclude via Theorem 2.1 that there are values $c_j^n > 0$ such that for each $\varepsilon > 0$ there is some $a(\varepsilon) > 0$ with

$$\phi_j^n(y) \geq a(\varepsilon) c_j^n (\lambda^*(\mathscr{L}_0))^{(1-\varepsilon)y}, \quad \text{all } y > 0.$$

To solve the system (N) for a given $\mathscr{L}_0$, we treat it for each fixed value of λ as a linear programming program (L_λ) and see if the optimal solution[3] has $c_1^2 > 0$. Now we numerically locate an approximation $\hat{\lambda}^*(\mathscr{L}_0)$ to the maximal value

[3]Since all the constraints in (L_λ) are homogeneous except the last constraint $c_1^2 \leq 1$, and since taking all $c_i^j = 0$ is always a feasible solution, the optimal solution has either $c_1^2 = 0$ or $c_1^2 = 1$.

$\lambda^*(\mathscr{L}_0)$ by a bisection search, starting from the a priori bounds $1 \le \lambda^* \le 2$, such that

$$\hat{\lambda}^*(\mathscr{L}_0) \le \lambda^*(\mathscr{L}_0) \le \hat{\lambda}^*(\mathscr{L}_0) + 10^{-6}. \tag{2.11}$$

The a priori upper bound $\lambda^* \le 2$ follows because a $3x+1$ tree has at most two branches at each node, hence no more than 2^k nodes at depth k; hence all $\phi_k^m(y) \le 2^y$. We discard any system $\mathscr{L}_0$ having $\lambda^* < 1$.

It remains to choose $\mathscr{L}_0$ to maximize $\lambda^*(\mathscr{L}_0)$, over all systems $\mathscr{L}_0$ derivable from Krasikov's inequalities $\mathscr{I}_k$. This seems to be a difficult problem which very likely does not have a nice answer. The splitting procedure and the μ-truncation operation interact in a complicated fashion, as we now show.

3. Solving linear programs

We consider several splitting procedures based on heuristic splitting rules.

The simplest case to consider is *No Splitting*: directly μ-truncate the original inequalities (2.6). The resulting values $\hat{\lambda}^*(\mathscr{L}_0)$ appear in Table 3.1, up to $k = 9$.

TABLE 3.1. Krasikov lower bounds: No Splitting

k	λ_k^*	γ_k^*
2	1.330924	0.412428
3	1.455956	0.541967
4	1.506537	0.591237
5	1.523923	0.607790
6	1.543372	0.626086
7	1.553768	0.635771
8	1.561429	0.642867
9	1.568114	0.649031

This table gives also the corresponding value for γ in (1.3), which is computed for $\lambda = \hat{\lambda}^*(\mathscr{L}_0)$ by

$$\gamma = \log_2(\lambda) = \frac{\log \lambda}{\log 2}. \tag{3.1}$$

In Table 3.2 we give an optimal solution to the linear program (L_λ) for $k = 2$ and 3, for $\lambda = \hat{\lambda}_0^*(\mathscr{L}_0)$.

TABLE 3.2. Optimal L. P. solution: No Splitting $(k = 2, 3)$

$c_2^2 = 1.771362$	$c_3^2 = 2.516443$	$c_3^5 = 1.037679$	$c_3^8 = 1.489515$
$c_2^5 = 1.000000$	$c_3^{11} = 2.119809$	$c_3^{14} = 1.187108$	$c_3^{17} = 2.679816$
$c_2^8 = 1.564538$	$c_3^{20} = 2.199682$	$c_3^{23} = 1.000000$	$c_3^{26} = 1.961256$
depth $k = 2$	depth $k = 3$		

A theoretical upper bound for the value of $\lambda^*(\mathscr{L})$ attainable using the No Splitting rule on $\mathscr{I}_k$ for any k is $\lambda \doteq 1.596823$, the positive root of

$$1 = \lambda^{-2} + \tfrac{1}{3}(\lambda^{\alpha-2} + 1), \quad \alpha = \log_2 3. \tag{3.2}$$

To show this, note that (2.9b) implies that

$$c_{k-1}^{n} \le \tfrac{1}{3}\left(c_k^n + c_k^{n+3^{k-1}} + c_k^{n+2\cdot 3^{k-1}}\right). \tag{3.3}$$

The No Splitting inequalities (2.9a) are

$$c_k^m \le c_k^{4m} + c_{k-1}^{(4m-2)/3}\lambda^{\alpha-2} \quad \text{if } m \equiv 2 \pmod 9, \tag{3.4a}$$

$$c_k^m \le c_k^{4m} \quad \text{if } m \equiv 5 \pmod 9, \tag{3.4b}$$

$$c_k^m \le c_k^{4m} + c_{k-1}^{(2m-1)/3} \quad \text{if } m \equiv 8 \pmod 9. \tag{3.4c}$$

Let

$$\tilde{c}_k = \sum_{m\in[3^k]} c_k^m.$$

Then, adding up all the inequalities (3.4) over $\{m\colon m \in [3^k]\}$ and substituting (3.3) on the right side of the resulting inequality yields

$$\tilde{c}_k \le \tilde{c}_k\lambda^{-2} + \tfrac{1}{3}\tilde{c}_k(\lambda^{\alpha-2}+1).$$

Since $\tilde{c}_k > 0$, the upper bound (3.2) on λ follows. The $k=9$ bound in Table 3.1 is quite close to the upper bound $\lambda \doteq 1.596823$.

Next we consider the effect of splitting some terms in (2.6). We start with *Advanced Splitting*: if a term $c_j^n(y')$ is *advanced*, i.e., $y' > y$, then split it. Do this until no more splitting is possible, which occurs when all remaining advanced terms are c_1^2. Advanced Splitting appears reasonable because μ-truncation only weakens advanced terms. The resulting optimal values $\hat{\lambda}^*(\mathscr{L}_0)$ and exponents γ for Advanced Splitting appear in Table 3.3. It shows that splitting terms helps in getting better exponent bounds, and these bounds exceed the theoretical limit possible using the No Splitting rule.

TABLE 3.3. Krasikov lower bounds: Advanced Splitting

k	λ_k^*	γ_k^*
2	1.330924	0.412428
3	1.454167	0.540193
4	1.533045	0.616400
5	1.598484	0.676704
6	1.651222	0.723534
7	1.688407	0.755663
8	1.716310	0.779311
9	1.738468	0.797817

In Table 3.4 we give optimal solutions to the linear program (L_λ) for $k=2$ and 3 for $\lambda = \hat{\lambda}^*(\mathscr{L}_0)$.

TABLE 3.4. Optimal L. P. solution: Advanced Splitting $(k=2,3)$

$c_2^2 = 1.771362$	$c_3^2 = 2.665871$	$c_3^5 = 1.193517$	$c_3^8 = 1.809290$
$c_2^5 = 1.000000$	$c_3^{11} = 2.114611$	$c_3^{14} = 1.260696$	$c_3^{17} = 2.661316$
$c_2^8 = 1.564538$	$c_3^{20} = 2.523814$	$c_3^{23} = 1.000000$	$c_3^{26} = 2.061603$
depth $k=2$	depth $k=3$		

TABLE 3.5. Krasikov lower bounds: 8 (mod 9) Splitting

k	λ_k^*	γ_k^*
2	1.353400	0.436589
3	1.527333	0.611015
4	1.583694	0.663294
5	1.641865	0.715335
6	1.674310	0.743566
7	1.702186	0.767388
8	1.727744	0.788890
9	1.746603	0.804552

This splitting rule is not optimal. In the case $k = 2$ it fails to do as well as Krasikov's bound $\gamma = .43$, which he analytically derived from the $k = 2$ inequalities.

We next consider 8 (mod 9) *Splitting*: split every term $c_j^n(y)$ having $n \equiv 8 \pmod 9$, and also split any advanced term that can be split. For $k = 2$ this agrees with the splitting rule that Krasikov [3] implicitly used. The bounds we obtain for $\hat{\lambda}^*(\mathscr{L}_0)$ for 8 (mod 9) Splitting are given in Table 3.5; they are superior to the Advanced Splitting bounds.

In Table 3.6 we give the optimal solutions for $k = 2$ and 3 for the linear program (L_λ) for $\lambda = \hat{\lambda}^*(\mathscr{L}_0)$, and in Table 3.7 we give the value for $k = 4$. We notice a regularity in these optimal solutions, namely that all $c_j^n = 1$ when $n \equiv 8 \pmod 9$. It seems nonintuitive that splitting all 8 (mod 9) functions, even when they have a retarded argument, yields a larger value of $\lambda^*(\mathscr{L}_0)$ than that obtained by not splitting terms with a retarded argument, but it so proves.

TABLE 3.6. Optimal L. P. solution: 8 (mod 9) Splitting $(k = 2, 3)$

depth $k=2$	depth $k=3$		
$c_2^2 = 1.831692$	$c_3^2 = 2.701465$	$c_3^5 = 1.064138$	$c_3^8 = 1.000000$
$c_2^5 = 1.000000$	$c_3^{11} = 2.332747$	$c_3^{14} = 1.158062$	$c_3^{17} = 1.000000$
$c_2^8 = 1.000000$	$c_3^{20} = 2.482365$	$c_3^{23} = 1.000000$	$c_3^{26} = 1.000000$

We experimented with *Partially Optimized Greedy Splitting*: for each given inequality, compute which single terms will increase λ^* when split individually, then split all of these simultaneously for all inequalities, and iterate until either λ^* does not increase or else no more single terms improve λ^* when split. In fact this procedure continued to improve λ^* in smaller and smaller increments with no sign of terminating, so we halted the process when λ^* increased by less than .0001 in one step. This method improves on 8 (mod 9) Splitting for all $k \geq 4$ that we tried. However, the regularity in an optimal solution of (L_λ) that all $c_j^n = 1$ when $n \equiv 8 \pmod 9$ does not hold. For $k = 9$ it gave the exponent $\gamma = .810454$ when we halted it. The resulting linear program[4] gives a proof of Theorem 1.1.

[4]Implementations of Partially Optimized Greedy Splitting are sensitive to roundoff error in implementing the decision rule for which terms to split, so that our particular computation may not be easily reproducible. The splitting rule used in 8 (mod 9) Splitting avoids this issue, allowing the exponent .804 to be more easily checked.

TABLE 3.7. Optimal L. P. solution: 8 (mod 9) Splitting ($k=4$)

$c_4^2 = 3.179753$	$c_4^5 = 1.313029$	$c_4^8 = 1.000000$
$c_4^{29} = 2.790878$	$c_4^{32} = 1.434117$	$c_4^{35} = 1.000000$
$c_4^{56} = 3.354117$	$c_4^{59} = 1.237611$	$c_4^{62} = 1.000000$
$c_4^{11} = 2.588165$	$c_4^{14} = 1.337320$	$c_4^{17} = 1.000000$
$c_4^{38} = 2.647338$	$c_4^{41} = 1.267799$	$c_4^{44} = 1.000000$
$c_4^{65} = 2.508088$	$c_4^{68} = 1.112751$	$c_4^{71} = 1.000000$
$c_4^{20} = 3.293192$	$c_4^{23} = 1.031927$	$c_4^{26} = 1.000000$
$c_4^{47} = 3.596892$	$c_4^{50} = 1.055520$	$c_4^{53} = 1.000000$
$c_4^{74} = 3.104038$	$c_4^{77} = 1.000000$	$c_4^{80} = 1.000000$
2 (mod 9)	5 (mod 9)	8 (mod 9)

Finally we considered *Ultimate Splitting*: continue splitting until all terms are c_j^2 for various values of j. At each level there remain theree variables $c_{j+1}^2, c_{j+1}^{2+3^j}, c_{j+1}^{2+2\cdot 3^j}$, and the latter two are then eliminated by substituting the inequalities (2.3) for them. In this way we get a linear program (L_λ) that involves only the k variables $\{c_j^2 \colon 1 \le j \le k\}$. Table 3.8 gives the values of $\hat{\lambda}^*(\mathscr{L})$ and γ obtained, up to $k=6$. It seems evident that the exponents γ are converging to a limit below 1. This procedure splits an exponential number of times and, empirically, Table 3.8 indicates that this discards too much information to get $\gamma \to 1$ as $k \to \infty$.

TABLE 3.8. Krasikov lower bounds: Ultimate Splitting

k	λ_k^*	γ_k^*
2	1.353400	0.436589
3	1.527463	0.611137
4	1.530090	0.613616
5	1.530094	0.613620
6	1.530094	0.613620

4. KRASIKOV'S CONJECTURE

Krasikov [3] conjectures that, for any $\varepsilon > 0$, bounds of the form

$$\pi_a(x) \ge x^{1-\varepsilon} \quad \text{for } x \ge x_0(a) \tag{4.1}$$

can be derived from the Krasikov inequalities (mod 3^k), for sufficiently large k. This seems undoubtedly true. The result could potentially be rigorously proved by guessing a feasible solution to a suitable family of linear programs (L_λ) derived by the method of §3. To do this, one hopes to find systems of inequalities (2.7) such that (L_λ) has regularities in the optimal solutions of such linear programs.

TABLE 4.1. Krasikov lower bounds: No truncation of advanced terms

k	λ_k^*	γ_k^*
2	1.353400	0.436589
3	1.527595	0.611262
4	1.612286	0.689108
5	1.662760	0.733580
6	1.694451	0.760818
7	1.720191	0.782569
8	1.744963	0.803196
9	1.761532	0.816831

What is the limit of the linear programming method using just the Krasikov inequalities $\mathscr{I}_k$ of level k ? Consider the following linear program $(\mathrm{L}_k^{\mathrm{NT}})$ which does no truncation:

$$(\mathrm{L}_k^{\mathrm{NT}}) \left\{ \begin{array}{ll} \text{maximize } c_1^2 & \\ c_k^m \leq c_k^{4m} \lambda^{-2} + c_{k-1}^{(4m-2)/3} \lambda^{\alpha-2} & \text{if } m \equiv 2 \pmod 9, \\ c_k^m \leq c_k^{4m} \lambda^{-2} & \text{if } m \equiv 5 \pmod 9, \\ c_k^m \leq c_k^{4m} \lambda^{-2} + c_{k-1}^{(2m-1)/3} \lambda^{\alpha-1} & \text{if } m \equiv 8 \pmod 9, \\ c_j^n \leq c_{j+1}^{n+l\cdot 3^j}, \quad \text{all } n \in [3^j],\ l=0,1,2;\ 1 \leq j \leq k-1, & \\ c_n^j \geq 0, \quad \text{all } n \in [3^j],\ 1 \leq j \leq k, & \\ c_1^2 \leq 1. & \end{array} \right\}$$

Now maximize λ where $(\mathrm{L}_k^{\mathrm{NT}})$ has a feasible solution with $c_1^2 = 1$. Approximations $\hat{\lambda}^*(\mathrm{L}_k^{\mathrm{NT}})$ to the resulting quantities $\lambda^*(\mathrm{L}_k^{\mathrm{NT}})$ are given in Table 4.1, for $2 \leq k \leq 9$. The values in Table 4.1 exceed all the lower bounds in §3.

It seems intuitively reasonable that the bounds $\lambda^*(\mathrm{L}_k^{\mathrm{NT}})$ should be theoretical upper bounds for the optimal value of λ for any linear program (L_λ) obtained by splitting from (2.6) with fixed k, *with no truncation done.* So far, we cannot prove this, although it is true on all examples we computed. However, we also have examples showing that, for linear programs (L_λ) derived by splitting alone, with no truncation done, splitting a term can sometimes *increase* λ^*. For definiteness we state a weaker conjecture.

Conjecture 4.1. *For any linear program* (L_λ) *derived by repeated splitting from the Krasikov inequalities* $\mathscr{I}_k$ *(possibly using also* $\mathscr{I}_l$ *for* $1 \leq l < k$*) and then truncating, one has*

$$\lambda^*(\mathrm{L}_k^{\mathrm{NT}}) \geq \lambda^*(\mathrm{L}_\lambda).$$

We note that $\lambda^*(\mathrm{L}_k^{\mathrm{NT}})$ are *strictly increasing in* k. This property is easy to prove, for a feasible solution to $(\mathrm{L}_k^{\mathrm{NT}})$ can be constructed from a feasible solution to $(\mathrm{L}_{k-1}^{\mathrm{NT}})$ by letting

$$c_k^m = c_k^{m+3^{k-1}} = c_k^{m+2\cdot 3^{k-1}} := c_{k-1}^m$$

for all $m \in [3^k]$. Furthermore, this feasible solution can be shown to be not optimal[5] for $(\mathrm{L}_k^{\mathrm{NT}})$, hence

$$\lambda^*(\mathrm{L}_k^{\mathrm{NT}}) > \lambda^*(\mathrm{L}_{k-1}^{\mathrm{NT}}). \tag{4.2}$$

The nontruncated linear program $(\mathrm{L}_k^{\mathrm{NT}})$ is of a particularly simple form. If Conjecture 4.1 is true, then a necessary condition for Krasikov's Conjecture to hold is

$$\lambda^*(\mathrm{L}_k^{\mathrm{NT}}) \to 2 \quad \text{as } k \to \infty. \tag{4.3}$$

Now consider $(\mathrm{L}_k^{\mathrm{NT}})$ and introduce the *averaged variables*:

$$\bar{c}_{j,k} := \frac{1}{3^{j-1}} \sum_{m \in [3^j]} c_j^m. \tag{4.4}$$

Adding up all the $\mathscr{I}_k$-equations in $(\mathrm{L}_k^{\mathrm{NT}})$ yields

$$\bar{c}_{k,k} \le \bar{c}_{k,k}\lambda^{-2} + \tfrac{1}{3}\bar{c}_{k-1,k}(\lambda^{\alpha-1} + \lambda^{\alpha-2}). \tag{4.5}$$

At an optimal solution of $(\mathrm{L}_k^{\mathrm{NT}})$, all of the $m \equiv 2, 5, 8 \pmod 9$ inequalities in $(\mathrm{L}_k^{\mathrm{NT}})$ must hold with equality; hence (4.5) then holds with equality. Conversely, if (4.5) holds with equality for a feasible solution of $(\mathrm{L}_k^{\mathrm{NT}})$, so must all of the $m \equiv 2, 5, 8 \pmod 9$ inequalities in (L_k^{NT}). It follows that a necessary and sufficient condition for (4.3) to hold is that $(\mathrm{L}_k^{\mathrm{NT}})$ have optimal solutions satisfying

$$\frac{\bar{c}_{k-1,k}}{\bar{c}_{k,k}} \to 1 \quad \text{as } k \to \infty. \tag{4.6}$$

Can any of the splitting methods of §3 be used to prove Krasikov's Conjecture? By (3.2), the *No Splitting* inequalities are not strong enough to yield (4.1). A proof of (4.1) definitely requires that some kind of nontrivial splitting rule be used. Both Advanced Splitting and 8 (mod 9) Splitting empirically appear to retain enough information to derive (4.1). However, there is no obvious pattern in the optimal solutions to such (L_λ).

One can experiment with splitting rules that yield optimal solutions to (L_λ) having a nice structure. For example, 8 (mod 9) Splitting had optimal solution with $c_j^m = 1$ for all $m \equiv 8 \pmod 9$. We checked that splitting all terms that were 5 or 8 (mod 9) and forcing the solutions to have $c_j^m = 1$ for all $m \equiv 5$ or $8 \pmod 9$ by adding extra equality constraints led to little loss on the exponent: we obtained $\gamma = .788$ for $k = 9$, compared with .804 for 8 (mod 9) Splitting. In this approach splitting is essentially being used to eliminate variables in the linear program. The results for Ultimate Splitting demonstrate that there are limitations to the amount of elimination of variables allowed using this approach.

These experiments show that the bounds implied by systems of difference inequalities *for nondecreasing functions* have a surprising complexity. It seems a fruitful area for further study.

---

[5]This holds because some inequality in $\mathscr{I}_k$ is strict for these values. Otherwise, all $c_{k-1}^m = c_{k-1}^{m+3^{k-2}} = c_{k-1}^{m+2\cdot 3^{k-2}}$, and by downward induction on k, all c_k^m are equal, which contradicts optimality for $k = 2$.

Bibliography

1. D. Applegate and J. C. Lagarias, *Density bounds for the* $3x + 1$ *problem,* I. *Tree-search method*, Math. Comp. **64** (1995), 411–426.

R. E. Crandall, *On the* " $3x + 1$ " *problem*, Math. Comp. **32** (1978), 1281–1292.

3. I. Krasikov, *How many numbers satisfy the* $3x + 1$ *conjecture?*, Internat. J. Math. Math. Sci. **12** (1989), 791–796.

J. C. Lagarias, *The* $3x+1$ *problem and its generalizations*, Amer. Math. Monthly **92** (1985), 3–21.

5. J. W. Sander, *On the* $(3N + 1)$*-conjecture*, Acta Arith. **55** (1990), 241–248.

6. G. Wirsching, *An improved estimate concerning* $3N + 1$ *predecessor sets*, Acta Arith. **63** (1993), 205–210.

AT&T Bell Laboratories, Murray Hill, New Jersey 07974
E-mail address, D. Applegate: `david@research.att.com`
E-mail address, J. C. Lagarias: `jcl@research.att.com`