

AN ANALYTIC METHOD FOR CONVERGENCE ACCELERATION OF CERTAIN HYPERGEOMETRIC SERIES

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ABSTRACT. A method is presented for convergence acceleration of the generalized hypergeometric series ${}_3F_2$ with the argument ± 1 , using analytic properties of their terms. Iterated transformation of the series is performed analytically, which results in obtaining new fast converging expansions for some special functions and mathematical constants.

1. INTRODUCTION

We consider convergence acceleration of some generalized hypergeometric series of the type

$$(1.1) \quad {}_3F_2(a, b, c; d, e; \varepsilon) \equiv {}_3F_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} \middle| \varepsilon \right) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n}{n! (d)_n (e)_n} \varepsilon^n,$$

where $\varepsilon = \pm 1$, and $(\alpha)_n := \alpha(\alpha + 1) \cdots (\alpha + n - 1)$ is Pochhammer's symbol. The function ${}_3F_2(a, b, c; d, e; 1)$ is one of the fundamental functions of applied mathematics. For special values of its parameters this function can be represented as a ratio of finite products of Gamma function evaluations. In the most general formulae of this type, due to Dixon, Whipple and Watson, two of the parameters a, b, c, d, e are linear combinations of the others [4, vol. 1, § 4.4, (5)–(7); 8, vol. 1, § 3.13, (6)–(8)]. Wimp [14] has shown that these formulae *cannot* be generalized to the case where *four* parameters are not interrelated. (Recently, Zeilberger [16] gave a short proof of Wimp's theorem.) As for the case of $\varepsilon = -1$, the most general results are: Kummer's formula for ${}_2F_1(a, b; 1 + a - b; -1)$ ([4, vol. 1, § 2.8, (47)]) and Bailey's formula for ${}_3F_2(a, b, 1 + \frac{1}{2}a; 1 + a - b, \frac{1}{2}a; -1)$ (cf. [4, vol. 1, § 4.5, (4)]).

Let us recall that many elementary functions ($\tan, \tanh, \cot, \coth, \csc, \csc^2, \dots$), as well as some special functions ($\psi, \psi', \text{Beta}, \dots$) and mathematical constants ($\pi, \pi^2, \text{Catalan's constant } G, \text{ the lemniscate constant } A, \log 2, \dots$) can be expressed in terms of the series of type (1.1).

For almost any set of the parameters the series (1.1) is an unsatisfactory way of computing ${}_3F_2(\varepsilon)$. Wimp [15, § 9.1] (see also [6]) has proposed an algorithm

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for computing ${}_3F_2(a, b, c; d, e; 1)$ numerically, which is based on a three-term recurrence relation for ${}_3F_2(a, b, c; d+m, e; 1)$ ($m = 0, 1, \dots$). In this paper we propose an *analytic method* for the acceleration of convergence of certain series of type (1.1). The term "analytic" means that

1° we use explicitly not only values of partial sums of the series (1.1) but also analytic properties of its terms,

2° we obtain an analytic form for the transformed series.

The effect of convergence acceleration is remarkable. The transformed series $\sum d_n$ is such that $d_{n+1}/d_n \rightarrow \lambda_\varepsilon$, where $\lambda_1 = \frac{1}{9}$, $\lambda_{-1} = -\frac{1}{8}$.

To give the main idea of the proposed approach, let us consider a general series

$$(1.2) \quad \sum_{n=0}^{\infty} \varepsilon^n a_n \quad (\varepsilon = \pm 1)$$

with the partial sums

$$s_n^{(0)} := \sum_{j=0}^{n-1} \varepsilon^j a_j \quad (n = 0, 1, \dots).$$

The series (1.2) is identical to

$$(1.3) \quad s_0^{(0)} + \sum_{n=0}^{\infty} \Delta s_n^{(0)}$$

(throughout the paper the forward difference operator Δ always acts on the subscript n).

A vast class of methods of accelerating the convergence of the series (1.2) can be considered as recurrent transformation of the sequence $\{s_n^{(0)}\}$ of partial sums of (1.2) to certain suitably chosen sequences $\{s_n^{(k)}\}$ ($k = 1, 2, \dots$), i.e., transformation of (1.3) to

$$(1.4) \quad s_0^{(k)} + \sum_{n=0}^{\infty} \Delta s_n^{(k)},$$

according to the formula

$$(1.5) \quad s_n^{(k)} := s_n^{(k-1)} + C_n^{(k-1)} \Delta s_n^{(k-1)} \quad (n = 0, 1, \dots, k = 1, 2, \dots).$$

The parameters $C_n^{(k-1)}$ should be such that for any $k > 0$, $C_n^{(k-1)} \Delta s_n^{(k-1)} \rightarrow 0$, when $n \rightarrow \infty$. In this case, all the series (1.4) and the series (1.2) have the same sum which we denote by s .

Of course, transforming the series (1.2) only makes sense when the rate of convergence of the sequence $\{\Delta s_n^{(k)}\}$ to 0 grows with k . Notice that the exact value of s can be obtained as a result of performing the first transformation $\{s_n^{(0)}\} \Rightarrow \{s_n^{(1)}\}$. Indeed, if we define

$$(1.6) \quad C_n^{(0)} := \frac{s - s_n^{(0)}}{\Delta s_n^{(0)}} = \frac{1}{\varepsilon^n a_n} \sum_{j=n}^{\infty} \varepsilon^j a_j \quad (n = 0, 1, \dots),$$

then $s_n^{(1)} = s$ for every n . A useful closed-form expression for the optimum factors $C_n^{(0)}$ can be given for trivial series only. In general, we are looking for

factors which accelerate the convergence of the series. For this purpose, we may use the following relation implied by (1.6):

$$(1.7) \quad \frac{C_n^{(0)} - 1}{C_{n+1}^{(0)}} = \varepsilon \frac{a_{n+1}}{a_n} \quad (n = 0, 1, \dots).$$

Let us assume that

$$(1.8) \quad \frac{a_{n+1}}{a_n} \sim 1 + \rho_1 n^{-1} + \rho_2 n^{-2} + \dots \quad (n \rightarrow \infty)$$

(this is the case for the series (1.1)). By virtue of a theorem given by Sidi (see [11, Theorem 6.1], or [12, Theorem 2.1]), $C_n^{(0)}$ has an asymptotic expansion of the form

$$(1.9) \quad C_n^{(0)} \sim n^\tau \sum_{k=0}^{\infty} c_k n^{-k} \quad (n \rightarrow \infty)$$

with $\tau = (1 + \varepsilon)/2$.

Methods presented in this paper are based on the following facts:

(i) Coefficients of the asymptotic expansion (1.9) of the optimum accelerating factor $C_n^{(0)}$ can be effectively calculated using (1.2) and (1.7).

(ii) With methods of Padé approximation applied to this expansion, good rational approximants to this factor can be effectively constructed. (In the simplest case these approximants are identical to those proposed by Knopp [7, § 35A, (153)].)

(iii) For some series (1.2), similar factors $C_n^{(k)}$ for $k > 0$ can be found provided the approximate expressions for $C_n^{(0)}$ are sufficiently simple. This gives a two-dimensional *table S* of approximants $s_n^{(k)}$ of the sum of the series.

(iv) To any *path* in the table *S* there corresponds a series $\sum d_n$, where $d_0, \Delta d_n$ are entries lying on the path, and analytic expressions for d_n are known. It is possible to find a path for which the corresponding series converges to s significantly faster than any series (1.4).

In this paper we show that the process (i)–(iv) can be actually performed for the series ${}_3F_2(1, b, c; d, e; 1)$ (see Theorem 2.2 in §2) and ${}_3F_2(1, b, c; d, e; -1)$ with $b + d = c + e$ (see Theorem 3.2 in §3). Some important particular cases are considered in the numerous corollaries and examples, in §2 for $\varepsilon = 1$, and in §3 for $\varepsilon = -1$. Obviously, the analytic convergence acceleration based on Theorems 2.2 and 3.2 can compete only in a certain limited class of series with purely numerical methods such as Brezinski's Θ -algorithm, ε -algorithm, or Levin's transforms (see [3, §§ 2.3, 2.7, 2.9]).

As a by-product, some identities for contiguous series (1.1) with $\varepsilon = 1$ are obtained (see Theorem 2.3).

2. SERIES FOR $\varepsilon = 1$

Assuming that $\varepsilon = 1$, we consider first how the identity (1.7), i.e.,

$$(2.1) \quad C_n - 1 = \frac{a_{n+1}}{a_n} C_{n+1}$$

(for simplicity, we drop the superscript in the symbols $C_n^{(0)}, C_{n+1}^{(0)}$), can be exploited. A general closed-form expression for C_n , applicable to any series

(1.2) for $\varepsilon = 1$, remains unknown. In the hypergeometric case (1.1) we have

$$C_n = {}_4F_3 \left(\begin{matrix} 1, n+a, n+b, n+c \\ n+1, n+d, n+e \end{matrix} \middle| 1 \right),$$

which does not seem to suggest any practical way of computing this factor.

Let C_n satisfy (2.1), and let (1.8) hold. Then a theorem due to Sidi (see [11, Theorem 6.1], or [12, Theorem 2.1]) applies, and we obtain a Poincaré-type asymptotic expansion

$$(2.2) \quad C_n \sim \sum_{k=-1}^{\infty} c_{k+1} n^{-k}.$$

The coefficients c_k can be computed relatively easily by use of the linear relation (2.1). Thereupon, we can introduce the formal power series

$$(2.3) \quad F(x) := \sum_{k=0}^{\infty} c_k x^k.$$

It is known that Padé approximation often gives good rational approximants to such a series, even if it is divergent. As we have $C_n = nF(n^{-1})$, any Padé approximant $[l/m]_F(x)$ gives an approximate value $C_n^{(0)} := n[l/m]_F(n^{-1})$ of the exact accelerating factor C_n . It can be checked that the equation $[l/m]_F(x) - F(x) = O(x^{l+m+1})$ implies the relation $\Delta s_n^{(1)} = \Delta s_n^{(0)} O(n^{-(l+m+1)})$, showing how much faster the transformed series (1.4) for $k = 1$ converges compared with the convergence of the original series (1.3). The most natural are perhaps approximants of the type $[m+1/m]$, as this is the only case where $C_n^{(0)}$ defined above is a rational function with respect to n , which has all the coefficients of the numerator and the denominator polynomials different from zero, in general.

The simplest transformation of the series (1.3) is based on using the approximant $[1/0]_F$ to the auxiliary series (2.3). In this case, we use the following

Lemma 2.1. *Let*

$$(2.4) \quad \begin{aligned} s_n^{(0)} &:= \sum_{j=0}^{n-1} a_j, \\ s_n^{(1)} &:= s_n^{(0)} + (\beta n + \gamma) \Delta s_n^{(0)}, \end{aligned}$$

where

$$(2.5) \quad \beta = -\frac{1}{\rho_1 + 1}, \quad \gamma = \frac{\rho_1 + \rho_2}{(\rho_1)_2},$$

and ρ_1, ρ_2 are the coefficients of the expansion (1.8). (We assume that $\rho_1 \neq 0, -1$). Then

$$\Delta s_n^{(1)} = \Delta s_n^{(0)} O(n^{-2}).$$

Proof. As

$$\Delta s_n^{(1)} = \Delta s_n^{(0)} + [\beta(n+1) + \gamma] \Delta s_{n+1}^{(0)} - (\beta n + \gamma) \Delta s_n^{(0)},$$

we have, in view of (1.8),

$$\begin{aligned} \Delta s_n^{(1)} &= \Delta s_n^{(0)} \left((\beta n + \beta + \gamma) \frac{\Delta s_{n+1}^{(0)}}{\Delta s_n^{(0)}} - \beta n - \gamma + 1 \right) \\ &\sim \Delta s_n^{(0)} [(\beta n + \beta + \gamma)(1 + \rho_1 n^{-1} + \rho_2 n^{-2} + \dots) - \beta n - \gamma + 1]. \end{aligned}$$

For β, γ given in (2.5) the expression in brackets is equal to $\delta_2 n^{-2} + \delta_3 n^{-3} + \dots = O(n^{-2})$. \square

Remark 1. Knopp [7, § 35A, (153)] observed that for $a_n = 1/(n + 1)^2$ it is useful to apply the Kummer transformation, i.e., to replace (1.2) (with $\varepsilon = 1$) by the series

$$qb + \sum_{n=0}^{\infty} \left(1 - q \frac{b_n}{a_n} \right) a_n \quad \left(b := \sum_{n=0}^{\infty} b_n, \quad q := \lim_{n \rightarrow \infty} \frac{a_n}{b_n} \right),$$

taking $b_n := (n + y)a_n - (n + y + 1)a_{n+1}$ with a suitably chosen y . The above lemma generalizes this observation in a substantial way.

Remark 2. In general, the transformation (2.4) may be iterated. Indeed,

$$\begin{aligned} \frac{\Delta s_{n+1}^{(1)}}{\Delta s_n^{(1)}} &\sim (1 + \rho_1 n^{-1} + \rho_2 n^{-2} + \dots) \frac{\delta_2(n+1)^{-2} + \delta_3(n+1)^{-3} + \dots}{\delta_2 n^{-2} + \delta_3 n^{-3} + \dots} \\ &\sim 1 + \bar{\rho}_1 n^{-1} + \bar{\rho}_2 n^{-2} + \dots. \end{aligned}$$

Now, the sequence $\{s_n^{(2)}\}$ can be constructed in an analogous way, provided $\bar{\rho}_1 \neq 0, -1$.

The following theorem shows that the transformation (2.4) can be effectively iterated for a certain four-parameter family of hypergeometric series.

Theorem 2.2. *The series*

$$(2.6) \quad {}_3F_2 \left(\begin{matrix} 1, b, c \\ d, e \end{matrix} \middle| 1 \right) = \sum_{n=0}^{\infty} \frac{(b)_n (c)_n}{(d)_n (e)_n} \quad (d, e \neq 0, -1, -2, \dots, \Re(t) > 1),$$

where $t := d + e - b - c$, can be transformed according to the following formulae applied for $n = 0, 1, \dots$:

$$(2.7) \quad s_n^{(0)} := \sum_{j=0}^{n-1} \frac{(b)_j (c)_j}{(d)_j (e)_j},$$

$$(2.8) \quad s_n^{(k)} := s_n^{(k-1)} + (\beta_k n + \gamma_k) \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

where

$$(2.9) \quad \beta_k := \frac{1}{2k + t - 3}, \quad \gamma_k := 1 + \frac{b + c}{2k + t - 3} + \frac{bc - (k + d - 1)(k + e - 1)}{(2k + t - 3)_2},$$

$$(2.10) \quad \Delta s_n^{(k)} = \frac{(d - b)_k (d - c)_k (e - b)_k (e - c)_k (b)_n (c)_n}{(t - 1)_{2k} (d)_{n+k} (e)_{n+k}} \quad (k = 0, 1, \dots).$$

The quantities $\Delta s_n^{(k)}$ are such that $\Delta s_n^{(k)} = O(n^{-2k-t})$ ($n \rightarrow \infty$).

If $d - e = b - c$, the formula (2.8) simplifies to

$$s_n^{(k)} := s_n^{(k-1)} + \frac{2n + 3k - c + d + 2e - 5}{2(2k + t - 3)} \Delta s_n^{(k-1)}.$$

Proof. Let $k = 0$. Formula (2.10) then agrees with (2.7). Assuming that $k > 0$, we obtain from (2.8)

$$(2.11) \quad \Delta s_n^{(k)} = [\beta_k(n+1) + \gamma_k] \Delta s_{n+1}^{(k-1)} - (\beta_k n + \gamma_k - 1) \Delta s_n^{(k-1)} = \Delta s_n^{(k-1)} g_n^{(k-1)},$$

where

$$g_n^{(k-1)} := [\beta_k(n+1) + \gamma_k] \sigma_n^{(k-1)} - (\beta_k n + \gamma_k - 1),$$

$$\sigma_n^{(k-1)} := \frac{\Delta s_{n+1}^{(k-1)}}{\Delta s_n^{(k-1)}} = 1 + \rho_1^{(k-1)} n^{-1} + \rho_2^{(k-1)} n^{-2} + \dots$$

According to (2.5) (Lemma 2.1), let

$$(2.12) \quad \beta_k := -\frac{1}{\rho_1^{(k-1)} + 1}, \quad \gamma_k := \frac{\rho_1^{(k-1)} + \rho_2^{(k-1)}}{(\rho_1^{(k-1)})^2}.$$

Then $g_n^{(k-1)} = O(n^{-2})$. Note that

$$\sigma_n^{(k)} = \sigma_n^{(k-1)} \frac{g_{n+1}^{(k-1)}}{g_n^{(k-1)}}.$$

It can be shown by induction with respect to k that

$$\sigma_n^{(k)} = \frac{(n+b)(n+c)}{(n+k+d)(n+k+e)},$$

$$(2.13) \quad \rho_1^{(k)} = -2k - t, \quad \rho_2^{(k)} = (2k+t)(2k+d+e) - (k+d)(k+e) + bc,$$

$$(2.14) \quad g_n^{(k)} = \frac{(k+d-b)(k+d-c)(k+e-b)(k+e-c)}{(2k+t-1)_2(n+k+d)(n+k+e)}.$$

Now, (2.9) follows from (2.12) and (2.13), while (2.10) follows from (2.11), (2.14) and the equation $\Delta s_n^{(0)} = (b)_n(c)_n / [(d)_n(e)_n]$. If $d - e = b - c$, then $t = 2(e - c)$, $bc - (k + d - 1)(k + e - 1) = \frac{1}{2}(2k + t - 2)(k + c + d - 1)$, and (2.9) implies the last formula given in the theorem for $s_n^{(k)}$. \square

The following remarks should be added:

Remark 1. If at least one of the differences $d - b$, $d - c$, $e - b$, $e - c$ is a negative integer, then all $\Delta s_n^{(k)}$ vanish for a certain k , so that the number $s_0^{(k)}$ is the sum of the series (2.6). Thus, the procedure described in Theorem 2.2 detects some cases where the sum is a rational expression given in terms of the parameters.

Remark 2. Any Saalschützian series ${}_3F_2(a, b, c; d, e; 1)$, i.e., such that $d + e = a + b + c + 1$, reduces to the form (2.6). This result follows from the identity (see [8, vol. 1, § 3.13, (10)])

$${}_3F_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} \middle| 1 \right) = \frac{\Gamma(d)\Gamma(e)\Gamma(t)}{\Gamma(a)\Gamma(t+b)\Gamma(t+c)} {}_3F_2 \left(\begin{matrix} t, d-a, e-a \\ t+b, t+c \end{matrix} \middle| 1 \right),$$

where $t := d + e - a - b - c$, $\Re(t) > 0$, $\Re(a) > 0$.

Obviously, the same remark applies to the series ${}_3F_2(a, b, c; a + 1, e; 1)$, where $\Re(e - b - c) > 0$.

Remark 3. Some series of the type ${}_4F_3(1)$ can be transformed to ${}_3F_2(1)$ of one of the above forms. In case of the Saalschützian ${}_4F_3(1, b, c, d; 2, f, g; 1)$ ($f + g = b + c + d$), it suffices to apply the identity (see [9, § 5.2, (15)])

$${}_4F_3 \left(\begin{matrix} 1, b, c, d \\ 2, f, g \end{matrix} \middle| 1 \right) = \frac{(f-1)(g-1)}{(b-1)(c-1)(d-1)} \left[{}_3F_2 \left(\begin{matrix} b-1, c-1, d-1 \\ f-1, g-1 \end{matrix} \middle| 1 \right) - 1 \right].$$

The series ${}_4F_3(a, b, c, d; e, c + 1, d + 1; 1)$ can be written as (see [9, § 5.2, (14)])

$${}_4F_3 \left(\begin{matrix} a, b, c, d \\ e, c + 1, d + 1 \end{matrix} \middle| 1 \right) = \frac{d(c-e)}{(c-d)e} {}_3F_2 \left(\begin{matrix} a, b, c \\ e + 1, c + 1 \end{matrix} \middle| 1 \right) - \frac{c(d-e)}{(c-d)e} {}_3F_2 \left(\begin{matrix} a, b, d \\ e + 1, d + 1 \end{matrix} \middle| 1 \right).$$

Now, it should be emphasized that the transformations of partial sums of the series (2.6), which are given in Theorem 2.2, are based on the use of some identities for hypergeometric functions of unit argument. Indeed, we have

$$s_0^{(k-1)} + \sum_{n=0}^{\infty} \Delta s_n^{(k-1)} = s_0^{(k)} + \sum_{n=0}^{\infty} \Delta s_n^{(k)}$$

for $k > 0$. Hence it follows from (2.8) for $n = 0$ that

$$(2.15) \quad \frac{1}{\Delta s_0^{(k-1)}} \sum_{n=0}^{\infty} \Delta s_n^{(k-1)} = \gamma_k + \frac{\Delta s_0^{(k)}}{\Delta s_0^{(k-1)}} \frac{1}{\Delta s_0^{(k)}} \sum_{n=0}^{\infty} \Delta s_n^{(k)}.$$

By virtue of (2.9),

$$\frac{\Delta s_{n+1}^{(k)}}{\Delta s_n^{(k)}} = \frac{(n+b)(n+c)}{(n+k+d)(n+k+e)},$$

$$\frac{\Delta s_0^{(k)}}{\Delta s_0^{(k-1)}} = \frac{(k+d-b-1)(k+d-c-1)(k+e-b-1)(k+e-c-1)}{(2k+t-3)_2(k+d-1)(k+e-1)};$$

thus (2.15) implies the identity

$$(2.16) \quad {}_3F_2 \left(\begin{matrix} 1, b, c \\ k+d-1, k+e-1 \end{matrix} \middle| 1 \right) = \gamma_k + \omega_k {}_3F_2 \left(\begin{matrix} 1, b, c \\ k+d, k+e \end{matrix} \middle| 1 \right)$$

with

$$\omega_k = \frac{(k+d-b-1)(k+d-c-1)(k+e-b-1)(k+e-c-1)}{(2k+t-3)_2(k+d-1)(k+e-1)}.$$

The above identity may be obtained also from some relations which hold for contiguous hypergeometric functions.

Theorem 2.3. *The following identities hold:*

$$(2.17) \quad w {}_3F_2 \left(\begin{matrix} a, b, c \\ d-1, e-1 \end{matrix} \middle| 1 \right) \\ = (d-a)(e-a) z {}_3F_2 \left(\begin{matrix} a-1, b, c \\ d, e \end{matrix} \middle| 1 \right) \\ + \{w + abc(t-1) - z[(d-a)(e-a) - bc]\} {}_3F_2 \left(\begin{matrix} a, b, c \\ d, e \end{matrix} \middle| 1 \right),$$

where $t := d + e - a - b - c$, $w := (t-2)_2 (d-1)(e-1)$, $z := (t-2)(d+e-2) - (d-1)(e-1) + ab + ac + bc$, and

$$(2.18) \quad (d+e-b-c-1)_2 de {}_3F_2 \left(\begin{matrix} 1, b, c \\ d, e \end{matrix} \middle| 1 \right) \\ = de[(d+e-b-c)(d+e-1) - de + bc] \\ + (d-b)(d-c)(e-b)(e-c) {}_3F_2 \left(\begin{matrix} 1, b, c \\ d+1, e+1 \end{matrix} \middle| 1 \right).$$

Notice that (2.18) is equivalent to (2.16) for $k = 1$.

Proof. Let the symbols F_{de} , F_a and F denote the three functions ${}_3F_2(1)$ occurring in (2.17). Further, let $F_+ := {}_3F_2(a+1, b+1, c+1; d+1, e+1; 1)$. The identities

$$(d-a)(e-a)F_a = [(d-a)(e-a) - bc]F + (t-1)F_+, \\ (t-2)(d-1)(e-1)F_{de} = [(t-2)(d-1)(e-1) + abc]F + zF_+$$

can be obtained using Bailey's technique of [2]. Eliminating F_+ from this system yields equation (2.17).

Letting $a = 1$ in (2.17), which reduces the second function ${}_3F_2$ to 1, and increasing d and e by 1, we obtain (2.18). \square

Example 2.4. The *lemniscate constant* A is defined in [13, (14.3-7)] by the formula

$$A := \int_0^1 (1-x^4)^{-1/2} dx = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(4n+1)(2n)!!} \\ = \frac{[\Gamma(\frac{1}{4})]^2}{4\sqrt{2\pi}} = 1.31102877714605990523$$

(see also [1, 18.14.7], where this name is given to $\tilde{\omega} := 2A$). Also, we have

$$(2.19) \quad A = {}_2F_1 \left(\begin{matrix} \frac{1}{4}, \frac{1}{2} \\ \frac{5}{4} \end{matrix} \middle| 1 \right) = {}_3F_2 \left(\begin{matrix} 1, \frac{1}{4}, \frac{1}{2} \\ 1, \frac{5}{4} \end{matrix} \middle| 1 \right).$$

Convergence of this series may be speeded up using Theorem 2.2. As $d - e = b - c$, the recurrence formula (2.8) simplifies to

$$s_n^{(k)} := s_n^{(k-1)} + \frac{2n+3k-2}{4k-3} \Delta s_n^{(k-1)}.$$

Moreover,

$$\Delta s_n^{(k)} = \frac{\left[\left(\frac{3}{4}\right)_k\right]^2 \left(\frac{1}{2}\right)_n}{4(k+1)_n \left(k + \frac{1}{2}\right)_k \left(n + \frac{1}{4}\right)_{k+1}}.$$

Transforming the series (1.3) into (1.4) for $k = 1, 2, \dots$ yields a two-dimensional table S of numbers $s_n^{(k)}$ (k is a column index, n is a row index). The r th ($r \geq 0$) antidiagonal of the table, formed by the entries $s_n^{(k)}$ with $n + k = r$, depends only on the terms a_0, a_1, \dots, a_r of the series (1.3) (and on their asymptotic properties). The sum s may be approximated by the partial sums of a fixed series (1.4). It is also possible to use the values $s_0^{(k)}$, lying in the first row of S . However, there is a better way of using the table. Indeed, in case of the series (2.19) all the $s_n^{(k)}$ are less than its sum and such that $s_n^{(k)} < s_n^{(k+1)}$ and $s_n^{(k)} < s_{n+1}^{(k)}$. Therefore, an optimum sequence of approximations lies on a staircase path in S along which the approximations grow most quickly. More specifically, we take $s_{n+1}^{(k)}$ as an approximation next to $s_n^{(k)}$ if

$$(2.20) \quad s_{n+1}^{(k)} - s_n^{(k)} \geq s_n^{(k+1)} - s_n^{(k)}$$

and $s_n^{(k+1)}$ otherwise. Since

$$s_{n+1}^{(k)} - s_n^{(k)} = \Delta s_n^{(k)}, \quad s_n^{(k+1)} - s_n^{(k)} = \frac{2n + 3(k + 1) - 2}{4(k + 1) - 3} \Delta s_n^{(k)},$$

the inequality (2.20) holds if and only if $k \geq 2n$. Thus we obtain the sequence

$$\begin{array}{ccccccc} s_0^{(0)} & & & & & & \\ s_1^{(0)} & s_1^{(1)} & & & & & \\ & s_2^{(2)} & s_2^{(3)} & & & & \\ & & s_3^{(4)} & \dots & & & \end{array}$$

Consequently, (2.19) implies the following series summing to A :

$$\begin{aligned} & s_0^{(0)} + \sum_{n=1}^{\infty} [(s_n^{(2n-2)} - s_{n-1}^{(2n-2)}) + (s_n^{(2n-1)} - s_n^{(2n-2)}) + (s_n^{(2n)} - s_n^{(2n-1)})] \\ &= \sum_{n=1}^{\infty} \left(\Delta s_{n-1}^{(2n-2)} + \frac{8n - 5}{8n - 7} \Delta s_n^{(2n-2)} + \frac{8n - 2}{8n - 3} \Delta s_n^{(2n-1)} \right) \\ &= \sum_{n=1}^{\infty} \left[\frac{\left(\frac{3}{4}\right)_{2n-2}^2 \left(\frac{1}{2}\right)_{n-1}}{4(2n-1)_{n-1} \left(2n - \frac{3}{2}\right)_{2n-2} \left(n - \frac{3}{4}\right)_{2n-1}} \right. \\ &\quad + \frac{\left(\frac{3}{4}\right)_{2n-2} \left(\frac{3}{4}\right)_{2n-1} \left(\frac{1}{2}\right)_n}{2(2n-1)_n \left(2n - \frac{3}{2}\right)_{2n-1} \left(n + \frac{1}{4}\right)_{2n-1}} \\ &\quad \left. + \frac{\left(\frac{3}{4}\right)_{2n-2} \left(\frac{3}{4}\right)_{2n-1} \left(\frac{1}{2}\right)_n}{3(8n-3)(2n)_n \left(2n - \frac{1}{2}\right)_{2n-2} \left(n + \frac{1}{4}\right)_{2n-1}} \right]. \end{aligned}$$

It may seem that this series has a complicated form. Notice, however, that it can be written as

$$A = \sum_{n=1}^{\infty} (p_n + q_n + r_n),$$

where $p_1 := 1$, and

$$\begin{aligned} q_n &:= \frac{(2n-1)(4n-3)(8n-5)}{2(3n-2)(8n-7)(12n-7)} p_n, \\ r_n &:= \frac{(2n-1)(4n-3)}{3(3n-1)(8n-3)} q_n, \\ p_{n+1} &:= \frac{8n-1}{6(12n+1)} r_n \end{aligned}$$

for $n = 1, 2, \dots$. Note that the quotients q_n/p_n , r_n/q_n , p_{n+1}/r_n tend to $\frac{1}{9}$ when $n \rightarrow \infty$, so that adding a new term increases the number of correct decimal digits of the approximate sum by approximately $\log_{10} 9 = 0.95$. Also, it is possible to use the Stirling formula to obtain better estimations of terms (as in Example 2.7).

The above series was summed in Turbo Pascal 5.0 on a microcomputer IBM PC in the extended arithmetic (precision of about 19 decimal digits). Using the terms up to $q_1, r_1, p_2, q_2, r_2, \dots, r_6$ we obtained approximate values of A with 2.1, 3.1, 4.0, 5.1, 6.1, 9.0, 12.0, 14.9, 17.9 correct digits, respectively. This agrees with theoretical estimations and shows that the sum may be computed to almost full machine precision. Notice that the regular analytic form of the series obtained enables one to accelerate its convergence, at least numerically. The same remark applies to other series obtained in this paper.

In [13, § 14.4], many numerical methods are used for accelerating the convergence of (2.19). The best result is obtained using Levin's u -transform, which for 11 terms of the series gives 11.3 digits of its sum. Other methods are much worse, being very sensitive to the influence of rounding errors, even in case where high-precision arithmetic (31–32 decimal digits) is used. We see that using analytic properties of a_{n+1}/a_n may increase the speed of convergence and enables us to control the precision of the result. \square

Example 2.5. If the hypergeometric series (2.6) does not satisfy the condition $|d - e| = |b - c|$, then, according to (2.9), the factor $\beta_k n + \gamma_k$ in the recurrence formula (2.8) depends on k in a more complex way than, e.g., in the preceding example. However, an optimum path in the table \mathbb{S} may be found also in this case, which is—except perhaps for a finite initial segment—as regular as the one obtained for the lemniscate constant A . Such a situation is observed, e.g., for the *Beta* function defined by

$$B(p, q) := \int_0^1 x^{p-1}(1-x)^{q-1} dx \quad (\Re(p), \Re(q) > 0).$$

It is known (see [5, 8.380.1 and 8.384.1], [8, vol. 2, § 13.6, (2)]) that

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = p^{-1} {}_2F_1 \left(\begin{matrix} p, 1-q \\ p+1 \end{matrix} \middle| 1 \right) = p^{-1} {}_3F_2 \left(\begin{matrix} 1, p, 1-q \\ 1, p+1 \end{matrix} \middle| 1 \right).$$

By virtue of Theorem 2.2, the value of $B(p, q)$ may be computed using the

formulae

$$s_n^{(0)} := \sum_{j=0}^{n-1} \frac{(1-q)_j}{j!(j+p)},$$

$$s_n^{(k)} := s_n^{(k-1)} + \frac{(2k+q-1)n + 3k^2 + (p+2q-4)k - q + 1}{(2k+q-2)_2} \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

$$\Delta s_n^{(k)} = \frac{(1-p)_k (p+q)_k (1-q)_n}{(n+p)_{k+1} (k+q)_k (k+1)_n} \quad (k = 0, 1, \dots).$$

The optimum path is defined by the following rule: $s_n^{(k)}$ should be followed by $s_{n+1}^{(k)}$ if $(k+q)^2 - p(k+1) > (2k+q+1)n$, and by $s_n^{(k+1)}$ otherwise. This path is very regular for small p, q . If, however, we take for instance $p = \frac{3}{4}, q = \frac{9}{2}$, it contains the approximants

$$s_0^{(0)}, \dots, s_4^{(0)}, s_4^{(1)}, s_4^{(2)}, s_5^{(2)}, s_5^{(3)}, s_5^{(4)}, s_6^{(4)}, s_6^{(5)}, s_6^{(6)}, s_7^{(6)} \text{ (or } s_6^{(7)}), \\ s_n^{(2n-7)}, s_n^{(2n-6)}, s_n^{(2n-5)} \quad (n = 7, 8, \dots).$$

The type of convergence of the final series is the same as in the preceding example. \square

If the hypergeometric series (2.6) is such that $d = b + 1, e = c + 1$, then $d - e = b - c$, so that we can apply the last part of Theorem 2.2. Dividing the series by bc , we obtain the series

$$(2.21) \quad \sum_{n=0}^{\infty} \frac{1}{(n+b)(n+c)} \quad (b, c \neq 0, -1, -2, \dots).$$

Certain mathematical constants as well as many elementary functions ($\tan, \tanh, \cot, \coth, \csc$ and \csc^2 [1, 4.3.91–93] and [5, § 1.42]) and special functions ($\psi(z), \psi'(z), \Re\psi(iy)$ and $\Im\psi(iy)$; see Example 2.9, or [1, §§ 6.3 and 6.4]) may be expressed in terms of such series.

Corollary 2.6. *The series (2.21) can be transformed according to the following formulae:*

$$s_n^{(0)} := \sum_{j=0}^{n-1} \frac{1}{(j+b)(j+c)},$$

$$(2.22) \quad s_n^{(k)} := s_n^{(k-1)} + \frac{2n + 3k + b + c - 2}{2(2k-1)} \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

$$(2.23) \quad \Delta s_n^{(k)} = \frac{k!(b-c+1)_k (c-b+1)_k}{2^k (2k-1)!! (n+b)_{k+1} (n+c)_{k+1}} \quad (k = 0, 1, \dots),$$

where $n = 0, 1, \dots$; this implies that $\Delta s_n^{(k)} = O(n^{-2k-2})$ ($n \rightarrow \infty$).

A simplified version of equation (2.23) for $b = c$ was obtained already by Knopp [7, (153)], who, however, did not exploit this result.

Example 2.7. Let us consider the series (2.21) for $b = c = 1$, i.e.,

$$(2.24) \quad \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{6}\pi^2.$$

We should take

$$s_n^{(k)} := s_n^{(k-1)} + \frac{2n+3k}{2(2k-1)} \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

$$\Delta s_n^{(k)} = \frac{(k!)^3}{2^k(2k-1)!![(n+1)_{k+1}]^2} \quad (k = 0, 1, \dots).$$

The optimum path is defined in such a way that $s_n^{(k)}$ is followed by $s_{n+1}^{(k)}$ provided $k > 2n + 1$, and by $s_n^{(k+1)}$ otherwise:

$$\begin{array}{ccccccc} s_0^{(0)} & s_0^{(1)} & s_0^{(2)} & & & & \\ & s_1^{(2)} & s_1^{(3)} & s_1^{(4)} & & & \\ & & s_2^{(4)} & s_2^{(5)} & s_2^{(6)} & & \\ & & & & \dots & & \end{array}$$

Therefore, (2.24) implies the expansion

$$s_0^{(0)} + \sum_{n=1}^{\infty} [(s_{n-1}^{(2n-1)} - s_{n-1}^{(2n-2)}) + (s_{n-1}^{(2n)} - s_{n-1}^{(2n-1)}) + (s_n^{(2n)} - s_{n-1}^{(2n)})]$$

$$= \sum_{n=1}^{\infty} \left(\frac{8n-5}{8n-6} \Delta s_{n-1}^{(2n-2)} + \Delta s_{n-1}^{(2n-1)} + \Delta s_{n-1}^{(2n)} \right)$$

and, finally, we have

$$\frac{1}{6}\pi^2 = \sum_{n=1}^{\infty} \left(\frac{(8n-5)[(2n-2)!]^3}{2^{2n-1}(4n-3)!![(n)_{2n-1}]^2} + \frac{[(2n-1)!]^3}{2^{2n-1}(4n-3)!![(n)_{2n}]^2} + \frac{[(2n)!]^3}{2^{2n}(4n-1)!![(n)_{2n+1}]^2} \right).$$

Let n be fixed. The first term of the expression in parentheses is asymptotically equal to $(4\sqrt{2}/27)\pi\sqrt{\pi}729^{1-n}/\sqrt{n}$, while the second term is asymptotically 9 times less than the first one, and the third 9 times less than the second one. Thus, the series is similar to the one obtained in Example 2.4. It can be checked that restricting ourselves to the terms for $n \leq 5$, we obtain π^2 with a relative precision of 14.8 decimal digits. Notice that the known series (see [7, (156)])

$$3 \sum_{n=0}^{\infty} \frac{n!}{(n+1)_{n+2}}$$

corresponds to the first row of the table S. It also converges qualitatively faster than the original series (each term is at least four times smaller than the preceding one), but it is worse than the series obtained in this example. \square

Example 2.8. The series

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)(2n+2)} = \log 2$$

differs by a constant factor from a series of the type (2.21). It follows from (2.22) and (2.23) that

$$s_n^{(0)} := \sum_{j=0}^{n-1} \frac{1}{(2j+1)(2j+2)},$$

$$s_n^{(k)} := s_n^{(k-1)} + \frac{2n+3k-\frac{1}{2}}{2(2k-1)} \Delta s_n^{(k-1)} \quad (k=1, 2, \dots),$$

$$\Delta s_n^{(k)} = \frac{(2k+1)!}{4^{2k+1}(n+\frac{1}{2})_{k+1}(n+1)_{k+1}} \quad (k=0, 1, \dots).$$

The best approximants from the antidiagonals of the table S are $s_0^{(0)}, s_0^{(1)}, s_1^{(1)}, s_1^{(2)}, s_1^{(3)}, s_2^{(3)}, s_2^{(4)}, s_2^{(5)}, \dots$. The series

$$\log 2 = \frac{11}{16} + \sum_{n=1}^{\infty} \frac{(2n)!}{16^n} \left(\frac{16n-1}{(4n-1)_{2n+2}} + \frac{16n+5}{4(4n+1)_{2n+2}} + \frac{1}{4(4n+4)_{2n+1}} \right),$$

which has such partial sums, behaves similarly as the series obtained in the preceding example. In particular, the approximant $s_5^{(10)}$ gives 15 significant decimal digits of $\log 2$. □

Example 2.9. The function $\psi(z) := (\log \Gamma(z))'$ (called *digamma*) and its derivative ψ' (known as *trigamma*) have the following expansions in the complex plane (see [4, vol. 1, § 1.7, (3) and § 1.9, (10)]):

$$\psi(z) = -\gamma + (z-1) \sum_{k=0}^{\infty} \frac{1}{(k+1)(k+z)},$$

$$\psi'(z) = \sum_{k=0}^{\infty} \frac{1}{(k+z)^2} \quad (z \neq 0, -1, -2, \dots).$$

Notice that both series are of the form (2.21). Reasoning as above, we can show that for $z \in [1, 2)$ the optimum transformation gives

$$\psi(z) = -\gamma + \sum_{n=0}^{\infty} \frac{n!(z-2n-1)_{3n+1}}{2^{2n+1}(4n+1)!!}$$

$$\times \left[\frac{8n+z+2}{(2n+1)_{n+1}(z+2n)_{n+1}} + \frac{(8n+z+5)(2n+2-z)}{2(4n+3)(2n+2)_{n+1}(z+2n+1)_{n+1}} \right.$$

$$\left. + \frac{(2n-z+2)(2n-z+3)}{2(4n+3)(2n+3)_{n+1}(z+2n+2)_{n+1}} \right],$$

$$\psi'(z) = \sum_{n=0}^{\infty} \frac{[(2n)!]^3}{2^{2n}(4n+1)!! [(n+z)_{2n+1}]^2}$$

$$\times \left\{ 4n+z+\frac{1}{2} + \frac{(2n+1)^3}{2(4n+3)(3n+z+1)^2} \left[4n+z+2 + \frac{4(n+1)^3}{(3n+z+2)^2} \right] \right\},$$

respectively. In both cases, adding a new term, i.e., passing through consecutive three elements of the table S, increases the number of correct decimal digits of the sum by $\log_{10} 729 = 2.86$ at least. This remains true for real and even

complex z lying outside the interval $[1, 2)$, though the optimum path may depend on z . \square

Example 2.10. In (2.22), (2.23), parameters b, c may have complex values. If, in particular, they are complex conjugate numbers, then the resulting formulae contain real quantities only. Let $b = i, c = -i$, which corresponds to the series $\sum 1/(n^2 + 1)$ summing to $\frac{1}{2}(\pi + 1) + \pi/(e^{2\pi} - 1) = 2.076674047468581$ [1, 6.3.13]. Then

$$s_n^{(k)} := s_n^{(k-1)} + \frac{2n + 3k - 2}{2(2k - 1)} \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

$$\Delta s_n^{(k)} = \frac{k! \prod_{j=1}^k (j^2 + 4)}{2^k (2k - 1)!! \prod_{j=n}^{n+k} (j^2 + 1)} \quad (k = 0, 1, \dots).$$

The optimum sequence of approximants $s_0^{(0)}, s_1^{(0)}, s_1^{(1)}, s_2^{(1)}, s_2^{(2)}, s_2^{(3)}, s_3^{(3)}, s_3^{(4)}, s_3^{(5)}, \dots$ is convergent to the sum of the series as quickly as in the previous examples. In particular, $s_8^{(14)}$ gives the result 2.07667404746858117, which has at least 16 correct digits. \square

Proceeding in an analogous way, one can obtain new expansions of $\coth x, \Im\psi(1 + ix)$ and some other functions, as, for instance, we have [1, 6.3.13]

$$\Im\psi(1 + ix) = -\frac{1}{2x} + \frac{1}{2}\pi \coth \pi x = x \sum_{n=1}^{\infty} \frac{1}{n^2 + x^2}.$$

Let us go back to the general case of series of ${}_3F_2(1, b, c; d; e; 1)$ discussed in Theorem 2.2. It can be deduced from the forms obtained for β_k, γ_k that the optimum path in the table \mathbb{S} contains, putting aside a finite initial segment, the entries

$$s_n^{(2n+m)}, s_n^{(2n+m+1)}, s_n^{(2n+m+2)} \quad (n = n_{\min}, n_{\min} + 1, \dots),$$

where m, n_{\min} are integers depending on b, c, d, e . This implies that the sum of the above series can be expressed as a finite sum corresponding to the initial segment of the path plus the sum of the following three series:

$$\sum_{n=n_{\min}}^{\infty} (\beta_{2n+m+1}n + \gamma_{2n+m+1}) \Delta s_n^{(2n+m)},$$

$$\sum_{n=n_{\min}}^{\infty} (\beta_{2n+m+2}n + \gamma_{2n+m+2}) \Delta s_n^{(2n+m+1)}, \quad \sum_{n=n_{\min}}^{\infty} \Delta s_n^{(2n+m+2)}.$$

It can be checked that the first two series are of the type ${}_{13}F_{12}(1, \dots; \dots; \frac{1}{729})$, and the last is of the type ${}_{11}F_{10}(1, \dots; \dots; \frac{1}{729})$. The parameters of these new series depend on b, c, d, e, m . In particular, if $|d - e| \neq |b - c|$, then as a rule, two numerator parameters and two denominator parameters of ${}_{13}F_{12}$ are nonrational functions of b, c, d, e, m . Obviously, the new series can be simplified considerably in particular cases. In general, however, the theorem relates ${}_3F_2(1, b, c; d, e; 1)$ to the abovementioned more complex series which are faster convergent.

Concluding the considerations related to the transformation (2.4), we recall that its parameters β , γ are determined using the asymptotic expansion of the quotient $\Delta s_{n+1}^{(0)}/\Delta s_n^{(0)} = a_{n+1}/a_n$. These parameters may also be determined numerically so that three consecutive elements $s_n^{(1)}$, $s_{n+1}^{(1)}$, $s_{n+2}^{(1)}$ (n fixed) of the sequence $\{s_n^{(1)}\}$ are identical (and equal to an approximation \tilde{s} of the exact sum s). We then have

$$\beta j + \gamma = \frac{\tilde{s} - s_j^{(0)}}{\Delta s_j^{(0)}} \quad (j = n, n+1, n+2).$$

Hence,

$$\Delta^2 \left(\frac{\tilde{s} - s_n^{(0)}}{\Delta s_n^{(0)}} \right) = 0.$$

Obviously, \tilde{s} depends on n , in general; thus we obtain the following transformation of the sequence $\{s_n^{(0)}\}$:

$$\tilde{s}_n^{(1)} := \Delta^2 \left(\frac{s_n^{(0)}}{\Delta s_n^{(0)}} \right) / \Delta^2 \left(\frac{1}{\Delta s_n^{(0)}} \right) \quad (n = 0, 1, \dots).$$

It is easy to verify the following:

(i) The above formula is identical to the first step of Brezinski's Θ -algorithm (see, e.g., [3, §2.9] and [13, (10.3-3)]), more specifically, to passing from the sequence $\{\Theta_0^{(n)}\}$, identical to $\{s_n^{(0)}\}$, to $\{\Theta_2^{(n)}\}$. Also, notice that the method defined by this formula is equivalent to the Lubkin W-method [13, §10.3].

(ii) For some series, $\{\tilde{s}_n^{(1)}\}$ is identical to $\{s_{n+1}^{(1)}\}$. Such a situation occurs in the case (2.19), where $\Delta^2(1/\Delta s_n^{(0)}) = 2$ for $n \geq 0$. In general, however, our analytic transformations differ from the Θ -transform (as well as from other known numerical transforms).

3. SERIES FOR $\varepsilon = -1$

In this section we consider series of the form

$$(3.1) \quad \sum_{n=0}^{\infty} (-1)^n a_n.$$

We assume that the a_n satisfy condition (1.8). Also in this case accelerating factors $C_n^{(k)}$ can be found which produce some new methods for convergence acceleration. The point of departure is, as in §2, equation (1.7), satisfied by the exact factor $C_n^{(0)}$, defined by (1.6). Applying Theorem 6.1 of [11] (see also [12, Theorem 2.1]), we obtain an expansion of this factor into a formal series $c_0 + c_1 n^{-1} + c_2 n^{-2} + \dots$, where $c_0 = \frac{1}{2}$. Having computed c_k , we can replace the exact $C_n^{(0)}$ by, e.g., $[m/m](n^{-1})$, where $[m/m]$ is a Padé approximant to the formal power series $c_0 + c_1 x + c_2 x^2 + \dots$.

In the simplest case, the exact factor $C_n^{(0)}$ is replaced by the number $\frac{1}{2}$ (i.e., the Padé approximant $[0/0]$). This transformation, defined by $s_n^{(1)} := s_n^{(0)} + \frac{1}{2} \Delta s_n^{(0)}$, is a typical step of the well-known Euler's transformation. More useful is the following transformation, which is based on the use of $[1/1]$.

Lemma 3.1. Let $s_n^{(0)}$ be the partial sum of the series (3.1),

$$s_n^{(0)} := \sum_{j=0}^{n-1} (-1)^j a_j,$$

and let

$$s_n^{(1)} := s_n^{(0)} + \frac{n + \zeta}{2n + \eta} \Delta s_n^{(0)},$$

where

$$\zeta = -\frac{1}{2} - \frac{\rho_2}{\rho_1} \quad (\text{provided } \rho_1 \neq 0), \quad \eta = 2\zeta + \rho_1.$$

Then we have $\Delta s_n^{(1)} = -\Delta s_n^{(0)} O(n^{-3})$.

Proof. In view of (1.8), we have

$$\begin{aligned} \Delta s_n^{(1)} &= \frac{(-1)^{n+1} a_n}{2n + \eta} \left[(n + \zeta + 1) \frac{2n + \eta}{2n + \eta + 2} \left(1 + \sum_{i=1}^{\infty} \rho_i n^{-i} \right) - n - \eta + \zeta \right] \\ &= \frac{(-1)^{n+1} a_n}{2n + \eta} \left[(n + \zeta + 1) \left(1 + \sum_{i=1}^{\infty} \tilde{\rho}_i n^{-i} \right) - n - \eta + \zeta \right], \end{aligned}$$

where $\tilde{\rho}_1 := \rho_1 - 1$, $\tilde{\rho}_2 := \rho_2 - \rho_1 + \frac{1}{2}\eta + 1$. The expression in brackets can be expanded into a series with respect to the powers of n^{-1} . For ζ , η given in Lemma 3.1 the coefficients of n^0 and n^{-1} vanish, hence the result. \square

Remark. Recall that in §2, when $\varepsilon = 1$, we obtained the formula $\Delta s_n^{(1)} = \Delta s_n^{(0)} O(n^{-2})$ (see Lemma 2.1). Thus, it might be thought that we obtain here a stronger acceleration effect. This is, however, not confirmed in the case when we choose an optimum sequence of approximations.

The next examples show that the parameters ζ and η can be computed effectively also for the further transformations of series, provided the way in which a_n depends on n is sufficiently simple. The most general hypergeometric series to which we are able to apply the proposed method is ${}_3F_2(1, b, c; d, e; -1)$, where $b + d = c + e$. Obviously, formula (3.4), given in the theorem below, is a consequence of Lemma 3.1.

Theorem 3.2. The series

$$(3.2) \quad {}_3F_2 \left(\begin{matrix} 1, b, c \\ d, e \end{matrix} \middle| -1 \right) = \sum_{n=0}^{\infty} (-1)^n \frac{(b)_n (c)_n}{(d)_n (e)_n}$$

$(d, e \neq 0, -1, -2, \dots, \quad b + d = c + e, \quad \Re(d - c) > \frac{1}{2})$

may be transformed using the following formulae:

$$(3.3) \quad s_n^{(0)} := \sum_{j=0}^{n-1} (-1)^j \frac{(b)_j (c)_j}{(d)_j (e)_j},$$

$$(3.4) \quad s_n^{(k)} := s_n^{(k-1)} + \frac{n + 2k + \frac{1}{2}(2e - b + d - 5)}{2n + k + b + d - 2} \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

$$(3.5) \quad \Delta s_n^{(k)} = (-1)^{n+k} \left(n + \frac{1}{2}(k + b + d - 1) \right) \times \frac{(d - c)_k \left(\frac{1}{2}(e - c + 1) \right)_k \left(\frac{1}{2}(d - b + 1) \right)_k (b)_n (c)_n}{(e)_{n+k} (d)_{n+k} \left(n + \frac{1}{2}(b + d - 1) \right)_{k+1}} \quad (k = 0, 1, \dots).$$

Here $n = 0, 1, \dots$

Proof. If $k = 0$, then (3.5) agrees with (3.3). In case $k > 0$, it is sufficient to show that (3.5) satisfies the identity

$$\Delta s_n^{(k)} = \frac{n + 2k + \frac{1}{2}(2e - b - d - 3)}{2n + k + b + d} \Delta s_{n+1}^{(k-1)} + \frac{n - k + \frac{1}{2}(2b - e + c + 1)}{2n + k + b + d - 2} \Delta s_n^{(k+1)},$$

which follows from (3.4). \square

Remark 1. If $b + d - 1 = -2m$ ($m = 0, 1, \dots$), then the denominator of (3.5) contains $(n - m)_{k+1}$, which vanishes for $n \in \{m, m - 1, \dots, \max(0, m - k)\}$. Thus the formula can be applied for $n > m$ only. The same remark applies to (3.4).

Remark 2. The recurrence relations given in Theorem 3.2 imply the following identity:

$$\begin{aligned} & {}_5F_4 \left(\begin{matrix} 1, b, c, \frac{1}{2}(b + d - 1), \frac{1}{2}(k + b + d) \\ k + e - 1, k + d - 1, k + \frac{1}{2}(b + d - 1), \frac{1}{2}(k + b + d) - 1 \end{matrix} \middle| -1 \right) \\ &= \frac{2k + \frac{1}{2}(3e + d - 5)}{k + b + d - 2} \\ & - \frac{(k + e - b - 1)(k + b + d - 1)(k + \frac{1}{2}(e - c - 1))(k + \frac{1}{2}(d - b - 1))}{(k + e - 1)(k + d - 1)(k + b + d - 2)(k + \frac{1}{2}(b + d - 1))} \\ & \times {}_5F_4 \left(\begin{matrix} 1, b, c, \frac{1}{2}(b + d - 1), \frac{1}{2}(k + b + d + 1) \\ k + e, k + d, k + \frac{1}{2}(b + d + 1), \frac{1}{2}(k + b + d - 1) \end{matrix} \middle| -1 \right). \end{aligned}$$

A regular part of the optimum path in the table S, corresponding to the function ${}_3F_2(1, b, c; d, e; -1)$, contains only the elements

$$s_n^{(n+m)}, s_n^{(n+m+1)} \quad (n = n_{\min}, n_{\min} + 1, \dots),$$

where m, n_{\min} depend on b, c, d, e . Thus, the sum of the above series can be written as a finite sum, corresponding to the irregular (initial) part of the path, plus

$$\sum_{n=n_{\min}}^{\infty} (\beta_{n+m+1} n + \gamma_{n+m+1}) \Delta s_n^{(n+m)} + \sum_{n=n_{\min}}^{\infty} \Delta s_{n+1}^{(n+m+1)}.$$

Note that both series are of the type ${}_8F_7(1, \dots; \dots; \frac{1}{64})$.

Remark 3. If the condition $\Re(d-c) > \frac{1}{2}$ is relaxed, the series (3.2) is divergent. However, if $d-c$, $\frac{1}{2}(e-c+1)$, or $\frac{1}{2}(d-b+1)$ is a nonnegative integer, then equations (3.4) and (3.5) still hold, and $\Delta s_n^{(k)}$ vanishes for sufficiently large values of k . Hence, the analytic transformation of the series (3.2) according to Theorem 3.2 may be viewed as a *summation method* for a divergent series.

Example 3.3. Theorem 3.2 is directly applicable to the series expansion of

$$\psi(z) + \gamma = \frac{2(z-1)}{z} {}_3F_2 \left(\begin{matrix} 1, 1, 2-z \\ 2, 1+z \end{matrix} \middle| -1 \right)$$

(see [10]). For the expansion of the right-hand side of the above equation, we have

$$s_n^{(k)} := s_n^{(k-1)} + \frac{n+2k+z-1}{2n+k+1} \Delta s_n^{(k-1)},$$

$$\Delta s_n^{(k)} = \frac{(-1)^k (2n+k+2)k! (z-n-1)_{n+k+1}}{(z+k)_{n+1} [(n+1)_{k+1}]^2}.$$

Let us assume that, for example, $z \in (1, 2)$. Then, given arbitrary k , $n \geq 0$, the approximants $s_n^{(k)}$ and $s_n^{(k+1)}$ lie on the opposite sides of the sum; the same property holds for the numbers $s_n^{(k)}$ and $s_{n+1}^{(k)}$, which means that the manner, in §2, for choosing the best approximants from the antidiagonals of table S should be reversed here: if $|s_n^{(k+1)} - s_n^{(k)}| < |s_{n+1}^{(k)} - s_n^{(k)}|$, i.e., $n \geq k+1$, then $s_n^{(k)}$ is followed by $s_n^{(k+1)}$, otherwise by $s_{n+1}^{(k)}$. This defines the best path: $s_0^{(0)}, s_1^{(0)}, s_1^{(1)}, s_2^{(1)}, \dots$ and the transformed form of $\psi(z)$:

$$\begin{aligned} \psi(z) + \gamma &= \sum_{n=0}^{\infty} [(s_{n+1}^{(n)} - s_n^{(n)}) + (s_{n+1}^{(n+1)} - s_{n+1}^{(n)})] \\ &= \sum_{n=0}^{\infty} \left(\Delta s_n^{(n)} + \frac{3n+z+2}{3n+4} \Delta s_{n+1}^{(n)} \right) = \sum_{n=0}^{\infty} (p_n - q_n), \end{aligned}$$

where $p_0 := 2(z-1)/z$, and

$$q_n := \frac{(n-z+2)(3n+z+2)}{4(3n+2)(2n+z+1)} p_n,$$

$$p_{n+1} := \frac{(n+1)(3n+5)(n+z)^2}{(2n+3)^2(2n+z+2)(3n+z+2)} q_n \quad (n=0, 1, \dots).$$

The speed of convergence of this series results from the obvious relations $q_n/p_n \rightarrow \frac{1}{8}$, $p_{n+1}/q_n \rightarrow \frac{1}{8}$. Thus, adding a new term to the series increases the number of exact digits of the result by approximately $\log_{10} 8 = 0.9$. If, for example, $z = \frac{5}{4}$, then the terms up to p_4 give the approximate value 0.34976213, where all the digits are correct. More precise asymptotic expressions for p_n , q_n can also be obtained. It is worthwhile to compare the above series with the results from Example 2.9. \square

It is obvious that any series ${}_2F_1(1, b; d; -1)$ may be written in the form (3.2) with $c = e$. Thus, a simple consequence of the Theorem 3.2 is

Corollary 3.4. *The series*

$$(3.6) \quad {}_2F_1 \left(\begin{matrix} 1, b \\ d \end{matrix} \middle| -1 \right) = \sum_{n=0}^{\infty} \frac{(-1)^n (b)_n}{(d)_n} \quad (d \neq 0, -1, -2, \dots, \Re(d-b) > 1)$$

may be transformed using for $n = 0, 1, \dots$ the formulae

$$s_n^{(0)} := \sum_{j=0}^{n-1} \frac{(-1)^j (b)_j}{(d)_j},$$

$$(3.7) \quad s_n^{(k)} := s_n^{(k-1)} + \frac{n+2k+d-\frac{5}{2}}{2n+k+b+d-2} \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

$$(3.8) \quad \Delta s_n^{(k)} = \frac{(-1)^{n+k} \left(\frac{1}{2}\right)_k (2n+k+b+d-1)(d-b)_{2k} (b)_n}{(d)_{n+k} (2n+b+d-1)_{2k+1}} \quad (k = 0, 1, \dots).$$

Remark 1. If $d-b$ is a negative integer, the series (3.6) diverges. However, $\Delta s_n^{(k)}$ defined by equation (3.8) vanishes for sufficiently large k , if $d-b$ is a negative integer. Thus, the transformation of the series (3.6) according to Corollary 3.4 is a *summation method* for a divergent hypergeometric series.

Remark 2. It is interesting that, given b, d , equation (3.7) may serve to accelerate the convergence of unique hypergeometric series, different from ${}_2F_1(1, b; d; -1)$, namely, ${}_2F_1(b+d-1, \frac{1}{2}(b-d)+1; \frac{1}{2}(b+3d)-1; -1)$. Note, however, that the sum of the latter series can be expressed in terms of the Gamma function [4, vol. 1, § 2.8, (47)].

Remark 3. Under the assumption that $d > b > 0$, we obtain the same optimum path as in Example 3.3.

Example 3.5. Corollary 3.4 enables us to obtain a rapidly convergent expansion for $\sqrt{2}$. It is known that

$$\begin{aligned} \sqrt{1+x} &= {}_1F_0 \left(\begin{matrix} -\frac{1}{2} \\ - \end{matrix} \middle| -x \right) = 1 + \frac{1}{2} x {}_2F_1 \left(\begin{matrix} 1, \frac{1}{2} \\ 2 \end{matrix} \middle| -x \right) \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} (2n-3)!!}{(2n)!!} x^n. \end{aligned}$$

Now, we can set for $x = 1$

$$s_n^{(0)} := 1 + \sum_{j=1}^{n-1} \frac{(-1)^{j-1} (2j-3)!!}{(2j)!!},$$

$$s_n^{(k)} := s_n^{(k-1)} + \frac{n+2k-\frac{3}{2}}{2n+k-\frac{3}{2}} \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

$$\Delta s_n^{(k)} = \frac{(-1)^{n+k-1} 2^k \left(\frac{1}{2}\right)_k \left(\frac{3}{2}\right)_{2k} (2n-3)!! (2n+k-\frac{1}{2})}{(2n-\frac{1}{2})_{2k+1} (2n+2k)!!} \quad (k = 0, 1, \dots).$$

The optimum sequence of approximations of $\sqrt{2}$ is: $s_1^{(0)}, s_2^{(0)}, s_2^{(1)}, s_3^{(1)}$,

$s_3^{(2)}, \dots$. Thus, we obtain the expansion

$$\sqrt{2} = 1 + 2 \sum_{n=1}^{\infty} \frac{(2n-3)!!(4n-3)!!}{32^n(n-1)!} \left(\frac{12}{(2n-\frac{1}{2})_{2n-1}} - \frac{6n-1}{n(2n+\frac{3}{2})_{2n-1}} \right),$$

where each term corresponds to a shift in the table **S** by two entries, i.e., to two terms of the original series. An asymptotic expression may be found for this term, like in the next example. Let us limit ourselves to the remark that the n th term is a difference of two numbers, g_n and h_n , say, such that the quotients g_n/h_n and h_n/g_{n-1} are both asymptotically equal to 8. Thus, adding a new term to the series increases the number of correct digits of the result approximately by $\log_{10} 64 = 1.8$. This is confirmed by computations performed for $n \leq 1, 2, 3, 4$, as we obtain 2.6, 4.5, 6.4, 8.3 decimal digits, respectively. \square

Example 3.6. Assume now that $d = b + 1$, which gives the series

$$(3.9) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{n+b}$$

having the sum $\frac{1}{2} [\psi(\frac{1}{2}(b+1)) - \psi(\frac{1}{2}b)] = \frac{1}{2}G(b)$ [4, vol. 1, §1.8]. Notice that (3.6) reduces in this case to a series which is equal to (3.9) multiplied by b . Using the formula (3.8), we should, therefore, divide its right-hand side by b . In particular, for $b = \frac{1}{3}$ (the sum of (3.9) is then $\pi/\sqrt{3} + \log 2$; see [5, 0.239.1]) and $b = \frac{1}{2}$ (the sum is $\pi/2$; [5, 0.232.2]) this gives the sequence $s_0^{(0)}, s_1^{(0)}, s_1^{(1)}, s_2^{(1)}, \dots$ corresponding to the series

$$\sum_{n=1}^{\infty} [(s_n^{(n-1)} - s_{n-1}^{(n-1)}) + (s_n^{(n)} - s_{n-1}^{(n)})] = \sum_{n=1}^{\infty} \left(\Delta s_{n-1}^{(n-1)} + \frac{3n+d-\frac{3}{2}}{3n+2d-1} \Delta s_n^{(n-1)} \right).$$

For $b = \frac{1}{2}$, this implies the expansion

$$(3.10) \quad \pi = \sum_{n=1}^{\infty} (n-1)! \left[\left(\frac{1}{2} \right)_{n-1} \right]^2 \left(\frac{3n-2}{[(n-\frac{1}{2})_n]^2 (n)_{n-1}} - \frac{3n-1}{[(n+\frac{1}{2})_n]^2 (n+1)_{n-1}} \right).$$

Its first three terms give 3.1415878..., which is a value of π correct to 5.8 decimal digits.

In case $b = \frac{2}{3}$ (sum of (3.9) is $\pi/\sqrt{3} - \log 2$; see [5,0.239.2]) and for $b = 1$ (the sum is $\log 2$) the optimum sequence is $s_0^{(0)}, s_0^{(1)}, s_1^{(1)}, s_1^{(2)}, \dots$. Hence, we obtain the series

$$\sum_{n=1}^{\infty} [(s_{n-1}^{(n)} - s_{n-1}^{(n-1)}) + (s_n^{(n)} - s_{n-1}^{(n)})] = \sum_{n=1}^{\infty} \left(\frac{3n+d-\frac{5}{2}}{3n+2d-3} \Delta s_{n-1}^{(n-1)} + \Delta s_n^{(n)} \right).$$

For $b = 1$ this gives the expansion

$$\log 2 = 3 \sum_{n=1}^{\infty} \frac{[(2n-2)!]^2}{4^n(4n-3)!} \left(1 - \frac{2n-1}{4(4n-1)} \right).$$

It can be verified that the n th term of the last series is asymptotically equal to $\frac{1}{16} \sqrt{2\pi} 64^{1-n} / \sqrt{n}$ (this applies also to the series (3.10)). In particular, the

approximant $s_7^{(8)}$ gives 14.2 decimal digits of $\log 2$. Note that this result is much better than the one obtained in §2. Indeed, the series, which has been transformed there, is obtained by summing pairs of subsequent terms of (3.9); to obtain 15 correct digits, we needed about 30 terms of (3.9). \square

Let us consider the series

$$(3.11) \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+b)(n+c)} = \frac{1}{bc} {}_3F_2 \left(\begin{matrix} 1, b, c \\ b+1, c+1 \end{matrix} \middle| -1 \right) \quad (b, c \neq 0, -1, -2, \dots).$$

Notice that the above ${}_3F_2$ is of the form (3.2) with $d = c + 1, e = b + 1$. Thus, we obtain the following.

Corollary 3.7. *The series (3.11) may be transformed using*

$$s_n^{(0)} := \sum_{j=0}^{n-1} \frac{(-1)^j}{(j+b)(j+c)},$$

$$s_n^{(k)} := s_n^{(k-1)} + \frac{n+2k+\frac{1}{2}(b+c)-1}{2n+k+b+c-1} \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

$$\Delta s_n^{(k)} = \frac{(-1)^{n+k} k! [n + \frac{1}{2}(k+b+c)] [\frac{1}{2}(b-c)+1]_k [\frac{1}{2}(c-b)+1]_k}{[n + \frac{1}{2}(b+c)]_{k+1} (n+b)_{k+1} (n+c)_{k+1}} \quad (k = 0, 1, \dots)$$

for $n = 0, 1, \dots$. If $b = c$, the last formula simplifies to

$$\Delta s_n^{(k)} = (-1)^{n+k} (n + \frac{1}{2}k + b) \left(\frac{k!}{(n+b)_{k+1}} \right)^3.$$

For $b \neq c$ the series (3.11) can be written as a combination of series from Example 3.6, i.e., in terms of the ψ function. If $b = c$, its sum is $\frac{1}{4} [\psi'(\frac{1}{2}b) - \psi'(\frac{1}{2}(b+1))]$; note that a series expansion for ψ' has been obtained in Example 2.9. However, it may be inferred that the optimum (in the sense defined) convergence acceleration of the series (3.11) would give simpler and better results than using the same procedure for each of the two series separately. Indeed, this is the case.

Example 3.8. If $b = c = 1$, the series (3.11) has the sum $\frac{1}{12}\pi^2$. The optimum sequence $s_0^{(0)}, s_1^{(0)}, s_1^{(1)}, s_2^{(1)}, \dots$ of approximants yields the series

$$\begin{aligned} & \sum_{n=1}^{\infty} [(s_n^{(n-1)} - s_{n-1}^{(n-1)}) + (s_n^{(n)} - s_n^{(n-1)})] \\ &= \frac{1}{2} \sum_{n=1}^{\infty} \left[(3n-1) \left(\frac{(n-1)!}{(n)_n} \right)^3 - 3n \left(\frac{(n-1)!}{(n+1)_n} \right)^3 \right] \\ &= \frac{1}{2} \sum_{n=1}^{\infty} (21n-8) \left(\frac{(n-1)!}{(n+1)_n} \right)^3. \end{aligned}$$

The n th term of the last series is asymptotically equal to $21\pi\sqrt{\pi}64^{-n}/\sqrt{n}$, i.e., the convergence acceleration has the same character as in the preceding example.

If $b = c = \frac{1}{2}$, the series (3.11) has the sum $4G$, where G is the so-called *Catalan's constant*, approximately equal to 0.915965594177219015 (see, e.g., [8, vol. 2, p.293]). Proceeding as before, we obtain the expansion

$$G = 1 - \frac{1}{8} \sum_{n=1}^{\infty} \left[3n - \frac{1}{2} - 8(3n+1) \left(\frac{n}{4n+1} \right)^3 \right] \left(\frac{(n-1)!}{(n+\frac{1}{2})_n} \right)^3.$$

As before, adding one term to a partial sum of this expansion, i.e., adding two terms of the original series, increases the accuracy of the result by $\log_{10} 64 = 1.8$ decimal digits at least.

The parameters b, c may have complex values. In case of $\sum (-1)^n / (n^2 + n + 1)$ we have $b + c = bc = 1$, hence

$$s_n^{(k)} := s_n^{(k-1)} + \frac{n + 2k - \frac{1}{2}}{2n + k} \Delta s_n^{(k-1)} \quad (k = 1, 2, \dots),$$

$$\Delta s_n^{(k)} = \frac{(-1)^{n+k} k! \left(n + \frac{1}{2}(k+1) \right) \prod_{h=1}^k (h^2 + \frac{3}{4})}{(n + \frac{1}{2})_{k+1} \prod_{h=n}^{n+k} (h^2 + h + 1)} \quad (k = 0, 1, \dots).$$

The optimum sequence $s_0^{(0)}, s_1^{(0)}, s_1^{(1)}, s_2^{(1)}, \dots$ converges to the sum

$$0.761310204001103486$$

as quickly as in the other cases. In particular, $s_8^{(8)}$ has 14 correct decimal digits. \square

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BIBLIOGRAPHY

1. M. Abramowitz and I. A. Stegun (eds.), *Handbook of mathematical functions*, Nat. Bureau of Standards, Washington, 1964.
2. W. N. Bailey, *Contiguous hypergeometric functions of the type ${}_3F_2(1)$* , Proc. Glasgow Math. Assoc. 2 (1954), 62–65.
3. C. Brezinski and M. Redivo Zaglia, *Extrapolation methods. Theory and practice*, North-Holland, Amsterdam, 1991.
4. A. Erdélyi (ed.), *Higher transcendental functions*, McGraw-Hill, New York, 1953.
5. I. S. Gradshteyn and I. M. Ryzhik, *Tables of integrals, series and products*, Academic Press, New York, 1980.
6. S. L. Kalla, S. Conde, and Y. L. Luke, *Integrals of Jacobi functions*, Math. Comp. 38 (1982), 207–214.
7. K. Knopp, *Theory and application of infinite series*, Hafner, New York, 1949.
8. Y. L. Luke, *The special functions and their approximations*, Academic Press, New York, 1969.
9. ———, *Mathematical functions and their approximations*, Academic Press, New York, 1975.
10. ———, *Rational approximations for the logarithmic derivative of the gamma function*, Appl. Anal. 1 (1979), 65–73.

11. A. Sidi, *Convergence properties of some nonlinear sequence transformations*, Math. Comp. **33** (1979), 315–326.
12. ———, *Analysis of convergence of the T-transformation for power series*, Math. Comp. **35** (1980), 833–850.
13. E. J. Weniger, *Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series*, Comp. Phys. Reports **10** (1989), 189–371.
14. J. Wimp, *Irreducible recurrences and representation theorems for ${}_3F_2(1)$* , Comput. Math. Appl. **9** (1983), 663–678.
15. ———, *Computation with recurrence relations*, Pitman, Boston-London, 1984.
16. D. Zeilberger, *Gauss's ${}_2F_1(1)$ cannot be generalized to ${}_2F_1(x)$* , J. Comput. Appl. Math. **39** (1992), 379–382.

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