A QUADRATURE FORMULA
WITH ZEROS OF BESSEL FUNCTIONS AS NODES

GEORGI R. GROZEV AND QAZI I. RAHMAN

ABSTRACT. A quadrature formula for entire functions of exponential type wherein the nodes are the zeros of the Bessel function of the first kind was recently obtained by C. Frappier and P. Olivier. Here the condition imposed on the function is relaxed. Some applications of the formula are also given.

1. INTRODUCTION AND STATEMENT OF RESULTS

If f is an entire function of exponential type $2\tau > 0$ belonging to $L^1(-\infty, \infty)$ then ([3])

$$(1) \qquad \int_{-\infty}^{\infty} f(x)\, dx = \frac{\pi}{\tau} \sum_{k=-\infty}^{\infty} f\left(\frac{(2k-1)\pi}{2\tau}\right),$$

where the series is absolutely convergent.

If f belongs to $L^1(-\infty, \infty)$, then so do the functions $(f(x)+f(-x))/2$ and $(f(x)-f(-x))/2$. Since $(f(x)-f(-x))/2$ is odd, $\int_{-\infty}^{\infty} f(x)dx$ is nothing but $\int_0^{\infty}(f(x)+f(-x))dx$. Hence (1) may be written as

$$(1') \quad \int_0^{\infty}(f(x)+f(-x))\, dx = \frac{\pi}{\tau} \sum_{k=1}^{\infty}\left(f\left(\frac{(2k-1)\pi}{2\tau}\right) + f\left(-\frac{(2k-1)\pi}{2\tau}\right)\right).$$

Applying (1) to the function $f(z) + f(-z)$, we see that in this form, the formula is valid even if $f(x) + f(-x)$ belongs to $L^1[0, \infty)$ without f being in $L^1(-\infty, \infty)$.

Let $J_\alpha(z)$ be the Bessel function of the first kind of order α. We shall denote by $j_1(\alpha), j_2(\alpha), \ldots, j_k(\alpha), \ldots$ the zeros of $\frac{J_\alpha(z)}{z^\alpha}$ in the right half-plane arranged in ascending order of magnitude.

Recently, Frappier and Olivier [4] have proved the following result which we find very interesting.

Theorem A. *If* $\Re e\,\alpha > -1$, *then the quadrature formula*

$$(2) \quad \int_0^\infty x^{2\alpha+1}(f(x)+f(-x))\,dx$$

$$= \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^\infty \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \left(f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right)$$

holds for every entire function f *of exponential type* 2τ *such that* $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$ *with* $\delta > 2\Re e\,\alpha + 2$.

Formula (2) reduces to ($1'$) when $\alpha = -\frac{1}{2}$ since

$$\frac{J_{-\frac{1}{2}}(\tau z)}{(\tau z)^{-\frac{1}{2}}} = \sqrt{\frac{2}{\pi}} \cos(\tau z).$$

However, in the case $\alpha = -\frac{1}{2}$, formula (2) holds for every entire function f of exponential type 2τ such that $x^{2\alpha+1}(f(x)+f(-x))$ belongs to $L^1[0,\infty)$. It is therefore natural to wonder if the same can be said for other values of α too. We shall show that this is indeed the case if α is real and > -1. To be precise, we have

Theorem 1. *If* $\alpha > -1$, *then* (2) *holds for every entire function* f *of exponential type* 2τ *such that* $x^{2\alpha+1}(f(x)+f(-x))$ *belongs to* $L^1[0,\infty)$. *Besides, the series on the right-hand side of* (2) *is absolutely convergent.*

It may be mentioned that the zeros $j_1(\alpha), j_2(\alpha), \ldots, j_k(\alpha), \ldots$ are all real when α is real and > -1.

If f is integrable on $[1, X]$ for all $X > 1$ and

$$\lim_{X\to\infty} \int_1^X f(x)\,dx$$

exists, then we denote the limit by $\int_1^{\to\infty} f(x)\,dx$ and say that f is integrable in the sense of Cauchy on $[1, \infty)$. If f is integrable on $[\xi, 1]$ for all $\xi \in (0, 1)$ and

$$\lim_{\xi\to 0} \int_\xi^1 f(x)\,dx$$

exists, then we denote the limit by $\int_{\to 0}^1 f(x)dx$ and say that f is integrable in the sense of Cauchy on $(0, 1]$. If f is integrable in the sense of Cauchy on $[1, \infty)$ as well as on $(0, 1]$, then we say that f is integrable (in the sense of Cauchy) on $(0, \infty)$ and denote the integral by

$$\int_{\to 0}^{\to\infty} f(x)\,dx \ .$$

The question arises if in Theorem 1 it would be enough to assume the function $x^{2\alpha+1}(f(x)+f(-x))$ to be integrable in the sense of Cauchy on $(0, \infty)$. The answer is no. Indeed, the function

$$f_*(z) = \frac{J_\alpha(\tau z)J_{\alpha+1}(\tau z)}{z^{2\alpha+1}},$$

which was considered in [4, §5.2] serves as a counterexample. This function is of exponential type 2τ and is integrable in the sense of Cauchy on $(0, \infty)$; the value of the integral is $\frac{1}{\tau}$ [8, p.406] whereas the right-hand side of (2) is 0. However, we shall prove the following

Theorem 2. *Let $\alpha > -1$ and f be an entire function of exponential type $\sigma < 2\tau$ such that $x^{2\alpha+1}(f(x) + f(-x))$ is integrable in the sense of Cauchy on $(0, \infty)$. Then*

$$(3) \qquad \int_{\to 0}^{\to \infty} x^{2\alpha+1}(f(x) + f(-x))\, dx$$

$$= \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^{\infty} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \left(f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right),$$

if the series on the right is convergent.

The following result was also proved in [4] to which we refer the reader for the definition of the nodal function ω.

Theorem B. *Let $\alpha > -1$. Among all quadrature formulae of the form*

$$\int_0^\infty x^{2\alpha+1}(f(x) + f(-x))\, dx = \sum_{k=-\infty}^{\infty} \lambda_k f(x_k)$$

having ω as nodal function, only one is valid for all entire functions of exponential type 2τ satisfying $f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, $\delta > 2\alpha+2$. This formula is (2) and the associated nodal function is

$$\omega_e(x) := \frac{(j_m(\alpha))^\alpha}{J_\alpha'(j_m(\alpha))} \frac{J_\alpha(\tau x)}{x^\alpha(\tau x - j_m(\alpha))}.$$

Moreover, ω_e minimizes the integral

$$\int_0^\infty x^{2\alpha+1}(\omega^2(x) + \omega^2(-x))\, dx$$

over all nodal functions ω.

The proof of Theorem B as given in [4] allows us, in view of Theorem 1, to state the following

Theorem 3. *Theorem B holds if instead of "$f(x) = O(|x|^{-\delta})$, $x \to \pm\infty$, $\delta > 2\alpha + 2$" we assume that $x^{2\alpha+1}(f(x) + f(-x))$ belongs to $L^1[0, \infty)$.*

As an application of Theorem 1 we prove

Theorem 4. *If f is an entire function of exponential type τ such that*

$$(4) \qquad\qquad |x|^{\alpha+\frac{1}{2}} f(x) \in L^2(\mathbb{R})$$

for some $\alpha > -1$, then

$$(5) \qquad \int_{-\infty}^{\infty} |x|^{2\alpha+1} |f(x)|^2\, dx = \frac{2}{\tau^{2\alpha+2}} \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{|j_k(\alpha)|^{2\alpha}}{(J_\alpha'(|j_k(\alpha)|))^2} \left| f\left(\frac{j_k(\alpha)}{\tau}\right) \right|^2,$$

where $j_{-k}(\alpha) = -j_k(\alpha)$ for all $k \neq 0$.

In the case $\alpha = -\frac{1}{2}$, (5) reduces to the following well-known formula of Plancherel and Pólya ([7, p.116, formula (52)])

$$\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{\pi}{\tau} \sum_{k=-\infty}^{\infty} \left| f\left(\frac{(2k-1)\pi}{2\tau}\right) \right|^2.$$

Here is an obvious corollary of Theorem 4.

Corollary 1. *If f is an entire function of exponential type π such that (4) holds for some $\alpha > -1$ and f vanishes at all the points*

$$\frac{j_k(\alpha)}{\pi} \quad , \quad k = \pm 1, \pm 2, \pm 3, \dots ,$$

then f is identically zero.

This result is to be compared with a classical theorem of F. Carlson according to which, if f is an entire function of exponential type $< \pi$ and vanishes at the points $n = 0, \pm 1, \pm 2, \dots$, then it is identically zero. We refer the reader to [2, Chapter 9] for various extentions of Carlson's theorem. It may be noted that in the above corollary the function is allowed to be of order 1 type π as long as (4) holds.

Remark 1. The example $\frac{J_\alpha(\pi z)}{(\pi z)^\alpha}$ shows that the conclusion of the above corollary may not hold if (4) is not satisfied.

Remark 2. If a function is of exponential type π satisfying (4), then it may not be identically zero unless it vanishes at *all* the points $\frac{j_k(\alpha)}{\pi}$, $k = \pm 1, \pm 2, \dots$. This is shown by the example

$$(6) \qquad\qquad f(z) := \frac{J_\alpha(\pi z)}{(\pi z)^\alpha (\pi z - j_n(\alpha))} \quad ,$$

where $n \in \{\pm 1, \pm 2, \pm 3, \dots\}$.

As a special case of Theorem 4 we have

Corollary 2. *If f satisfies the conditions of Theorem 4, then*

$$\left| f\left(\frac{j_n(\alpha)}{\tau}\right) \right| \leq \frac{\tau^{\alpha+1}}{\sqrt{2}} \frac{|J'_\alpha(j_n(\alpha))|}{|j_n(\alpha)|^\alpha} \left(\int_{-\infty}^{\infty} |x|^{2\alpha+1} |f(x)|^2 \, dx \right)^{1/2} \quad (n = 1, 2, \dots) .$$

The inequality is sharp; indeed, it becomes an equality for the function defined in (6).

2. Auxiliary Results

Let $\phi(t)$ be a nonnegative, nondecreasing, and convex (nonconcave) function of $\log t$, with $\phi(0+) = 0$ but $\phi(t)$ not identically zero. Further, let $\phi(e^{-t}) = O(t^{-1/2})$ as $t \to +\infty$. It was proved by Boas [1, Theorem 2] that if f is regular and of exponential type c in the closed right half-plane such that for some $\mu > 0$, $\lambda > 2$,

$$\int_0^\infty \{\phi(|f(x)|^\mu)\}^\lambda \, dx < \infty ,$$

then for an arbitrary increasing sequence of positive numbers $\lambda_0, \lambda_1, \lambda_2, \dots$ with $\lambda_{k+1} - \lambda_k \geq 2\delta > 0$ we have

$$\sum_{k=0}^{\infty} \left\{\phi(e^{-c'\mu\delta}|f(\lambda_k)|^\mu)\right\}^\lambda < \infty ,$$

if $c' > c$. Going through the proof of this result, the reader will notice an obvious misprint in [1, Lemma 4] concerning the assumption " $\phi(e^{-t}) = O(t^{-1/2})$ as $t \to +\infty$ ". The conditions imposed on ϕ are clearly satisfied by $\phi(t) = t$.

Given a sequence $\lambda_1, \lambda_2, \lambda_3, \ldots$ such that $\lambda_1 > 0$, $\lambda_{k+1} - \lambda_k \geq 2\delta > 0$, and a function f regular and of exponential type in the open right half-plane we may apply the above result of Boas to the function $f(z + \frac{1}{2}\lambda_1)$ taking $\phi(t) = t$, $\mu = \frac{1}{3}$, $\lambda = 3$ to obtain

Lemma 1. *Let $\{\lambda_k\}$ be an increasing sequence of positive numbers with $\lambda_{k+1} - \lambda_k \geq 2\delta > 0$. If f is regular and of exponential type in the open right half-plane such that*

$$\int_0^\infty |f(x)|\, dx < \infty ,$$

then

$$\sum_{k=1}^\infty |f(\lambda_k)| < \infty .$$

Hereafter, let

$$\varphi_h(z) := \left(\frac{\sin hz}{hz} \right)^2$$

where $h > 0$.

Lemma 2. *We have*

(7)
$$\int_{-\infty}^\infty |\varphi_h'(x)|\, dx \leq 2\pi .$$

Besides, for $\alpha > -1$ and all $\gamma > 0$

$$\sum_{k=1}^\infty |\varphi_h(\gamma j_{k+1}(\alpha)) - \varphi_h(\gamma j_k(\alpha))| \leq \pi .$$

Proof. Since φ_h is an entire function of exponential type $2h$ belonging to $L^1(-\infty, \infty)$, we have [2, Theorem 11.3.1]

$$\int_{-\infty}^\infty |\varphi_h'(x)|\, dx \leq 2h \int_{-\infty}^\infty |\varphi_h(x)|\, dx = 2\pi .$$

Further, using the fundamental theorem of integral calculus, we have

$$\sum_{k=1}^\infty |\varphi_h(\gamma j_{k+1}(\alpha)) - \varphi_h(\gamma j_k(\alpha))| = \gamma \sum_{k=1}^\infty \left| \int_{j_k(\alpha)}^{j_{k+1}(\alpha)} \varphi_h'(\gamma x)\, dx \right|$$

$$\leq \frac{1}{2} \int_{-\infty}^\infty |\varphi_h'(x)|\, dx \leq \pi . \quad \square$$

Lemma 3 [5, §2.2]. *Let f be regular and of exponential type in the open right half-plane. If f is integrable in the sense of Cauchy on $[1, \infty)$, then $f(x) \to 0$ as $x \to \infty$, and so $f(x)$ is bounded for $x \geq 1$.*

Lemma 4 [5, Lemma 1]. *If f is integrable in the sense of Cauchy on $[1, \infty)$, then $\int_1^{\to\infty} \varphi_h(x) f(x)\, dx$ exists for all $h > 0$ and*

$$\lim_{h \to 0} \int_1^{\to\infty} \varphi_h(x) f(x)\, dx = \int_1^{\to\infty} f(x)\, dx$$

provided f is bounded on $[1, \infty)$.

Lemma 4′. *If* f *is integrable in the sense of Cauchy on* $(0, 1]$, *then*

$$\int_{\to 0}^{1} \varphi_h(x) f(x) \, dx$$

exists for all $h > 0$ *and*

(8) $$\lim_{h \to 0} \int_{\to 0}^{1} \varphi_h(x) f(x) \, dx = \int_{\to 0}^{1} f(x) \, dx .$$

Proof. For every $\varepsilon > 0$, there exists $X_\varepsilon > 0$ such that

$$\left| \int_{x_1}^{x_2} f(x) \, dx \right| < \varepsilon \qquad \text{for } 0 < x_1 < x_2 \leq X_\varepsilon .$$

Hence, if $F(x) := \int_x^{X_\varepsilon} f(t) dt$ $(0 < x \leq X_\varepsilon)$, then

(9) $$|F(x)| < \varepsilon \qquad \text{for } 0 < x \leq X_\varepsilon .$$

Let $0 < x_1 < x_2 \leq X_\varepsilon$. Integration by parts gives

$$\int_{x_1}^{x_2} \varphi_h(x) f(x) \, dx = -\varphi_h(x_2) F(x_2) + \varphi_h(x_1) F(x_1) + \int_{x_1}^{x_2} \varphi_h'(x) F(x) \, dx ,$$

which by virtue of (7) and (9) implies

(10) $$\left| \int_{x_1}^{x_2} \varphi_h(x) f(x) \, dx \right| \leq 2(1 + \pi)\varepsilon ,$$

i.e., $\int_{\to 0}^{1} \varphi_h(x) f(x) \, dx$ exists for all $h > 0$.

Note that (10) holds for all h and so, using (9), we obtain

$$\left| \int_{\to 0}^{X_\varepsilon} \varphi_h(x) f(x) \, dx - \int_{\to 0}^{X_\varepsilon} f(x) \, dx \right| \leq (3 + 2\pi)\varepsilon .$$

Since

$$\left| \int_{X_\varepsilon}^{1} \varphi_h(x) f(x) \, dx - \int_{X_\varepsilon}^{1} f(x) \, dx \right| < \varepsilon$$

for all sufficiently small h, we conclude that (8) holds. □

Remark 3. The above proof differs only in some details from that of Lemma 4 as presented in [5].

Lemma 5. *If* f *satisfies the conditions of Theorem* 1, *then the series on the right-hand side of* (2) *is absolutely convergent.*

Proof. Let

$$\lambda_k := \frac{1}{\tau} j_k(\alpha) \qquad (k = 1, 2, \dots) .$$

Since $j_1(\alpha), j_2(\alpha), \dots, j_k(\alpha), \dots$, are all simple, positive and (see [8, p. 506])

(11)
$$j_k(\alpha) = \left(k + \frac{1}{2}\alpha - \frac{1}{4} \right) \pi - \frac{4\alpha^2 - 1}{8\pi(k + \frac{1}{2}\alpha - \frac{1}{4})}$$
$$- \frac{(4\alpha^2 - 1)(28\alpha^2 - 31)}{384\pi^3(k + \frac{1}{2}\alpha - \frac{1}{4})^3} - \cdots ,$$

there exists a positive number δ such that $\lambda_{k+1} - \lambda_k \geq 2\delta$.

According to hypothesis we have

$$\int_0^\infty |x|^{2\alpha+1} |f(x) + f(-x)| \, dx \; < \; \infty$$

and so by Lemma 1

(12)
$$\sum_{k=1}^\infty \left(\frac{1}{\tau} j_k(\alpha)\right)^{2\alpha+1} \left| f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right|$$

$$= \sum_{k=1}^\infty |\lambda_k|^{2\alpha+1} |f(\lambda_k) + f(-\lambda_k)| \; < \; \infty \, .$$

Now we recall the formula ([9, p. 368], [6, p. 198 (Lemma 14)])

$$J_\alpha'(z) \; = \; -\left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{|z|^{3/2}}\right) \, , \; z \in \mathbb{R}, \quad z \to \infty \, .$$

It follows from (11) that for all large k

$$\left| \sin\left(j_k(\alpha) - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right) \right| \; > \; \frac{1}{\sqrt{2}}.$$

So, if k is sufficiently large (say $k \geq K_1$), then

(13)
$$|J_\alpha'(j_k(\alpha))| \; \geq \; \left(\frac{1}{\pi j_k(\alpha)}\right)^{1/2} - O\left(\left(\frac{1}{j_k(\alpha)}\right)^{3/2}\right)$$

$$> \; \frac{1}{\sqrt{2}\sqrt{\pi j_k(\alpha)}} \, ,$$

and

$$\frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \left| f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right|$$

$$< \; 2\pi (j_k(\alpha))^{2\alpha+1} \left| f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right| \, .$$

Besides, $J_\alpha'(j_k(\alpha)) \neq 0$ for all k since the positive zeros of J_α are all simple. Hence, in view of (12), the lemma holds. □

3. PROOFS OF THE THEOREMS

Proof of Theorem 1. Without loss of generality we may assume $\tau = \pi$. Then

$$\psi_\varepsilon(z) := \varphi_\varepsilon(z) \cdot (f(z) + f(-z)) \qquad (\varepsilon > 0)$$

is an *even* entire function of exponential type $2(\pi + \varepsilon)$.

The function $z^{2\alpha+1}(f(z) + f(-z))$ is regular and of exponential type in $\mathfrak{Re}\, z > 0$. Besides, according to our assumption, it belongs to $L^1[0, \infty)$ and so by Lemma 3

$$|x|^{2\alpha+1} |f(x) + f(-x)| \; \to \; 0 \qquad \text{as} \;\; x \to \infty.$$

Hence,

$$|f(x) + f(-x)| = O\left(|x|^{-(2\alpha+1)}\right) \qquad \text{as} \ x \to \pm\infty,$$

which implies that

$$|\psi_\varepsilon(x)| = O\left(|x|^{-(2\alpha+3)}\right) \qquad \text{as} \ x \to \pm\infty.$$

Thus the function $\psi_\varepsilon(z)$ satisfies the conditions of Theorem A with $2(\pi + \varepsilon)$ instead of 2τ. We therefore have

$$(14) \qquad \int_0^\infty x^{2\alpha+1}(2\psi_\varepsilon(x)) \, dx = \frac{2}{(\pi+\varepsilon)^{2\alpha+2}} \sum_{k=1}^\infty \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \cdot 2\psi_\varepsilon\left(\frac{j_k(\alpha)}{\pi+\varepsilon}\right).$$

Next we show that for each $\delta > 0$ there exists K_0 such that

$$(15) \qquad \frac{2}{(\pi+\varepsilon)^{2\alpha+2}} \sum_{k=K_0}^\infty \left| \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \psi_\varepsilon\left(\frac{j_k(\alpha)}{\pi+\varepsilon}\right) \right| < \delta$$

if $\varepsilon \in [0, \frac{1}{2}]$. In view of (11) we can choose an integer K_2 such that for $\varepsilon \in [0, \frac{1}{2}]$,

$$(16) \qquad \frac{1}{\pi+\varepsilon}(j_{k+1}(\alpha) - j_k(\alpha)) > \frac{1}{2} \qquad \text{for } k \ge K_2.$$

Let $F(x) := x^{2\alpha+1}(f(x) + f(-x))$ and $\xi_k \ (1 \le k < \infty)$ be the smallest number in $[k - \frac{1}{2}, k + \frac{1}{2}]$ such that

$$|F(\xi_k)| = \max_{k-\frac{1}{2} \le x \le k+\frac{1}{2}} |F(x)|.$$

The points ξ_k which lie in $[k - \frac{1}{2}, k)$ form a subsequence $\{\xi_n'\}$ and those which lie in $[k, k + \frac{1}{2}]$ form another subsequence $\{\xi_n''\}$. Obviously $\xi_{n+1}' - \xi_n' \ge \frac{1}{2}$ and $\xi_{n+1}'' - \xi_n'' \ge \frac{1}{2}$ and so by Lemma 1

$$\sum_{n=1}^\infty |F(\xi_n')| < \infty, \qquad \sum_{n=1}^\infty |F(\xi_n'')| < \infty.$$

As such,

$$\sum_{k=1}^\infty |F(\xi_k)| < \infty.$$

Therefore there exists $K_0 > \max\{K_1, K_2\}$ such that

$$8 \sum_{k=[\frac{6}{7}K_0]}^\infty |F(\xi_k)| < \delta.$$

By (13),

$$\left| \frac{2}{(\pi+\varepsilon)^{2\alpha+2}} \sum_{k=K_0}^\infty \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \cdot \psi_\varepsilon\left(\frac{j_k(\alpha)}{\pi+\varepsilon}\right) \right|$$

$$\le \frac{4\pi}{(\pi+\varepsilon)} \sum_{k=K_0}^\infty \left| \frac{j_k(\alpha)}{\pi+\varepsilon} \right|^{2\alpha+1} \left| f\left(\frac{j_k(\alpha)}{\pi+\varepsilon}\right) + f\left(-\frac{j_k(\alpha)}{\pi+\varepsilon}\right) \right|$$

$$= \frac{4\pi}{\pi+\varepsilon} \sum_{k=K_0}^\infty \left| F\left(\frac{j_k(\alpha)}{\pi+\varepsilon}\right) \right|.$$

Because of (16), intervals of the form $[k - \frac{1}{2}, k + \frac{1}{2}]$ can contain at most two points of the sequence $\left\{ \frac{j_k(\alpha)}{\pi + \varepsilon} \right\}_{k \geq K_2}$. Hence,

$$\frac{4\pi}{\pi + \varepsilon} \sum_{k=K_0}^{\infty} \left| F\left(\frac{j_k(\alpha)}{\pi + \varepsilon} \right) \right| \leq 4 \times 2 \sum_{k=[\frac{4}{5}K_0]}^{\infty} |F(\xi_k)| < \delta ,$$

i.e., (15) holds.

It is clear that as $\varepsilon \to 0$ the function

$$\frac{1}{(\pi + \varepsilon)^{2\alpha+2}} \psi_\varepsilon(z) \quad \text{tends to} \quad \frac{1}{\pi^{2\alpha+2}} (f(z) + f(-z))$$

uniformly on all compact subsets of $\mathbb{C}$. Hence, for any $\delta > 0$ and any $K \in \mathbb{N}$ there exists a positive ε_0 depending on δ and K such that

$$(17) \quad \left| \frac{2}{\pi^{2\alpha+2}} \sum_{k=1}^{K} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \left(f\left(\frac{j_k(\alpha)}{\pi} \right) + f\left(-\frac{j_k(\alpha)}{\pi} \right) \right) \right.$$
$$\left. - \frac{2}{(\pi + \varepsilon)^{2\alpha+2}} \sum_{k=1}^{K} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \psi_\varepsilon \left(\frac{j_k(\alpha)}{\pi + \varepsilon} \right) \right| < \delta$$

if $\varepsilon \in [0, \varepsilon_0]$.

Now let

$$I(f) := \int_0^\infty x^{2\alpha+1} (f(x) + f(-x)) \, dx$$

and

$$Q_\tau(f) := \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^{\infty} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \left(f\left(\frac{j_k(\alpha)}{\tau} \right) + f\left(-\frac{j_k(\alpha)}{\tau} \right) \right) .$$

Then

$$\begin{aligned} |I(f) - Q_\pi(f)| &= |I(f) + I(\psi_\varepsilon) - I(\psi_\varepsilon) - Q_\pi(f)| \\ &\leq |I(f) - I(\psi_\varepsilon)| + |Q_\pi(f) - I(\psi_\varepsilon)| \\ &= |I(f) - I(\psi_\varepsilon)| + |Q_\pi(f) - Q_{\pi+\varepsilon}(\psi_\varepsilon)| , \end{aligned}$$

where in the last step we have used (14). By Lemmas 3, 4 and 4', for each $\delta > 0$ there exists $\varepsilon_1 > 0$ such that

$$|I(f) - I(\psi_\varepsilon)| < \delta \qquad \text{for } 0 < \varepsilon < \varepsilon_1 .$$

Further, by (15), (17) and Lemma 5 we conclude that

$$|Q_\pi(f) - Q_{\pi+\varepsilon}(\psi_\varepsilon)| < 3\delta$$

for all $\varepsilon \in (0, \min\{\varepsilon_0, \frac{1}{2}\})$. Hence, for $\varepsilon \in (0, \min\{\varepsilon_0, \frac{1}{2}, \varepsilon_1\})$ we have

$$|I(f) - Q_\pi(f)| < 4\delta .$$

Since δ is an arbitrary positive number, we must have $I(f) = Q_\pi(f)$. This completes the proof of Theorem 1, since the absolute convergence of the series on the right-hand side of (2) has already been proved in Lemma 5. $\square$

Proof of Theorem 2. The function

$$\psi_\varepsilon(z) := \varphi_\varepsilon(z)(f(z) + f(-z))$$

satisfies the conditions of Theorem 1 for all $\varepsilon \in (0, \tau - \frac{\sigma}{2})$ and therefore $I(\psi_\varepsilon) = Q_\tau(\psi_\varepsilon)$. From Lemmas 3, 4 and 4' it follows that

$$\int_{\to 0}^{\to \infty} x^{2\alpha+1} \psi_\varepsilon(x)\, dx \longrightarrow \int_{\to 0}^{\to \infty} x^{2\alpha+1}(f(x) + f(-x))\, dx \qquad \text{as } \varepsilon \to 0.$$

Let δ be an arbitrary positive number and choose the smallest N_δ such that

$$\left| \sum_{k=N_\delta}^{n} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \left(f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right) \right| < \delta$$

for all $n \geq N_\delta$. Setting

$$S_n := \sum_{k=N_\delta}^{n} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \left(f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right),$$

we see that for $n > N_\delta$ we have

$$\left| \sum_{k=N_\delta}^{n} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \psi_\varepsilon\left(\frac{j_k(\alpha)}{\tau}\right) \right|$$

$$= \left| \sum_{k=N_\delta}^{n} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \varphi_\varepsilon\left(\frac{j_k(\alpha)}{\tau}\right) \left(f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right) \right|$$

$$= \left| \varphi_\varepsilon\left(\frac{j_n(\alpha)}{\tau}\right) \cdot S_n + \sum_{k=N_\delta}^{n-1} \left(\varphi_\varepsilon\left(\frac{j_k(\alpha)}{\tau}\right) - \varphi_\varepsilon\left(\frac{j_{k+1}(\alpha)}{\tau}\right) \right) \cdot S_k \right|$$

$$< \left(\left| \varphi_\varepsilon\left(\frac{j_n(\alpha)}{\tau}\right) \right| + \sum_{k=N_\delta}^{\infty} \left| \varphi_\varepsilon\left(\frac{j_k(\alpha)}{\tau}\right) - \varphi_\varepsilon\left(\frac{j_{k+1}(\alpha)}{\tau}\right) \right| \right) \cdot \delta$$

$$< (1 + \pi)\delta$$

by Lemma 2. Therefore,

$$\left| \sum_{k=N_\delta}^{\infty} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \psi_\varepsilon\left(\frac{j_k(\alpha)}{\tau}\right) \right| < (1 + \pi)\delta.$$

The remainder of the proof is analogous to that of Theorem 1. □

Proof of Theorem 4. If f satisfies the conditions of Theorem 4, then applying Theorem 1 to the function $f(z)\overline{f(-\bar{z})}$, we get

$$
(18) \qquad
\begin{aligned}
\int_0^\infty & x^{2\alpha+1} \Re e(f(x)\overline{f(-x)})\, dx \\
&= \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^{\infty} \frac{(j_k(\alpha))^{2\alpha}}{(J_\alpha'(j_k(\alpha)))^2} \Re e\left(f\left(\frac{j_k(\alpha)}{\tau}\right) \overline{f\left(-\frac{j_k(\alpha)}{\tau}\right)} \right),
\end{aligned}
$$

where the series on the right-hand side of (18) is absolutely convergent.

Again, formula (2) applied to the function $(f(z) + f(-z))(\overline{f(\bar{z}) + f(-\bar{z})})$ gives

$$\int_0^\infty x^{2\alpha+1} |f(x) + f(-x)|^2 \, dx$$

$$= \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^\infty \frac{(j_k(\alpha))^{2\alpha}}{(J'_\alpha(j_k(\alpha)))^2} \left| f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right|^2 .$$

Now writing $|f(x) + f(-x)|^2$ as $|f(x)|^2 + |f(-x)|^2 + 2\Re e(f(x)\overline{f(-x)})$ and

$$\left| f\left(\frac{j_k(\alpha)}{\tau}\right) + f\left(-\frac{j_k(\alpha)}{\tau}\right) \right|^2$$

as

$$\left| f\left(\frac{j_k(\alpha)}{\tau}\right) \right|^2 + \left| f\left(-\frac{j_k(\alpha)}{\tau}\right) \right|^2 + 2\Re e\left(f\left(\frac{j_k(\alpha)}{\tau}\right) \overline{f\left(-\frac{j_k(\alpha)}{\tau}\right)} \right)$$

and using (18), we obtain

$$\int_0^\infty x^{2\alpha+1} \left(|f(x)|^2 + |f(-x)|^2 \right) \, dx$$

$$= \frac{2}{\tau^{2\alpha+2}} \sum_{k=1}^\infty \frac{(j_k(\alpha))^{2\alpha}}{(J'_\alpha(j_k(\alpha)))^2} \left(\left| f\left(\frac{j_k(\alpha)}{\tau}\right) \right|^2 + \left| f\left(-\frac{j_k(\alpha)}{\tau}\right) \right|^2 \right) ,$$

which is equivalent to (5). $\square$

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DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE, UNIVERSITÉ DE MONTRÉAL, MONTRÉAL, QUÉBEC, CANADA H3C 3J7
E-mail address, G. Grozev: grozevg@ere.umontreal.ca
E-mail address, Q. Rahman: rahmanqi@ere.umontreal.ca