MATHEMATICS OF COMPUTATION
VOLUME 64, NUMBER 210
APRIL 1995, PAGES 501-513

THE K-OPERATOR AND THE GALERKIN METHOD FOR STRONGLY ELLIPTIC EQUATIONS ON SMOOTH CURVES: LOCAL ESTIMATES

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ABSTRACT. Superconvergence in the L^2-norm for the Galerkin approximation of the integral equation $Lu = f$ is studied, where L is a strongly elliptic pseudodifferential operator on a smooth, closed or open curve. Let u_h be the Galerkin approximation to u. By using the K-operator, an operator that averages the values of u_h, we will construct a better approximation than u_h itself. That better approximation is a legacy of the highest order of convergence in negative norms. For Symm's equation on a slit the same order of convergence can be recovered if the mesh is suitably graded.

1. INTRODUCTION

In this paper we shall study a way of increasing the order of *local convergence* in the L^2-norm for the Galerkin approximation to the solution of strongly elliptic pseudodifferential equations on a smooth, closed or open curve in $\mathbb{R}^2$. This kind of integral equation is of importance in solving interior or exterior boundary value problems of potential theory. The most common example is Symm's first-kind integral equation with logarithmic kernel (see [6, 7, 12]). Other applications are hypersingular integral equations and singular integral equations of Cauchy type, which occur, e.g., in elasticity (see [18]).

For general Petrov-Galerkin methods when smoothest splines are used as trial and test functions, local error estimates were proved in [11] for smooth closed curves and in [16] for smooth open curves. Consider, for example, Symm's equation. With piecewise constant functions used as trial and test functions, it was proved that the local L^2-error converges with order $O(h)$ in the case of smooth closed curves [11] and with order $O(h^{1/2})$ in the case of smooth open curves [16]. However, it is well known that the highest orders of global convergence achieved (in negative norms) are $O(h^3)$ for the closed smooth case [5] and $O(h)$ for the open smooth case [4, 13]. The purpose of this article is to construct, from the Galerkin solution, a better approximate solution which inherits the highest possible orders of global convergence to give best local convergence in the L^2-norm (e.g., in the example mentioned above, order $O(h^3)$ for the closed case and $O(h)$ for the open case can be achieved locally in the L^2-norm). That better approximation is constructed by averaging the values of the Galerkin solution, using the K-*operator*.

Received by the editor October 26, 1993.
1991 *Mathematics Subject Classification*. Primary 65N38; Secondary 41A15.

The K-operator was proved to be an effective postprocessing method in the finite element environment [1, 2, 14]. Its salient features are well elucidated in [17]. It produces an easily computed new approximant in the form of a convolution of the Galerkin solution with a special kind of spline with small support. That spline function is a combination of B-splines chosen so that it reproduces certain polynomials under convolution.

In the boundary integral equation literature, the K-operator method has been used for the qualocation approximation to strongly elliptic integral equations on smooth closed curves [15]. In that paper, the *global* L^2-error is investigated with an essential condition namely that the mesh be *uniform*. Uniformity is an indispensable requirement because the method relies on, besides other factors, the translational invariance of the trial space.

In the present article, we will only require that the mesh be uniform in some subarc of the curve (closed or open) and can be quasi-uniform (or even graded in the case of open curves) on the remainder of the curve. What we will examine in this situation is the *local* L^2-error. This study is particularly interesting when the curve is open. It is known that solutions of Symm's equation on the interval $[-1, 1]$ do not belong to $L^2(-1, 1)$ and that (because of this defect) the highest order of convergence (which occurs in the H^{-1}-norm) is no greater than $O(h)$. It is also known that by using mesh grading one can recover convergence of order $O(h^{3/2})$ in the energy norm (i.e., $H^{-1/2}$-norm) [10, 19]. We will deduce from that result (with a suitable mesh grading) a convergence of order $O(h^3)$ in the H^{-2}-norm, and therefore by using the K-operator will be able to obtain local convergence of the same $O(h^3)$ order in the L^2-norm.

It is worth noting that for Fredhom integral equations of the second kind, Chandler [3] has used a method anologous to the K-operator (which he referred to as 'superinterpolation') to obtain superconvergence. The mesh used there is uniform and only global errors were investigated.

This paper contains five sections. Section 2 gives some notations to be used and a review of the global and local properties of the Galerkin approximation. The definition and some properties of the K-operator are also recalled here. Its application to the case of smooth, closed curves can then be found in §3. Section 4 is devoted to a consideration of a special kind of equation on open curves: Symm's equation on a slit. Both quasi-uniform and graded meshes (graded at the ends of the slit) are discussed in this section. A numerical experiment is considered in §5.

2. Notations and some preliminaries

Notations introduced in this section are to be used in §3 for the study of the smooth, closed curve case. Let Γ be a plane smooth, closed curve given by a parametric representation $\gamma\colon [0, 1] \to \mathbb{R}^2$. In boundary element methods, Γ is the boundary of a given domain associated with some boundary value problem. Via the parametrization we have one-to-one correspondence between functions on Γ and 1-periodic functions. We thus restrict ourselves without loss of generality to equations of the form

$$(*) \qquad Lu = f,$$

where u and f are 1-periodic functions. Each periodic function u has a

Fourier expansion

$$u(x) \sim \sum_{n\in\mathbb{Z}} \hat{u}(n)e^{2\pi inx},$$

where the Fourier coefficients are given by the formula

$$\hat{u}(n) = \int_0^1 u(x)e^{-2\pi inx}\,dx,$$

provided u is in $L^1(0, 1)$. For $s \in \mathbb{R}$ we define the norm

$$\|u\|_s^2 = |\hat{u}(0)|^2 + \sum_{n\neq 0} |n|^{2s}|\hat{u}(n)|^2.$$

The Sobolev space H_p^s consists of all periodic distributions u for which the norm $\|u\|_s$ is finite. If I' is an open subset of $I = [0, 1]$, we also consider the space $H^s(I')$ with norm denoted by $\|\cdot\|_{s,I'}$ (see, e.g., [9]). The inner product in $L^2(I)$ is denoted by $\langle\cdot, \cdot\rangle$.

The operator L is assumed to be of the form

$$L = L_0 + L_1,$$

where the principal part L_0 is defined by

$$L_0u(x) := \sum_{n\in\mathbb{Z}} [n]_\alpha \hat{u}(n)e^{2\pi inx}, \tag{2.1}$$

with $\alpha \in \mathbb{R}$ and $[n]_\alpha$ defined either by

$$[n]_\alpha := \begin{cases} 1 & \text{for } n = 0, \\ |n|^{2\alpha} & \text{for } n \neq 0, \end{cases} \tag{2.1a}$$

or by

$$[n]_\alpha := \begin{cases} 1 & \text{for } n = 0, \\ (\operatorname{sign} n)|n|^{2\alpha} & \text{for } n \neq 0. \end{cases} \tag{2.1b}$$

In either case, L_0 is a pseudodifferential operator of order 2α, and is an isometry from H_p^s to $H_p^{s-2\alpha}$ for all $s \in \mathbb{R}$. The operator L_1 is assumed to be bounded from H_p^s to $H_p^{s-2\alpha+\beta}$ for all $s \in \mathbb{R}$ and some positive number β to be specified later. We then have $L_0^{-1}L_1$ bounded from H_p^s to $H_p^{s+\beta}$ and compact on H_p^s for all $s \in \mathbb{R}$. We also assume that L is 1-1, and thus by the Fredholm alternative

$$(I + L_0^{-1}L_1)^{-1} : H_p^s \longrightarrow H_p^s$$

is bounded for all $s \in \mathbb{R}$.

Since the boundary integral operators associated with regular elliptic boundary value problems on smooth closed curves are pseudodifferential operators of integer order (see [18, Theorem 2.1]), we assume for simplicity in the sequel that the operator L has integer order 2α, even though our results are still correct for any real α.

Let $I_0, \dots, I_4$, I_* and I^* be intervals such that $I_i \Subset I_{i+1} \Subset I_* \Subset I^* \Subset I = [0, 1]$, where $i = 0, \dots, 3$, and $X \Subset Y$ means the closure of X is contained in the interior of Y. Let $\Delta = \{x_k\}$, $x_k < x_{k+1}$ for $k \in \mathbb{Z}$, be a set of points on the real axis such that $x_{k+N} = x_k + 1$ for some $N \in \mathbb{N}$ and all $k \in \mathbb{Z}$. We consider for $r \geq 2$ the space S_h^r of 1-periodic splines $\varphi \in S_h^r$ such that φ is a polynomial of degree at most $r-1$ in every subinterval (x_k, x_{k+1}) having continuous derivatives up to order $r-2$. Here, h is the maximum value of the stepsizes. The space S_h^1 means the space of 1-periodic piecewise constant functions. The order r is assumed to be chosen so that the conformity condition $S_h^r \subset H_p^\alpha$ is satisfied, i.e., $\alpha < r-1/2$, and so that u, the exact solution to $(*)$, belongs to H_p^r. We shall also consider the following spaces:

$$S_h^{r,0}(I_i) = \{\varphi \in S_h^r : \text{ supp } (\varphi|_I) \subset I_i\},$$
$$S_h^r(I_i) = \{v \in H_p^\alpha : v|_{I_i} = \phi|_{I_i} \text{ for some } \phi \in S_h^r\}, \quad i = 0, \dots, 4.$$

We shall assume that the mesh is uniform in the interval I^*. Then there exists an $h_0 > 0$ such that for any $h \in (0, h_0]$, for $i = 0, \dots, 3$, and for $j = 1, \dots, r-2\alpha$

$$T_{\pm h}^j \varphi \in S_h^{r,0}(I_{i+1}) \quad \forall \varphi \in S_h^{r,0}(I_i), \tag{2.2}$$

where T_h denotes the translation operator $T_h v(x) = v(x+h)$ and $T_h^j v = T_h(T_h^{j-1} v)$ for $j = 2, \dots, r-2\alpha$.

Let $u_h \in S_h^r$ satisfy

$$\langle Lu_h, \phi\rangle = \langle Lu, \phi\rangle \quad \text{for any } \phi \in S_h^r. \tag{2.3}$$

It is known that [5] for $2\alpha - r \le t \le s \le r$ and $t < r - \frac{1}{2}$

$$\|u_h - u\|_t \le ch^{s-t}\|u\|_s. \tag{2.4}$$

In particular, in the L^2-norm we have

$$\|u_h - u\|_0 \le ch^r\|u\|_r,$$

whereas in the most extreme negative norm we can obtain

$$\|u_h - u\|_{2\alpha - r} \le ch^{2(r-\alpha)}\|u\|_r. \tag{2.5}$$

Instead of u_h, we will now define $K_h * u_h$ as an approximation to u (where $*$ denotes the convolution, and the function K_h is to be defined later) in such a way that if $2\alpha - r < 0$ and if u is smoother than previously assumed in some subinterval of I, i.e., $u \in H^{r_1}(I_*) \cap H_p^r$, for some $r_1 > r$ to be specified later, then

$$\|K_h * u_h - u\|_{0, I_0} \le ch^{2(r-\alpha)}(\|u\|_{r_1, I_*} + \|u\|_r).$$

The function K_h is defined as follows: Let $\psi^{(l)}$ be the B-spline of order l symmetric about 0, and with integer or half-integer knots, i.e.,

$$\psi^{(l)} = \chi * \chi * \cdots * \chi \quad \text{with } l-1 \text{ times of convolution, } l \ge 1,$$

where

$$\chi(x) = \begin{cases} 1 & \text{if } -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{2} & \text{if } x = \pm\frac{1}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Let q, l be arbitrary but fixed positive integers. For $0 < h < 1$ we define

$$K_h(x) = K_{h,l}^q(x) = \frac{1}{h} \sum_{j=-(q-1)}^{q-1} k_j \psi^{(l)}\left(\frac{x}{h} - j\right), \tag{2.6}$$

with k_j, for $j = -(q-1), \dots, q-1$, suitably chosen such that for any v

$$\|v - K_h * v\|_{0,I_0} \le ch^s \|v\|_{s,I_*} \quad \text{for } 0 \le s \le 2q. \tag{2.7}$$

For more details of the definition and properties of the K-operator, the reader is referred to [1, 2, 14, 15, 17].

In the next section we will need the following version of local estimates, which is slightly different from that of [11].

Theorem A. *Let $v \in H_p^\alpha \cap H^s(I_*)$, $\alpha \le s \le r$, and $v_h \in S_h^r(I_*)$ satisfy*

$$\langle L(v_h - v), \varphi\rangle = 0 \quad \textit{for all } \varphi \in S_h^{r,0}(I_*). \tag{2.8}$$

Then for any $\eta \le \alpha$, with η arbitrary but fixed, we have

$$\|v_h - v\|_{t,I_0} \le c(h^{s-t}\|v\|_{s,I_*} + h^\sigma \|v_h - v\|_\eta),$$

with $2\alpha - r \le t \le s \le r$, $t < r - \frac{1}{2}$ and

$$\sigma = \begin{cases} 0 & \textit{if } t \le \alpha, \\ \alpha - t & \textit{if } \alpha < t. \end{cases}$$

This result was proved in [16]. It is different from the local estimates of Saranen [11] in that now v_h needs to be a spline only on the subinterval I_*; therefore, the defining equation (2.8) is just a kind of *'interior equation'*. This modification is necessary in the application of the K-operator, since we will consider the translate of the Galerkin solution u_h, which is no longer a spline on the whole interval I, whereas it is a spline on the subinterval I_* if the mesh is uniform on I^*. Theorem A is useful not only in the application of the K-operator but also in the analysis of local estimates for integral equations on open curves [16].

Throughout this paper, c denotes a generic constant which can take different values at different occurrences.

3. The case of smooth closed curves

We shall in this section exploit the highest order of convergence in negative norm given by (2.5) to further develop the order in the L^2-norm. It is therefore sensible to consider only the case $r - 2\alpha > 0$.

Theorem 3.1. *Let the mesh Δ be uniform in the interval I^*. Assume that $u \in H^{r_1}(I_*) \cap H_p^r$ with $r_1 \ge 2(r-\alpha)$. Assume further that L_1 is bounded from H_p^s to $H_p^{s-2\alpha+\beta}$ for any $s \in \mathbb{R}$ and for some $\beta \ge r - 2\alpha$. If K_h is defined by* (2.6) *and* (2.7) *with $l \ge r - 2\alpha$ and $q \ge r - \alpha$, then there exists an $h_0 > 0$ such that for $h \in (0, h_0]$*

$$\|K_h * u_h - u\|_{0,I_0} \le ch^{2(r-\alpha)}(\|u\|_{r_1,I_*} + \|u\|_r). \tag{3.1}$$

Proof. By the triangle inequality we have

$$\|K_h * u_h - u\|_{0,I_0} \le \|K_h * u - u\|_{0,I_0} + \|K_h * (u_h - u)\|_{0,I_0} = \mathrm{I} + \mathrm{II}. \tag{3.2}$$

We will prove separately that I and II satisfy (3.1). The result for the first term comes easily from (2.7), the conditions $q \ge r - \alpha$ and $r_1 \ge 2(r-\alpha)$:

$$\mathrm{I} \le ch^{2(r-\alpha)}\|u\|_{2(r-\alpha),I_*} \le ch^{2(r-\alpha)}\|u\|_{r_1,I_*}. \tag{3.3}$$

For the second term, since $l \geq r - 2\alpha$, it is possible to differentiate K_h up to the order $r - 2\alpha$. Therefore, by using [2, Lemmas 2.2 and 5.3] we are able to go from the L^2-norm down to the $H^{2\alpha-r}$-norm and then obtain

$$\begin{aligned}
\mathrm{II} &\leq c \sum_{j=0}^{r-2\alpha} \|D^j K_h * (u_h - u)\|_{2\alpha-r,I_1} \leq c \sum_{j=0}^{r-2\alpha} \|\partial_h^j (u_h - u)\|_{2\alpha-r,I_1} \\
&\leq c \sum_{j=0}^{r-2\alpha} \|\partial_h^j (u_h - u + L_0^{-1} L_1 (u_h - u))\|_{2\alpha-r,I_1} \\
&\quad + c \sum_{j=0}^{r-2\alpha} \|\partial_h^j L_0^{-1} L_1 (u_h - u)\|_{2\alpha-r,I_1} \\
&= \mathrm{III} + \mathrm{IV}.
\end{aligned} \tag{3.4}$$

Here, ∂_h^j is the forward difference operator defined by $\partial_h^j v = (\frac{T_h - I}{h})^j v$. The term IV is easily estimated by noting that $\|\partial_h^j v\|_s \leq \|v\|_{s+j}$ for any v and that $L_0^{-1} L_1$ is bounded from H_p^s to $H_p^{s+\beta}$ for any $s \in \mathbb{R}$:

$$\mathrm{IV} \leq c\|L_0^{-1} L_1 (u_h - u)\|_0 \leq c\|u_h - u\|_{-\beta}.$$

Since $\beta \geq r - 2\alpha$ we then deduce from (2.5)

$$\mathrm{IV} \leq c\|u_h - u\|_{2\alpha-r} \leq ch^{2(r-\alpha)}\|u\|_r. \tag{3.5}$$

For the term III let us note that from (2.3) we have

$$\langle L_0(u_h - u + L_0^{-1} L_1 (u_h - u)), \varphi\rangle = 0 \quad \text{for any } \varphi \in S_h^{r,0}(I_2). \tag{3.6}$$

We shall prove that for any $j = 0, \ldots, r - 2\alpha$

$$\langle L_0 \partial_h^j (u_h - u + L_0^{-1} L_1 (u_h - u)), \varphi\rangle = 0 \quad \text{for any } \varphi \in S_h^{r,0}(I_1). \tag{3.7}$$

From the definitions of ∂_h^j, L_0 and T_h and from (3.6) there follows

$$\begin{aligned}
&\langle L_0 \partial_h^j (u_h - u + L_0^{-1} L_1 (u_h - u)), \varphi\rangle \\
&\quad = \frac{1}{h^j} \sum_{i=0}^{j} \binom{j}{i} \langle L_0 T_h^i (u_h - u + L_0^{-1} L_1 (u_h - u)), \varphi\rangle \\
&\quad = \frac{1}{h^j} \sum_{i=0}^{j} \binom{j}{i} \langle T_h^i L_0 (u_h - u + L_0^{-1} L_1 (u_h - u)), \varphi\rangle \\
&\quad = \frac{1}{h^j} \sum_{i=0}^{j} \binom{j}{i} \langle L_0 (u_h - u + L_0^{-1} L_1 (u_h - u)), T_{-h}^i \varphi\rangle.
\end{aligned}$$

Equation (3.7) now follows from (2.2) and (3.6). It follows in turn that for any $\zeta \in S_h^r$ we have

$$\langle L_0\{[\partial_h^j u_h - \zeta] - [\partial_h^j (u - L_0^{-1} L_1 (u_h - u)) - \zeta]\}, \varphi\rangle = 0 \quad \forall \varphi \in S_h^{r,0}(I_1).$$

This equation, together with the boundedness of $L_0^{-1} L_1$ from H_p^s to $H_p^{s+\beta}$ with $\beta > 0$ and the condition (2.2), assure that we can use Theorem A with

$v = \partial_h^j(u + L_0^{-1}L_1(u - u_h)) - \zeta$, $v_h = \partial_h^j u_h - \zeta$, $t = 2\alpha - r$ and $s = r - 1/2 - \epsilon$ with $\epsilon > 0$, to obtain for the jth term III_j of III

$$\begin{aligned}
\mathrm{III}_j &= \|[\partial_h^j u_h - \zeta] - [\partial_h^j(u - L_0^{-1}L_1(u_h - u)) - \zeta]\|_{2\alpha - r, I_1} \\
&\leq c\{h^{2(r-\alpha)-1/2-\epsilon}\|\partial_h^j(u - L_0^{-1}L_1(u_h - u)) - \zeta\|_{r-1/2-\epsilon, I_2} \\
&\qquad + \|\partial_h^j(u_h - u + L_0^{-1}L_1(u_h - u))\|_\eta\} \\
&\leq c\{h^{2(r-\alpha)-1/2-\epsilon}(\|\partial_h^j u - \zeta\|_{r-1/2-\epsilon, I_2} + \|\partial_h^j L_0^{-1}L_1(u_h - u)\|_{r-1/2-\epsilon, I_2}) \\
&\qquad + \|\partial_h^j(u_h - u + L_0^{-1}L_1(u_h - u))\|_\eta\} \\
&\leq c\{h^{2(r-\alpha)-1/2-\epsilon}(\|\omega_2\partial_h^j u - \zeta\|_{r-1/2-\epsilon} + \|L_0^{-1}L_1(u_h - u)\|_{r-1/2-\epsilon+j}) \\
&\qquad + \|u_h - u + L_0^{-1}L_1(u_h - u)\|_{\eta+j}\} \quad \text{for arbitrary } \zeta \in S_h^r,
\end{aligned}$$

where ω_2 is a cutoff function satisfying $\omega_2 \in C_0^\infty(I_3)$ and $\omega_2 \equiv 1$ on I_2. Therefore,

$$\begin{aligned}
\mathrm{III}_j \leq c\{&h^{2(r-\alpha)-1/2-\epsilon}(\|\omega_2\partial_h^j u - \zeta\|_{r-1/2-\epsilon} + \|u_h - u\|_{r-1/2-\epsilon}) \\
&+ \|u_h - u\|_{\eta+r-2\alpha}\} \quad \text{for any } j = 0, \dots, r - 2\alpha.
\end{aligned}$$

Here again we have used the boundedness of $L_0^{-1}L_1$ from H_p^s to $H_p^{s+\beta}$ for any $s \in \mathbb{R}$ with $\beta \geq r - 2\alpha$. Lemma 2.5 of [11] assures us that we can choose ζ so that

$$\begin{aligned}
\|\omega_2\partial_h^j u - \zeta\|_{r-1/2-\epsilon} &\leq ch^{1/2+\epsilon}\|\partial_h^j u\|_{r, I_3} \leq ch^{1/2+\epsilon}\|u\|_{r+j, I_3} \\
&\leq ch^{1/2+\epsilon}\|u\|_{2(r-\alpha), I_3} \quad \text{for } j = 0, \dots, r - 2\alpha.
\end{aligned}$$

The estimate (2.4) and the assumption $r_1 \geq 2(r - \alpha)$ then imply, for any $j = 0, \dots, r - 2\alpha$,

$$\mathrm{III}_j \leq c\{h^{2(r-\alpha)}(\|u\|_{r_1, I_*} + \|u\|_r) + \|u_h - u\|_{\eta+r-2\alpha}\}. \tag{3.8}$$

Summing up the result in (3.8), combining with inequalities (3.2)–(3.5), we infer

$$\|K_h * u_h - u\|_{0, I_0} \leq c\{h^{2(r-\alpha)}(\|u\|_{r_1, I_*} + \|u\|_r) + \|u_h - u\|_{\eta+r-2\alpha}\}.$$

Let $\eta = 4\alpha - 2r$. The desired result then follows from (2.5). □

Remark 1. If L is the operator associated with Symm's equation on a smooth closed curve, then L_0 is given by (2.1) and (2.1a) with $\alpha = -1/2$, and L_1 is bounded from H_p^s to H_p^t for any $s, t \in \mathbb{R}$ (see e.g. [12]). The condition of Theorem 3.1 on L_1 is obviously satisfied. If L is the operator associated with the Dirichlet boundary value problem for the Helmholtz equation, then L_1 is only bounded from H_p^s to H_p^{s+3} for any $s \in \mathbb{R}$ (see [8]). Nevertheless, if we use piecewise constant functions to approximate the solution u, the condition on L_1 is satisfied with $\beta = 2$ (since $2\alpha = -1$), and hence the K-operator method is applicable to this problem.

Remark 2. As can be seen from the proof of Theorem 3.1, the parameter l in the definition of the function K_h is determined by the order of the Sobolev norm which gives best convergence order for the Galerkin approximation (i.e., $2\alpha - r$ for the smooth case discussed above, or -1 for Symm's equation on a slit) whereas the parameter q is determined (via (2.7)) by the rate of convergence to be achieved for the K-operator.

4. The case of smooth open curves

In this section we will study a special case of equations on open curves: Symm's equation on the interval $\Gamma = [-1, 1]$. The method is, however, applicable to any pseudodifferential equations on any smooth open curve, provided that negative norm error estimates are available. The equation is defined as

$$V_\Gamma \psi(x) := -\frac{1}{\pi}\int_\Gamma \log|x-y|\psi(y)\,ds(y) = 2g(x) \quad \text{for } x \in \Gamma. \tag{4.1}$$

A physical interpretation of ψ is that it is the jump in the normal derivative of the solution of a Dirichlet problem for the Laplacian in $\mathbb{R}^2 \setminus \bar{\Gamma}$ with boundary values g on Γ and vanishing at infinity [13]. It is known that [13, Theorem 1.5] $V_\Gamma: \widetilde{H}^\tau(\Gamma) \to H^{\tau+1}(\Gamma)$ is a continuous bijective mapping for $-1 < \tau < 0$, where $\widetilde{H}^s(\Gamma)$ is the dual space of $H^{-s}(\Gamma)$ for $s < 0$.

Let $\Delta = \{x_i\}$, with $x_i < x_{i+1}$, $i = 1, \dots, N$, $N \in \mathbb{N}$, be a mesh on Γ. Let $S_h = S_h(\Delta)$ be the space of piecewise constants on Γ with breakpoints Δ, where $h = 1/N$. The Galerkin approximation for the solution of equation (4.1) is defined as: $\psi_h \in S_h$ such that

$$\langle V_\Gamma \psi_h, \varphi\rangle_{L^2(\Gamma)} = \langle g, \varphi\rangle_{L^2(\Gamma)} \quad \text{for any } \varphi \in S_h. \tag{4.2}$$

Quasi-uniform mesh. If the mesh Δ is quasi-uniform, the following global error estimates hold (see [4,13]):

$$\|\psi - \psi_h\|_{\widetilde{H}^t(\Gamma)} \le ch^{\tau-t}\|\psi\|_{\widetilde{H}^\tau(\Gamma)} \quad \text{for } -1 < t \le \tau < 0.$$

The condition $\tau < 0$ is necessary because in general $\psi \notin H^0(\Gamma) = L^2(\Gamma)$. Therefore, for any $\epsilon > 0$, provided that the boundary data are sufficiently smooth, we have

$$\|\psi - \psi_h\|_{\widetilde{H}^{-1}(\Gamma)} \le ch^{1-2\epsilon}\|\psi\|_{\widetilde{H}^{-\epsilon}(\Gamma)}. \tag{4.3}$$

As proved in [16], the local L^2-error converges as

$$\|\psi - \psi_h\|_{L^2(\Gamma_0)} \le ch^{1/2-\epsilon}(\|\psi\|_{H^{1/2}(\Gamma_*)} + \|\psi\|_{\widetilde{H}^{-\epsilon}(\Gamma)}) \tag{4.4}$$

for some $\epsilon > 0$, even though the global L^2-norm of $\psi - \psi_h$ is not defined. Here and in the sequel we use nested subintervals

$$\Gamma_i \Subset \Gamma_{i+1} \Subset \Gamma_* \Subset \Gamma^* \Subset \Gamma \quad \text{for } i = 0, 1, 2.$$

We are led by Remark 2 following Theorem 3.1 to use a K_h spline of order 1, i.e., $l = 1$, in the hope that $K_h * u_h$ is an approximation to ψ which gives local convergence of order $O(h)$ in the L^2-norm. That function K_h, defined by (2.6) and (2.7), is

$$K_h(x) = K^1_{h,1}(x) = \frac{1}{h}\chi\left(\frac{x}{h}\right).$$

To define the convolution, we extend each function v on Γ by 0 onto $\mathbb{R} \setminus \Gamma$ and denote it by $\tilde{v}$. The K-operator acting on ψ_h is now given by

$$K_h(\psi_h) = K_h * \tilde{\psi}_h. \tag{4.5}$$

Theorem 4.1. *Assume that the mesh is uniform on Γ^* and that $\psi \in H^{1-\epsilon}(\Gamma_*) \cap \widetilde{H}^{-\epsilon}(\Gamma)$ for some $\epsilon > 0$. Let $h_0 > 0$ be such that $T_{\pm h_0}(\Gamma_1) = \{x \pm h_0) : x \in \Gamma_1\} \subset \Gamma_*$. Then for $h \in (0, h_0]$*

$$\|K_h(\psi_h) - \psi\|_{L^2(\Gamma_0)} \leq ch^{1-2\epsilon}(\|\psi\|_{H^{1-\epsilon}(\Gamma_*)} + \|\psi\|_{\widetilde{H}^{-\epsilon}(\Gamma)}). \tag{4.6}$$

Proof. Following the line of the proof of Theorem 3.1, we shall prove (4.6) by using (2.7) and the fact that the forward difference of ψ_h approximates that of ψ (in some sense). However, now that $\tilde{\psi} \notin L^2(\mathbb{R})$ it is not useful to define $K_h(\psi)$ as in (4.5). We will make use of the function $\psi_* = \omega_* \tilde{\psi}$, where ω_* is a cutoff function satisfying

$$\omega_* \equiv 1 \text{ on } \Gamma_* \quad \text{and} \quad \omega_* \in C_0^\infty(\Gamma^*).$$

By noting that $\psi_* = \psi$ on Γ_0 and using the triangle inequality, we obtain

$$\|K_h(\psi_h) - \psi\|_{L^2(\Gamma_0)} \leq \|K_h * \psi_* - \psi_*\|_{L^2(\Gamma_0)} + \|K_h * (\tilde{\psi}_h - \psi_*)\|_{L^2(\Gamma_*)} = \mathrm{I} + \mathrm{II}. \tag{4.7}$$

That I satisfies (4.6) comes from the local smoothness of ψ and inequality (2.7). To obtain the same estimate for II, again we use [2, Lemmas 2.2 and 5.3], so obtaining, with (4.3),

$$\begin{aligned} \mathrm{II} &\leq c(\|\tilde{\psi}_h - \psi_*\|_{H^{-1}(\Gamma_1)} + \|\partial_h(\tilde{\psi}_h - \psi_*)\|_{H^{-1}(\Gamma_1)}) \\ &\leq c(\|\psi_h - \psi\|_{H^{-1}(\Gamma_1)} + \|\partial_h(\tilde{\psi}_h - \tilde{\psi})\|_{H^{-1}(\Gamma_1)}) \\ &\leq c(h^{1-2\epsilon}\|\psi\|_{\widetilde{H}^{-\epsilon}(\Gamma)} + \|\partial_h(\tilde{\psi}_h - \tilde{\psi})\|_{H^{-1}(\Gamma_1)}) \quad \text{for } 0 < h \leq h_0. \end{aligned} \tag{4.8}$$

In the second-to-the-last step we have used the assumption $T_{\pm h}(\Gamma_1) \subset \Gamma_*$ to obtain $\partial_h \psi_* = \partial_h \tilde{\psi}$. To estimate the last term of (4.8), let $\tilde{\Gamma}$ be a smooth closed curve containing the interval $[-2, 2]$ and define $V_{\tilde{\Gamma}}$ by (4.1) with Γ replaced by $\tilde{\Gamma}$. We extend $\tilde{\psi}_h - \tilde{\psi}$ and $\partial_h(\tilde{\psi}_h - \tilde{\psi})$ by 0 onto $\tilde{\Gamma} \setminus [-2, 2]$. Then by using the equation (4.2), and by noting that $\tilde{\psi}$ and $\tilde{\psi}_h$ vanish outside $\Gamma = [-1, 1]$, we obtain, for any $\varphi \in S_h^0(\Gamma_2)$ and $h \in (0, h_0]$ with $h < 1$,

(4.9)
$$\begin{aligned} \langle V_{\tilde{\Gamma}} \partial_h(\tilde{\psi}_h - \tilde{\psi}), \varphi\rangle_{L^2(\tilde{\Gamma})} &= \frac{1}{h}\{\langle V_{\tilde{\Gamma}} T_h(\tilde{\psi}_h - \tilde{\psi}), \varphi\rangle_{L^2(\tilde{\Gamma})} - \langle V_{\tilde{\Gamma}}(\tilde{\psi}_h - \tilde{\psi}), \varphi\rangle_{L^2(\tilde{\Gamma})}\} \\ &= -\frac{1}{\pi h}\int_{-2}^{2}\int_{-2}^{2} \log|x-y|(\tilde{\psi}_h - \tilde{\psi})(y+h)\varphi(x)\,dydx \\ &= -\frac{1}{\pi h}\int_{-2}^{2}\int_{-2}^{2} \log|x-y|(\tilde{\psi}_h - \tilde{\psi})(y)\varphi(x-h)\,dydx \\ &= \frac{1}{h}\langle V_{\tilde{\Gamma}}(\tilde{\psi}_h - \tilde{\psi}), T_{-h}\varphi\rangle_{L^2(\tilde{\Gamma})} \\ &= \frac{1}{h}\langle V_\Gamma(\psi - \psi_h), T_{-h}\varphi\rangle_{L^2(\Gamma)}. \end{aligned}$$

Since the mesh is uniform on Γ^*, we have $T_{-h}\varphi \in S_h^0(\Gamma_3) \subset S_h$ for any $\varphi \in S_h^0(\Gamma_2)$. Equations (4.2) and (4.9) then imply

$$\langle V_{\tilde{\Gamma}} \partial_h(\tilde{\psi}_h - \tilde{\psi}), \varphi\rangle_{L^2(\tilde{\Gamma})} = 0 \quad \text{for any } \varphi \in S_h^0(\Gamma_2).$$

The one-to-one correspondence between functions on $\tilde{\Gamma}$ and 1-periodic functions (via an appropriate parametrization of $\tilde{\Gamma}$) now allows us to use Theorem

A to obtain

$$\begin{aligned}\|\partial_h(\tilde{\psi}_h - \tilde{\psi})\|_{H^{-1}(\Gamma_1)} &\le c(h^{1-2\epsilon}\|\partial_h\tilde{\psi}\|_{H^{-\epsilon}(\Gamma_*)} + \|\partial_h(\tilde{\psi}_h - \tilde{\psi})\|_{H^{\eta}(\tilde{\Gamma})})\\ &\le c(h^{1-2\epsilon}\|\tilde{\psi}\|_{H^{1-\epsilon}(\Gamma_*)} + \|\tilde{\psi}_h - \tilde{\psi}\|_{H^{\eta+1}(\tilde{\Gamma})})\\ &\le c(h^{1-2\epsilon}\|\psi\|_{H^{1-\epsilon}(\Gamma_*)} + \|\psi_h - \psi\|_{\tilde{H}^{\eta+1}(\Gamma)}).\end{aligned}$$

Choosing $\eta = -2$ and using (4.3) again, we get the desired estimate and hence the theorem is proved. □

Mesh grading. To recover the order $O(h^3)$ of the smooth case, mesh grading is necessary. We note that in the proof of Theorem A, the mesh is required to be quasi-uniform only in some subinterval of I (e.g., on I^*). Hence a consideration of mesh grading on $\Gamma \setminus \Gamma^*$ is permissible. For example, in the case $\Gamma = [-1, 1]$, we can define a mesh which is uniform on $[-3/4, 3/4]$ and graded on the other subintervals. More precisely, we can define $\Delta = \{x_k : k = 0, \dots, N\}$ as

$$x_k = \begin{cases} -1 + 4^{\varrho-1}(kh)^{\varrho} & \text{if } 0 \le k \le N/8 - 1, \\ -1 + kh & \text{if } N/8 \le k \le 7N/8 - 1, \\ 1 - 4^{\varrho-1}(2 - kh)^{\varrho} & \text{if } 7N/8 \le k \le N, \end{cases}$$

where $N = 8n$, $n \in \mathbb{N}$, $h = 2/N$ and $\varrho \ge 1$. Note that $x_{N/8} = -3/4$, $x_{7N/8} = 3/4$ and that the mesh is uniform when $\varrho = 1$.

For the application of the K-operator in this case, the availability of the error estimate in the deepest negative norm is necessary. By slightly modifying a result of von Petersdorff [10, Satz 3.7] and using the *a priori* estimates given in [13, Theorem 2.3], we obtain

Lemma 4.2. *Let $\epsilon > 0$ be given. Then*

$$\|\psi - \psi_h\|_{\tilde{H}^{-1/2}(\Gamma)} \le \begin{cases} ch^{\varrho/2}\|g\|_{H^{(\varrho+1)/2}(\Gamma)} & \text{if } 1 < \varrho < 3, \\ ch^{3/2}(\log \frac{1}{h})^{1/2}\|g\|_{H^{2+\epsilon}(\Gamma)} & \text{if } \varrho = 3, \\ ch^{3/2}\|g\|_{H^{2+\epsilon}(\Gamma)} & \text{if } \varrho > 3. \end{cases}$$

Following the line of reasoning in [4] in using Nitsche's trick, we can prove the following:

Lemma 4.3. *Let $\epsilon > 0$ be given. Then for $1/2 \le s \le \min(\frac{\varrho+1}{2}, 2 + \epsilon)$ there holds*

$$\|\psi - \psi_h\|_{\tilde{H}^{-s}(\Gamma)} \le \begin{cases} ch^{s+(\varrho-1)/2} & \text{if } 1 \le \varrho < 3, \\ ch^{3(1+\theta)/2}(\log \frac{1}{h})^{(1+\theta)/2} & \text{if } \varrho = 3, \\ ch^{3(1+\theta)/2} & \text{if } \varrho > 3, \end{cases}$$

where $\theta = 2(s - \frac{1}{2})/(3 + 2\epsilon)$. In particular, when $\varrho \ge 3$ we have

$$\|\psi - \psi_h\|_{\tilde{H}^{-2}(\Gamma)} \le ch^{3-\epsilon}. \tag{4.10}$$

Following the Remark 2 coming after Theorem 3.1, we will now use the spline $K_h = K^q_{h,l}$ with $l = 2$ and $q = 2$ to establish the new approximant. That function K_h has the form (see [1, 2, 15])

$$K_h(x) = \frac{1}{h}\left\{-\frac{1}{12}\{\psi^{(2)}(x/h-1)+\psi^{(2)}(x/h+1)\}+\frac{7}{6}\psi^{(2)}(x/h)\right\}. \tag{4.11}$$

Replacing (4.3) by (4.10) and using the same argument as in the proof of Theorem 4.1, we can prove

Theorem 4.4. *Let* $\varrho \geq 3$. *Assume that* $\psi \in H^{3-\epsilon}(\Gamma_*)\cap \widetilde{H}^{-\epsilon}(\Gamma)$ *for some* $\epsilon > 0$. *Let* $h_0 > 0$ *be such that* $T_{\pm 2h_0}(\Gamma_1) \subset \Gamma_*$. *Then for* $h \in (0, h_0]$

$$\|K_h(\psi_h) - \psi\|_{L^2(\Gamma_0)} = O(h^{3-\epsilon}).$$

5. Numerical experiments

Experiment 1. We tested the K-operator method when L is the logarithmic-kernel integral operator arising from the boundary value problem

$$\begin{aligned} \Delta U &= 0 \quad \text{in } \Omega, \\ U &= F \quad \text{on } \Gamma, \end{aligned} \tag{5.1}$$

where $\Gamma = \partial\Omega$ is the ellipse $16x^2 + 64y^2 = 1$ and $F(t) = t_1 + t_2$ with $t = (t_1, t_2)$. It is known that by using the direct method the problem (5.1) can be reformulated as

$$-\frac{1}{\pi}\int_\Gamma \log|t-s|z(s)\,dl_s = F(t) - \frac{1}{\pi}\int_\Gamma \left(\frac{\partial}{\partial n_s}\log|t-s|\right)F(s)\,dl_s, \quad t\in\Gamma, \tag{5.2}$$

where $z = \partial U/\partial n$ is the directional derivative of U with respect to the outward normal vector n. Using a parametrization $\gamma\colon [0, 1] \to \mathbb{R}^2$ for the curve Γ, we can rewrite (5.2) in the form

$$Lu(x) = f(x) \quad \text{for } x \in [0, 1], \tag{5.3}$$

where

$$\begin{aligned} u(x) &= (2\pi)^{-1} z[\gamma(x)]|\gamma'(x)|, \\ Lu(x) &= -2\int_0^1 \log(|\gamma(x)-\gamma(y)|)u(y)\,dy, \end{aligned} \tag{5.4}$$

and where f is obtained from the right side of (5.2) by using the parametrization. It is known that (see, e.g., [12]) $L = L_0 + L_1$ with L_0 expressible as

$$L_0u(x) = \hat{u}(0) + \sum_{n\neq 0}\frac{1}{|n|}\hat{u}(n)e^{2\pi i n x},$$

and with L_1 bounded from H_p^s to H_p^t for any $s, t \in \mathbb{R}$.

There being no need to consider a nonuniform mesh, we used a uniform mesh and investigated the global errors in this example. We chose piecewise constants as test and trial functions in the Galerkin approximation for (5.3), and used $K_h = K_{h,2}^2$ given by (4.11) to average the values of u_h (see Theorem 3.1 and Remark 2). The empirical orders of convergence obtained for $\|u-u_h\|_0$ and for $\|K_h * u_h - u\|_0$ were $O(h)$ and $O(h^3)$, respectively (see Table 1 on next page), which match the analysis.

TABLE 1. Errors and empirical orders of convergence for Experiment 1

N	$\|u_h - u\|_0$		$\|K_h * u_h - u\|_0$	
8	9.04e-02		4.49e-03	
16	4.49e-02	1.00	4.51e-04	3.31
32	2.24e-02	1.00	4.98e-05	3.18
64	1.12e-02	1.00	5.83e-06	3.09
128	5.60e-03	1.00	7.05e-07	3.05

TABLE 2. Errors and empirical orders of convergence for Experiment 2

	$\|e\|_{L^2(\Gamma_0)}$		$\|E\|_{L^2(\Gamma_0)}$							
N	$\varrho = 1$		$\varrho = 1$		$\varrho = 3$		$\varrho = 3.2$		$\varrho = 3.5$	
8	1.1e-1		1.7e-2		4.1e-2		4.1e-2		4.1e-2	
16	5.4e-2	1.00	5.8e-3	1.58	3.0e-3	3.77	3.1e-3	3.70	3.4e-3	3.60
32	2.7e-2	1.00	2.8e-3	1.06	8.2e-5	5.18	4.9e-5	5.98	7.9e-5	5.40
64	1.4e-2	1.00	1.3e-3	1.06	1.5e-5	2.42	9.0e-6	2.46	1.4e-5	2.51
128	6.8e-3	1.00	6.5e-4	1.03	2.8e-6	2.44	1.5e-6	2.61	1.6e-6	3.15
256	3.4e-3	1.00	3.2e-4	1.02	4.8e-7	2.57	2.4e-7	2.61	1.7e-7	3.18
512	1.7e-3	1.00	1.6e-4	1.01	7.5e-8	2.67	3.7e-8	2.70	2.1e-8	3.02

Experiment 2. We considered in this experiment the weakly singular integral equation (4.1) with $g(x) = x$ and tested the local convergence on $\Gamma_0 = (-1/2, 1/2)$ for the errors $e = \psi - \psi_h$ and $E = \psi - K_h(\psi_h)$ with various values of ϱ. When $\varrho = 1$ (uniform mesh) we achieved convergence of apparent order $O(h)$ for both errors (see Table 2), instead of the predicted orders of $O(h^{1/2})$ for $\|e\|_{L^2(\Gamma_0)}$ and $O(h)$ for $\|E\|_{L^2(\Gamma_0)}$. However, one can see that $\|E\|_{L^2(\Gamma_0)}$ is smaller than $\|e\|_{L^2(\Gamma_0)}$ by an order of magnitude. When $\varrho = 3$ or $\varrho = 3.2$ almost nothing changed for $\|e\|_{L^2(\Gamma_0)}$ whereas the empirical rate of convergence for $\|E\|_{L^2(\Gamma_0)}$ is slowly asymptotic to $O(h^3)$. When ϱ is increased to 3.5 the asymptotic $O(h^3)$ order is obtained much more quickly (see Table 2).

ACKNOWLEDGMENT

I wish to thank Professor Lars B. Wahlbin for instigating the study of local error estimates for the K-operator. I also would like to thank Professor Ian H. Sloan and Dr. William McLean for their helpful comments and continual encouragement.

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